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Dynamics of Functionally Separable Solutions for a Class of Nonlinear Wave Equations*

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Abstract. Using the method of planar dynamical systems to a nonlinear wave equations having functionally separable solutions, the existence of all smooth and non-smooth bounded solutions is recognized. In different regions of the parametric space, the sufficient conditions to guarantee the existence of the above solutions are given. The existence of uncountably infinite many breaking bounded wave solutions is discussed. Some exact explicit parametric representations of functionally separable solutions are given.

AMS Subject Classifications: 34C25-28, 58F05, 58F14, 58F30

Keywords: Functionally separable solution; Periodic wave solution; Smoothness of wave; Nonlinear wave equation

1. Introduction

In 2002, P.G.Estevez and C.Z.Qu [2] used the generalized conditional symmetry (GCS) approach to study the functional separation of variables for the nonlinear wave equation

$$u_{tt} = (B(u)u_x)_x + A(u), \qquad B(u) \neq 0. \tag{1.1}$$

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The authors of [2] have established that the equation

$$u_{tt} = (u^p u_x)_x + au + bu^{p+1} \tag{1.2}$$

admits the product separable solution

$$u(x,t) = \phi(t)\psi(x), \tag{1.3}$$

where $\phi(t)$ and $\psi(x)$ satisfy respectively the following two systems

$$\phi'' - a\phi - \lambda\phi^{p+1} = 0, \tag{1.4$_a$}$$

$$(\psi^p\psi')' + b\psi^{p+1} - \lambda\psi = 0, \tag{1.4$_b$}$$

λ denotes the separation constant, $p \in \mathbf{R}$ is a constant and a, b are arbitrary real parameters.

By using some integral formulas, the solutions of ϕ and ψ have been written implicitly in [2]. What are the dynamical behavior of ϕ and ψ? Can we have the exact explicit parametric representations for some bounded solutions of (1.2)? To our knowledge, it has not been considered before. We notice that the approach of the bifurcation theory of dynamical systems can provide clear understanding for all bounded solutions with the form (1.3) of (1.2). We are considering physical models in the 4-parameter space (a, b, p, λ) where only bounded solutions are meaningful. In this paper, we shall study of all possible bounded solutions for (1.4), which will be characterized in the parameter space.

We first notice that

(1) if $p = -1$ or $p = 0$, then (1.4_a) is a linear system. When $p = -1, a < 0$, there exists a family of periodic solutions of (1.4_a) with the parametric representation $\phi(t) = (-\frac{\lambda}{a}) + A\cos\sqrt{-a}t$, which surrounds the equilibrium point $(-\frac{\lambda}{a}, 0)$ in the $(\phi, \phi')-$phase plane, where A is an arbitrary real constant. When $p = 0$, $(a + \lambda) < 0$, there is a family of periodic solutions of (1.4_a) with the parametric representation $\phi(t) = A\cos\sqrt{-(a+\lambda)}t$,, which surrounds the equilibrium point $(0, 0)$ in the $(\phi, \phi')-$phase plane.

(2) If $p = 0$, then (1.4_b) is a linear system. When $(b - \lambda) < 0$, there is a family of periodic solutions of (1.4_b) with the parametric representation $\psi(x) = A\cos\sqrt{\lambda - b}x$,, which surrounds the equilibrium point $(0, 0)$ in the $(\psi, \psi')-$phase plane.

We next consider the case $p \neq -1$ or $p \neq 0$ for (1.4_a) and $p \neq 0$ for (1.4_b).

Equations (1.4_a) and (1.4_b) are respectively equivalent to the following two-dimensional systems

$$\begin{cases} \frac{d\phi}{dt} = y, & \frac{dy}{dt} = \frac{\lambda\phi + a\phi^{q+1}}{\phi^q}, \quad -q = p < -1; \\ \frac{d\phi}{dt} = y, & \frac{dy}{dt} = \phi^{1-q}(\lambda + a\phi^q), \quad -1 < p = -q < 0; \\ \frac{d\phi}{dt} = y, & \frac{dy}{dt} = a\phi + \lambda\phi^{p+1}, \quad p > 0 \end{cases} \tag{1.5}$$

and

$$\begin{cases} \frac{d\psi}{dx} = z, & \frac{dz}{dx} = \frac{qz^2 - b\psi^2 + \lambda\psi^{q+2}}{\psi}, \quad -q = p < 0; \\ \frac{d\psi}{dx} = z, & \frac{dz}{dx} = \frac{-p\psi^p z^2 + \lambda\psi^2 - b\psi^{p+2}}{\psi^{p+1}}, \quad 0 < p < 1; \\ \frac{d\psi}{dx} = z, & \frac{dz}{dx} = \frac{-p\psi^{p-1} z^2 + \lambda\psi - b\psi^{p+1}}{\psi^{p-1}}, \quad 1 \leq p < 2; \\ \frac{d\psi}{dx} = z, & \frac{dz}{dx} = \frac{-p\psi^{p-2} z^2 + \lambda - b\psi^{p}}{\psi^{p-1}}, \quad p \geq 2. \end{cases} \tag{1.6}$$

