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Periodicity in Neutral Functional Differential Equations by Direct Fixed Point Mapping

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Abstract. Burton-Kirk's fixed point theorem or degree theory is used to study the existence of periodic solutions in neutral functional differential equations by constructing a homotopy which is a combination of a contraction mapping and compact mapping. The construction of such a homotopy is very difficult in practice for nonlinear equations. In this paper, we use the direct fixed point mapping technique to link the homotopy to the right hand side of the equation directly and avoid those difficulties.

AMS Subject Classifications: 34K13, 34K40

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1. Introduction

We consider the system of neutral functional differential equations

$$\frac{d}{dt}\Big(x(t) - \int_0^\infty [dE(s)]x(t-s)\Big) = F(t, x_t) \tag{1.1}$$

where $x(t) \in R^n$, $F : R \times C \to R^n$ is continuous with C being the Banach space of bounded continuous functions $\phi : (-\infty, 0] \to R^n$ with the supremum norm $\|\cdot\|$ and $F(t, \phi)$ is T-periodic in t for each $\phi \in C$. Here $E : R^+ \to R^{n\times n}$ is continuous to the left and of bounded variation on R^+. The assumption on E allows for $\int_0^\infty [dE(s)]x(t-s)$ to include such forms as

$$Dx(t) + Kx(t-r) + \int_{-\infty}^t G(t-s)