System (1.5) has the first integral for $p \neq -2$,

$$H(\phi, y) = y^2 - a\phi^2 - \frac{2\lambda}{p+2}\phi^{p+2} = h_1 \tag{1.7}$$

and for $p = -2$,

$$H(\phi, y) = y^2 - a\phi^2 - 2\lambda \ln \phi = h_1. \tag{1.8}$$

System (1.6) has the first integral for $p \neq -1, -2$,

$$H(\psi, z) = \psi^{2p} z^2 + \frac{b}{p+1}\psi^{2(p+1)} - \frac{2\lambda}{p+2}\psi^{p+2} = h_2, \tag{1.9}$$

for $p = -1$,

$$H(\psi, z) = (z^2 + 2b\psi^2 \ln \psi - 2\lambda\psi^3)/\psi^2 = h_2 \tag{1.10}$$

and for $p = -2$,

$$H(\psi, z) = (z^2 - b\psi^2 - 2\lambda\psi^4 \ln \psi)/\psi^4 = h_2. \tag{1.11}$$

Systems (1.5) and (1.6) are planar dynamical systems defined in the 3-parameter spaces (a, λ, p) and (b, λ, p), respectively. Because the phase orbits defined by the vector fields of (1.5) and (1.6) determine all functionally separable solutions of (1.2), we will investigate bifurcations of phase portraits of these system, as their parameters are varied. Of course, we only concern with their bounded solutions.

We notice that when $p < -1$ (or $p \neq 0$), the right-hand side of the first equation in (1.5) (or (1.6)) is not continuous when $\phi = 0$ ($\psi = 0$). In other words, on such straight lines in the phase plane of (ϕ, y) ((ψ, z)), the function ϕ_t'' (ψ_x'') is not defined. The occurrence of breaking in the solutions of ((1.5) ((1.6)) (i.e., the phenomenon that a solution remains bounded but its slope becomes unbounded in finite "time" ("space coordinate")) is a very interesting and important problem. This phenomenon has been considered before (see [4-7]).

The rest of this paper is organized as follows. In Section 2,3, we discuss bifurcations of phase portraits of (1.5) and (1.6) for the parameter group (p, λ, a) and (p, λ, b) are varied respectively. Explicit parametric conditions will be derived. In Section 4, for some fixed p and the values of the parameters a, b, λ, we give some explicit exact parametric representations for the solutions of (1.2).

2. Bifurcations of phase portraits of system (1.5)

In this section, we discuss the dynamical behavior for the system (1.5). First, we consider the case $q = -p > 1$ in system (1.5). The system has the same phase orbits as the following system

$$\frac{d\phi}{d\tau} = y\phi^q, \quad \frac{dy}{d\tau} = \lambda\phi + a\phi^{q+1}, \quad q > 1, \tag{2.1}$$

except for the straight line $\phi = 0$, where $dt = \phi^q d\tau,\ \phi \neq 0$. Now, $\phi = 0$ is a integral straight line of (2.1).

When $a\lambda < 0$, depending to the different values of $q \in \mathbf{R}^+$, (2.1) has two (or three) equilibrium points at $O(0,0),\ E(\phi_1, 0)$ (or O, E and $E_1(-\phi_1, 0)$), where $\phi_1 = (-\frac{\lambda}{a})^{\frac{1}{q}} = (-\frac{a}{\lambda})^{\frac{1}{p}}$. In fact, if q is an irrational number, we only can consider $\phi > 0$. If $q = \frac{2k+1}{m} > 1,\ k, m \in \mathbf{N}$, then there is not the equilibrium point E_1.

Let $M(\phi_e, y_e)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point (ϕ_e, y_e). Then we have

$$J(0,0) = detM(0,0) = 0, \quad J(\phi_1, 0) = detM(\phi_1, 0) = -\frac{q\lambda^2}{a}.$$

By the theory of planar dynamical systems, we know that for an equilibrium point of a planar integrable system, if $J < 0$, then the equilibrium point is a saddle point; if $J > 0$ and $Trace(M(\phi_e, y_e)) = 0$ then it is a center point; if $J > 0$ and $(Trace(M(\phi_e, y_e))^2 - 4J(\phi_e, y_e) > 0$, then it is a node; if $J = 0$ and the index of the equilibrium point is 0, then it is a cusp. If the index of the equilibrium point is not 0 or ± 1, then it is a high-order critical point.

We see from the above fact that for $a > 0$ (< 0), the equilibrium point E is a saddle point (center point). The equilibrium point $(0,0)$ is a high-order critical point.

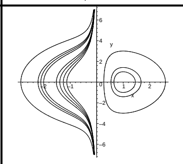
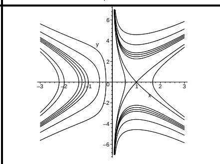
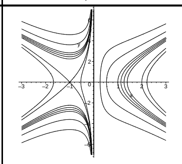
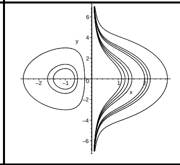
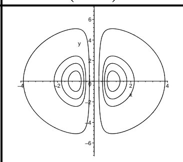
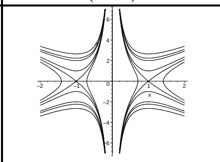
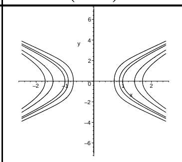
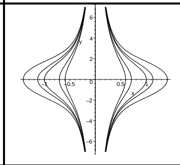
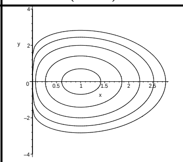
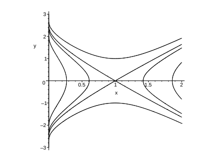
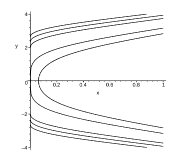
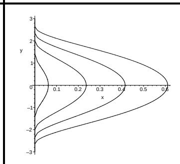
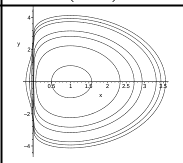
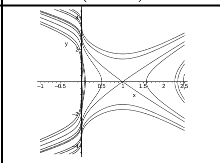
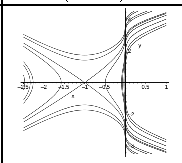
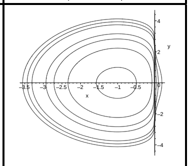
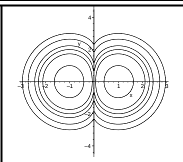
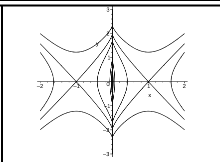
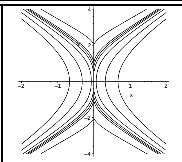
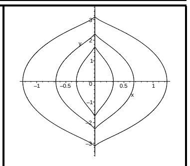
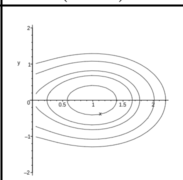
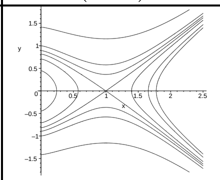
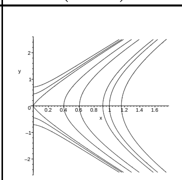
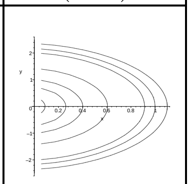
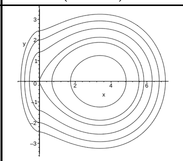
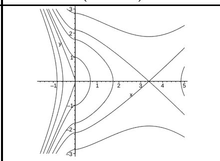
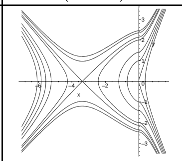
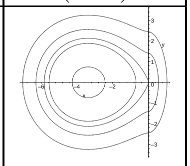
Denote that

$$h_{11} = H(\phi_1, 0) = \begin{cases} \frac{aq}{2-q}\left(-\frac{\lambda}{a}\right)^{\frac{2}{q}}, & q \neq 2, \\ \lambda\left(1 - 2\ln\left(-\frac{\lambda}{a}\right)\right), & q = 2. \end{cases}$$

For a fixed h, the level curve $H(\phi, y) = h$ defined by (1.7) and (1.8) determines a set of invariant curves of (2.1), which contains different branches of curves. As h is varied, it defines different families of orbits of (2.1), with different dynamical behaviors (see the discussion below).

By using the above discussion, we obtain the phase portraits of (2.1) shown in Fig. 1 (1-1)-(1-20), where $k, m \in \mathbf{N}$. We emphasize that if q is an irrational number, then we have the phase portraits in the right phase plane $\phi > 0$ in Fig.1.

Secondly, we consider the second and third systems in (1.5), we have the phase portraits of (2.1) shown in Fig. 1 (1-21)-(1-40), where $k, m \in \mathbf{N}$.

$p\ (q=-p)$	$a<0,\ \lambda>0$	$a>0,\ \lambda<0$	$a>0,\ \lambda>0$	$a<0,\ \lambda<0$
$q=2k+1>2$				
	(1-1)	(1-2)	(1-3)	(1-4)
$q=2k\geq 2$				
	(1-5)	(1-6)	(1-7)	(1-8)
$1<q=\frac{2m+1}{2k}<2$				
	(1-9)	(1-10)	(1-11)	(1-12)
$1<q=\frac{2m+1}{2k+1}<2$				
	(1-13)	(1-14)	(1-15)	(1-16)
$1<q=\frac{2m}{2k+1}<2$				
	(1-17)	(1-18)	(1-19)	(1-20)
$0<q=\frac{2m+1}{2k}<1$				
	(1-21)	(1-22)	(1-23)	(1-24)
$0<q=\frac{2m+1}{2k+1}<1$				
	(1-25)	(1-26)	(1-27)	(1-28)

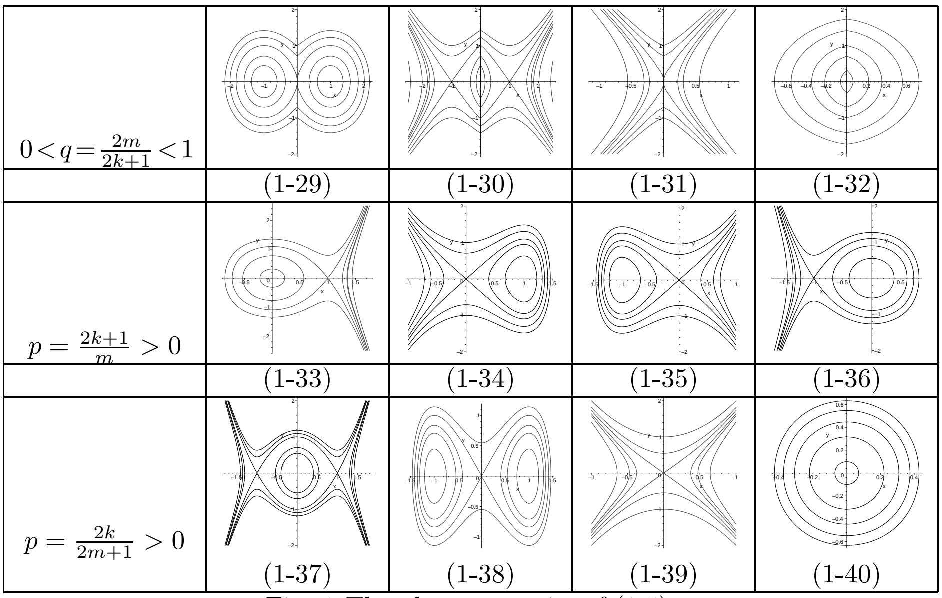


Fig. 1 The phase portraits of (1.5).

3. Bifurcations of phase portraits of system (1.6)

In this section, we discuss the dynamical behavior for the system (1.6). System (1.6) has the same phase orbits as the following system

$$\frac{d\psi}{d\zeta}=z\psi,\quad \frac{dz}{d\zeta}=qz^2-b\psi^2+\lambda\psi^{q+2},\qquad q=-p>0, \tag{3.1}$$

$$\frac{d\psi}{d\zeta}=\psi^{p+1}z,\quad \frac{dz}{d\zeta}=-p\psi^{p}z^2+\lambda\psi^2-b\psi^{p+2},\quad 0<p<1, \tag{3.2}$$

$$\frac{d\psi}{d\zeta}=\psi^{p}z,\quad \frac{dz}{d\zeta}=-p\psi^{p-1}z^2+\lambda\psi-b\psi^{p+1},\quad 1\le p<2, \tag{3.3}$$

$$\frac{d\psi}{d\zeta}=\psi^{p-1}z,\quad \frac{dz}{d\zeta}=-p\psi^{p-2}z^2+\lambda-b\psi^{p},\qquad p\ge 2, \tag{3.4}$$

except for the straight line $\psi=0$, where $dx=\psi d\zeta$ for (3.1), $dx=\psi^{p+1}d\zeta$ for (3.2), $dx=\psi^{p}d\zeta$ for (3.3) and $dx=\psi^{p-1}d\zeta$ for (3.4). Now, $\psi=0$ is a integral straight line of (3.1)-(3.4).

The equilibrium points of (3.1)-(3.4) at the origin $O(0,0)$ and $S_1(\psi_1,0)$, $\psi_1=\left(\frac{\lambda}{b}\right)^{\frac{1}{p}}$. If $p=2k,\ k\in\mathbf{N}$, then $S_2(-\psi_1,0)$ is also an equilibrium points of (3.1)-(3.4).

Let $M(\psi_e,y_e)$ be the coefficient matrix of the linearized system of (3.1)-(3.4) at an equilibrium point (ϕ_e,y_e). Then we have

$$J(0,0)=detM(0,0)=0,$$

and

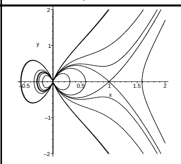
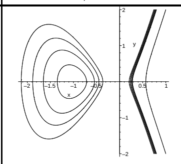
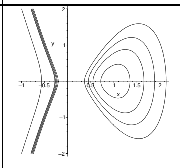
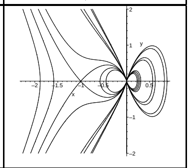
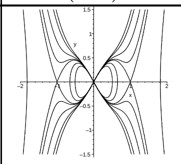
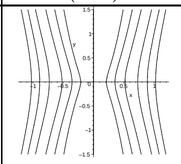
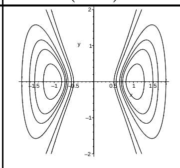
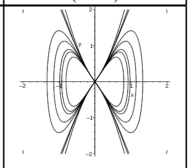
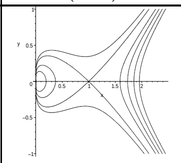
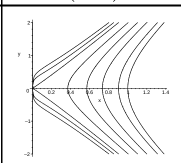
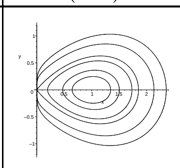
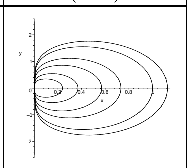
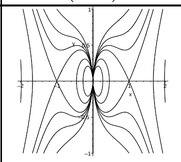
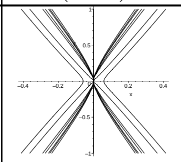
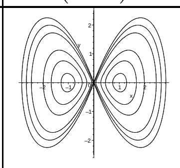
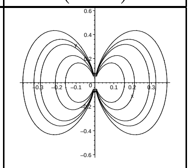
$$J(\phi_1, 0) = detM(\psi_1, 0) = -qb\psi_1^2, \quad -\frac{p\lambda^2}{b}\psi_1^2, \quad -\frac{p\lambda^2}{b}, \quad bp\psi_1^{2(p-1)},$$

respectively, for (3.1)-(3.4).

Denote that

$$h_{21} = H(\psi_1, 0) = \begin{cases} -\frac{p\lambda}{(p+1)(p+2)}\left(\frac{\lambda}{b}\right)^{\frac{p+2}{p}}, & p \neq -1, \ -2, \\ -2b\left(1 - \ln\left(\frac{b}{\lambda}\right)\right), & p = -1, \\ -\lambda\left(1 + \ln\left(\frac{b}{\lambda}\right)\right), & p = -2. \end{cases}$$

By using the above information, we have the bifurcations of phase portraits of (1.6) shown in Fig.2 (2-1)-(2-40).

$p\ (q=-p)$	$b>0,\ \lambda>0$	$b<0,\ \lambda>0$	$b<0,\ \lambda<0$	$b>0,\ \lambda<0$
$q=\frac{2m+1}{2k+1}\geq 1$				
	(2-1)	(2-2)	(2-3)	(2-4)
$q=\frac{2m}{2k+1}\geq 1$				
	(2-5)	(2-6)	(2-7)	(2-8)
$0<q=\frac{2m+1}{2k+1}<1$				
	(2-9)	(2-10)	(2-11)	(2-12)
$0<q=\frac{2m}{2k+1}<1$				
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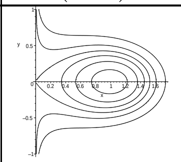
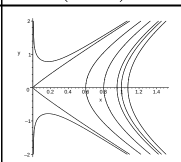
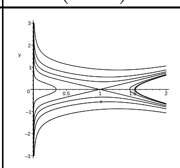
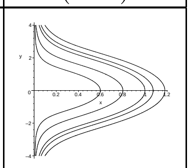
$0<p=\frac{2m+1}{2k+1}<2$				
	(2-17)	(2-18)	(2-19)	(2-20)

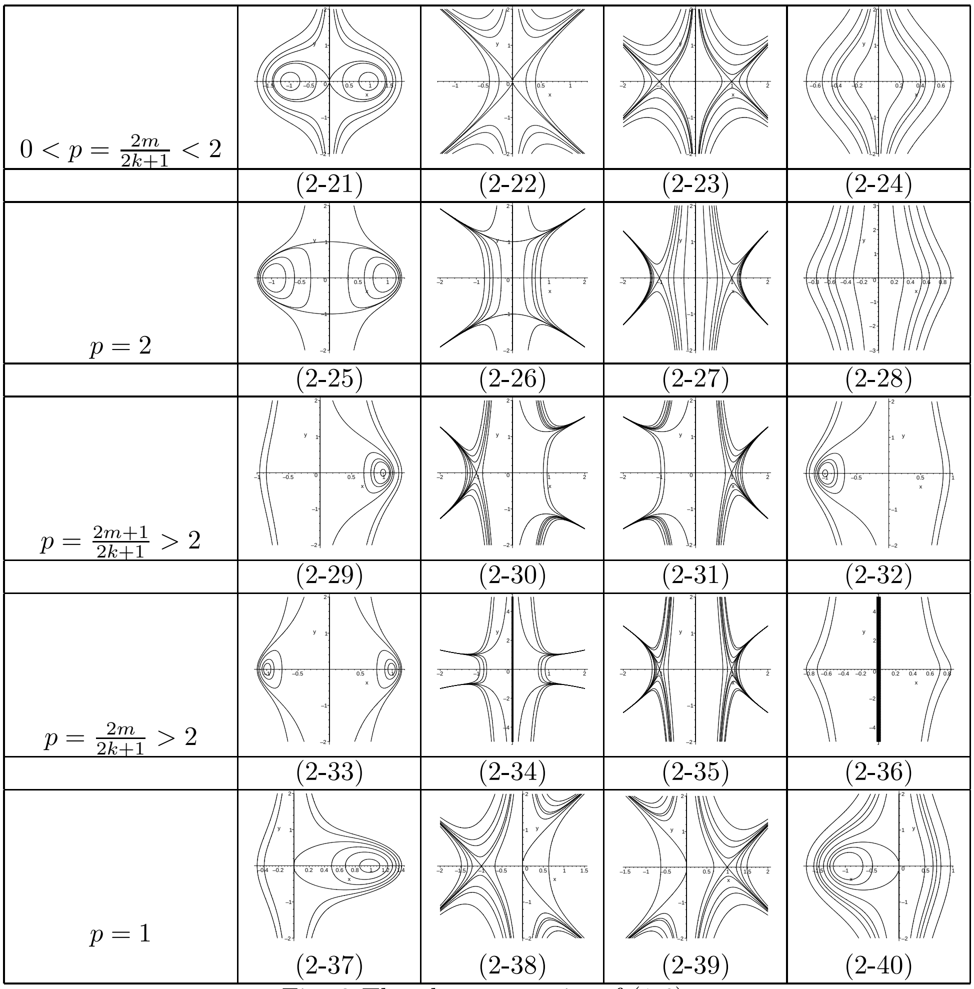


Fig. 2 The phase portraits of (1.6).

4. Exact explicit solutions and breaking bounded solutions of (1.2)

Generally, for any given $p \in \mathbf{R}$, except $p = -1, 0, 1, 2$, and for a given arbitrary parameter group (a, b, λ), we can not obtain the exact explicit parametric representations of $\phi(t)$ and $\psi(x)$. But, by using the phase portraits in Fig.1 and Fig.2, we know the information of all bounded solutions of (1.2).

We next consider some special case of p.

1. The case $p = 0$, i.e., we consider the linear partial differential equation

$$u_{tt} = u_{xx} + (a + b)u. \tag{4.1}$$

From section 1, we know that for any real number $A > 0$, the solutions of (4.1) have the explicit parametric representations:

$$u(x,t) = A\cos(\sqrt{-(a+\lambda)}t\cos(\sqrt{\lambda - b}x), \tag{4.2}$$

if $a + \lambda < 0$ and $\lambda - b > 0$.

2. The case $p = 1$, i.e., we consider the non-linear partial differential equation

$$u_{tt} = (uu_x)_x + au + bu^2 \tag{4.3}$$

and the corresponding two ordinary differential equations

$$\phi'' - a\phi - \lambda\phi^2 = 0, \qquad (\psi\psi')' - \lambda\psi + b\psi^2 = 0. \tag{4.4}$$

By using the third systems of (1.5) and (1.6) to calculate, we have the following conclusions.

(1) When $a < 0,\ \lambda > 0$, the homoclinic orbit defined by $H(\phi, y) = -\frac{a^3}{3\lambda^2} = h_{11}$ in Fig.1 (1-33) has the parametric representation

$$\phi(t) = \left(-\frac{a}{\lambda}\right)\left[1 - \frac{3}{2}sech^2\frac{\sqrt{-a}}{2}t\right]. \tag{4.5}$$

We see from (1.7) with $p = 1$ that $y^2 = h_1 + a\phi^2 + \frac{2\lambda}{3}\phi^3$. For $h_1 \in (0, -\frac{a^3}{3\lambda^2})$, it can be written as $y^2 = \frac{2\lambda}{3}\left[\frac{3h_1}{2\lambda} + \frac{3a}{2\lambda}\phi^2 + \phi^3\right] = \frac{2\lambda}{3}\left[(\alpha - \phi)(\beta - \phi)(\phi - \gamma)\right]$. Thus, the family of periodic orbits defined by $H(\phi, y) = h_1$ in Fig.1 (1-33) has the parametric representation

$$\phi(t) = \gamma + (\beta - \gamma)sn^2\left(\sqrt{\frac{\lambda(\alpha - \gamma)}{6}}t, \sqrt{\frac{\beta - \gamma}{\alpha - \gamma}}\right). \tag{4.6}$$

(2) When $a > 0,\ \lambda < 0$, the homoclinic orbit defined by $H(\phi, y) = 0$ in Fig.1 (1-34) has the parametric representation

$$\phi(t) = \left(-\frac{3a}{2\lambda}\right)sech(\sqrt{a}t). \tag{4.7}$$

For $h_1 \in (-\frac{a^2}{3\lambda^2}, 0)$, from (1.7) with $p = 1$, we have

$$y^2 = \frac{2(-\lambda)}{3}\left[\frac{3h_1}{2(-\lambda)} + \frac{3a}{2(-\lambda)}\phi^2 - \phi^3\right] = \frac{2\lambda}{3}\left[(\alpha - \phi)(\phi - \beta)(\phi - \gamma)\right].$$

Thus, the family of periodic orbits defined by $H(\phi, y) = h_1$ in Fig.1 (1-34) has the parametric representation

$$\phi(t) = \alpha - (\alpha - \beta)sn^2\left(\sqrt{\frac{(-\lambda)(\alpha - \gamma)}{6}}t, \sqrt{\frac{\alpha - \beta}{\alpha - \gamma}}\right). \tag{4.8}$$

(3) When $a > 0,\ \lambda > 0$, the homoclinic orbit defined by $H(\phi, y) = 0$ in Fig.1 (1-35) has the same parametric representation as (4.7). The family of periodic orbits defined by $H(\phi, y) = h_1,\ h_1 \in (h_{11}, 0)$ in Fig.1 (1-35) has the same parametric representation as (4.6).

(4) When $a < 0,\ \lambda < 0$, the homoclinic orbit defined by $H(\phi, y) = h_{11}$ in Fig.1 (1-36) has the same parametric representation as (4.5). The family of periodic orbits defined by $H(\phi, y) = h_1,\ h_1 \in (0, h_{11})$ in Fig.1 (1-36) has the same parametric representation as (4.8).

(5) When $b > 0,\ \lambda > 0$ or $b > 0,\ \lambda < 0$, the oval orbit defined by $H(\psi, z) = 0$ in Fig.2(2-37) or (2-40) has the following parametric representation

$$\psi(x) = \frac{2\lambda}{3b}\left(1 + \cos\sqrt{\frac{b}{2}}x\right). \tag{4.9}$$

The family of periodic orbits defined by $H(\psi, z) = h_2,\ h_2 \in (h_{21}, 0)$ can not have explicit parametric representation, where $h_{21} = -\frac{\lambda^4}{6b^3}$.

To sum up, we obtain

Proposition 4.1 *(i) For $p = 1,\ a < 0,\ b > 0,\ \lambda > 0$, we have the following two forms of the parametric representations of bounded solutions of (4.3):*

$$u(x,t) = \left(-\frac{2a}{3b}\right)\left(1 + \cos\sqrt{\frac{b}{2}}x\right)\left[1 - \frac{3}{2}sech^2\frac{\sqrt{-a}}{2}t\right]. \tag{4.10}$$

$$u(x,t) = \left(\frac{2\lambda}{3b}\right)\left(1 + \cos\sqrt{\frac{b}{2}}x\right)\left(\gamma + (\beta - \gamma)sn^2\left(\sqrt{\frac{\lambda(\alpha - \gamma)}{6}}t, \sqrt{\frac{\beta - \gamma}{\alpha - \gamma}}\right)\right). \tag{4.11}$$

(ii) For $p = 1,\ a > 0,\ b > 0,\ \lambda > 0$, we have the following two forms of the parametric representations of bounded solutions of (4.3)

$$u(x,t) = \left(-\frac{2a}{3b}\right)\left(1 + \cos\sqrt{\frac{b}{2}}x\right)sech(\sqrt{a}t). \tag{4.12}$$

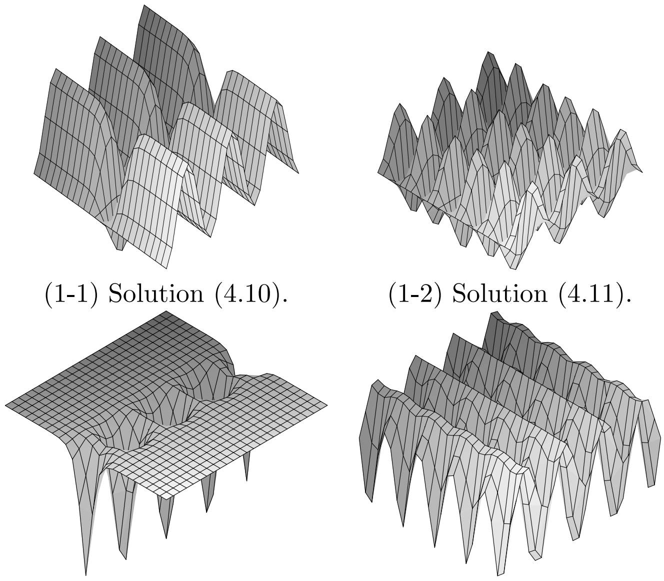
and (4.11).

(iii) For $p = 1,\ a > 0,\ b > 0,\ \lambda < 0$, we have the following two forms of the parametric representations of bounded solutions of (4.3)

$$u(x,t) = \left(\frac{2\lambda}{3b}\right)\left(1 + \cos\sqrt{\frac{b}{2}}x\right)\left(\beta + (\alpha - \beta)cn^2\left(\sqrt{\frac{\lambda(\alpha - \gamma)}{6}}t, \sqrt{\frac{\alpha - \beta}{\alpha - \gamma}}\right)\right) \tag{4.13}$$

and (4.12).

Fig.3 shows the three dimensional pictures for the above solutions.



(1-1) Solution (4.10). (1-2) Solution (4.11).

(1-3) Solution (4.12). (1-4) Solution (4.13).

Fig. 3 The pictures of solutions of (4.3).

3. The case $p = 2$, i.e., we consider the non-linear partial differential equation

$$u_{tt} = (u^2 u_x)_x + au + bu^3 \tag{4.14}$$

and the corresponding two ordinary differential equations

$$\phi'' - a\phi - \lambda\phi^3 = 0, \qquad (\psi^2\psi')' - \lambda\psi + b\psi^3 = 0. \tag{4.15}$$

Similar to the discussion in **2**, we have the following conclusions.

(1) When $a < 0,\ \lambda > 0$, the heteroclinic orbits defined by $H(\phi, y) = \frac{a^2}{2\lambda} = h_{11}$ in Fig.1 (1-37) have the parametric representations

$$\phi(t) = \pm\sqrt{\frac{-a}{\lambda}}\tanh\sqrt{\frac{\lambda}{2}}t. \tag{4.16}$$

We see from (1.7) with $p = 2$ that $y^2 = h_1 + a\phi^2 + \frac{\lambda}{2}\phi^4$. For $h_1 \in (0, \frac{a^2}{2\lambda})$, it can be written as $y^2 = \frac{\lambda}{2}\left[\frac{2h_1}{\lambda} + \frac{2a}{\lambda}\phi^2 + \phi^4\right] = \frac{\lambda}{2}\left[(\alpha^2 - \phi^2)(\beta^2 - \phi^2)\right]$, where $\alpha^2 = \frac{1}{\lambda}[-a + \sqrt{a^2 - 2h_1\lambda}]$, $\beta^2 = \frac{1}{\lambda}[-a - \sqrt{a^2 - 2h_1\lambda}]$. Thus, the family of periodic orbits defined by $H(\phi, y) = h_1$ in Fig.1 (1-37) has the parametric representation

$$\phi(t) = \beta sn\left(\sqrt{\frac{\lambda}{2}}\alpha t, \frac{\beta}{\alpha}\right). \tag{4.17}$$

(2) When $a > 0,\ \lambda < 0$, two homoclinic orbits defined by $H(\phi, y) = 0$ in Fig.1 (1-38) have the parametric representations

$$\phi(t) = \pm\sqrt{\frac{2a}{-\lambda}}sech\sqrt{\frac{a(-\lambda)}{2}}t. \tag{4.18}$$

For $h_1 \in (\frac{a^2}{2\lambda}, 0)$, from (1.7) with $p = 2$, we have $y^2 = \frac{(-\lambda)}{2}\left[\frac{2h_1}{(-\lambda)} + \frac{2a}{(-\lambda)}\phi^2 - \phi^4\right] = \frac{(-\lambda)}{2}\left[(\alpha^2 - \phi^2)(\phi^2 - \beta^2)\right]$. Thus, two families of periodic orbits defined by $H(\phi, y) = h_1$ in Fig.1 (1-38) have the parametric representations

$$\phi(t) = \pm\alpha dn\left(\sqrt{\frac{(-\lambda)}{2}}\alpha t, \frac{\sqrt{\alpha^2 - \beta^2}}{\alpha}\right). \tag{4.19}$$

The family of periodic orbits defined by $H(\phi, y) = h,\ h \in (0, \infty)$ enclosing three equilibrium points has the parametric representation as the following (4.20).

(3) When $a < 0,\ \lambda < 0$, for $h_1 \in (0, \infty)$, the family of periodic orbits defined by $H(\phi, y) = h_1$ i.e. $y^2 = \frac{(-\lambda)}{2}\left[\frac{2h_1}{(-\lambda)} - \frac{2a}{\lambda}\phi^2 - \phi^4\right] = \frac{(-\lambda)}{2}\left[(\alpha^2 + \phi^2)(\beta^2 - \phi^2)\right]$ in Fig.1 (1-40) has the parametric representation

$$\phi(t) = \beta cn\left(\sqrt{\frac{(-\lambda)(\alpha^2 + \beta^2)}{2}}t, \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}\right). \tag{4.20}$$

(4) When $b > 0,\ \lambda > 0$ the two arch orbits defined by $H(\psi, z) = 0$ in Fig.2(2-25) have the following parametric representations

$$\psi(x) = \pm\sqrt{\frac{3\lambda}{2b}}\cos\left(\sqrt{\frac{b}{3}}x\right), \quad -\sqrt{\frac{3}{b}}\frac{\pi}{2} < x < \sqrt{\frac{3}{b}}\frac{\pi}{2}. \tag{4.21}$$

Two families of periodic orbits defined by $H(\psi, z) = h_2,\ h_2 \in (h_{21}, 0)$ can not have explicit parametric representation, where $h_{21} = -\frac{\lambda^3}{6b^2}$.

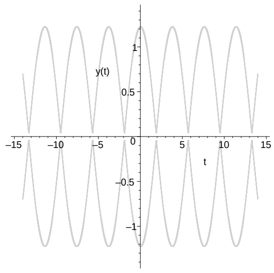
Proposition 4.2 *For $p = 2,\ a < 0,\ b > 0,\ \lambda > 0$, we have the following two forms of the parametric representations of bounded solutions of (4.14):*

$$u(x,t) = \pm\sqrt{\frac{3(-a)}{2b}}\cos\left(\sqrt{\frac{b}{3}}x\right)\tanh\sqrt{\frac{\lambda}{2}}t, \quad t \in \mathbf{R}, \ -\sqrt{\frac{3}{b}}\frac{\pi}{2} < x < \sqrt{\frac{3}{b}}\frac{\pi}{2} \tag{4.22}$$

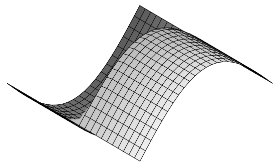
and

$$u(x,t) = \pm\beta\sqrt{\frac{3\lambda}{2b}}\cos\left(\sqrt{\frac{b}{3}}x\right)sn\left(\sqrt{\frac{\lambda}{2}}\alpha t, \frac{\beta}{\alpha}\right), \quad t \in \mathbf{R}, \ -\sqrt{\frac{3}{b}}\frac{\pi}{2} < x < \sqrt{\frac{3}{b}}\frac{\pi}{2} \tag{4.23}$$

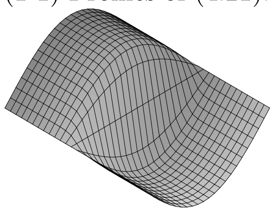
We notice that the two profiles defined by (4.21) are shown in Fig.4 (4-1). Because $\psi = 0$ is a singular straight line of the system (1.6). Therefore, two arches correspond to two cusp wave solutions. Fig.4 (4-2)-(4-4) show some bounded solutions of (4.14).



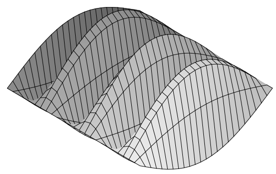
(1-1) Profiles of (4.21).



(1-2) Solution (4.22+).



(1-3) Solution (4.22-).



(1-4) Solution (4.23).

Fig. 4 The pictures of solutions of (4.14).

4. The case $p = -1$, i.e., we consider the non-linear partial differential equation

$$u_{tt} = \left(\frac{u_x}{u}\right)_x + au + b \tag{4.24}$$

and the corresponding two ordinary differential equations

$$\phi'' - a\phi - \lambda = 0, \qquad \left(\frac{\psi'}{\psi}\right)' - \lambda\psi + b = 0. \tag{4.25}$$

For $a < 0$, the first equation of (4.25) has a family of periodic solutions

$$\phi(t) = (-\frac{\lambda}{a}) + A\cos\sqrt{-a}t. \tag{4.26}$$

Unfortunately, we see from (1.10) that we can not calculate the explicit parametric representations of the solutions of the second equation of (4.25). From Fig.2 (2-1)-(2-4), we know that there exist uncountable many bounded periodic solutions or breaking solutions. For example, when $b > 0,\ \lambda > 0$, for $h_2 \in (-\infty, h_{21})$, the curve family defined by

$$z^2 = \frac{h_2 - [2b\psi^2 \ln\psi - 2\lambda\psi^3]}{\psi^2} \tag{4.27}$$

and connecting to the origin (see Fig.2 (2-1)) determine uncountable many bounded breaking solutions. The profiles are shown in Fig.5. If we denote formally these solutions as $\Psi(x)$. Then, we have the formal bounded solutions of (4.24) as follows:

$$u(x,t) = \Psi(x)\left[\left(-\frac{\lambda}{a}\right) + A\cos\sqrt{-a}t\right]. \tag{4.28}$$

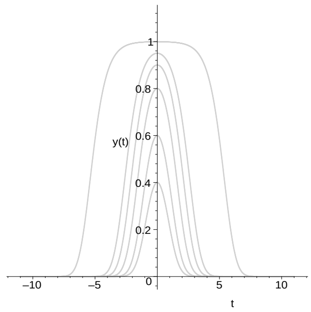


Fig. 5 The pictures of breaking solutions defined by (4.27).

5. The case $p = n \in \mathbf{N}^+$. Suppose that $a = \left(\frac{2}{n}\right)^2$, $\lambda = -\frac{2}{n}\left(\frac{n+2}{n}\right) < 0$. In this case, it is easy to check that equation $(1.4)_a$ has a explicit homoclinic solution having the parametric representation

$$\phi(t) = (sech(t))^{\frac{2}{n}}. \tag{4.29}$$

For the equation $(1.4)_b$, we see from Fig. 2 (2-31),(2-32),(2-35) and (2-36) that when $\lambda < 0$, there exist uncountable many bounded periodic solutions or breaking solutions written by $\Psi(x)$. Thus, we have the formal bounded solutions of (1.2) as follows:

$$u(x,t) = \Psi(x)(sech(t))^{\frac{2}{n}}. \tag{4.30}$$

Geometrically, for a given p and given parameter group (a, b, λ), we always can use the phase portraits given by Fig.1 and Fig.2 to do composition in order to get all bounded solutions of (1.2).

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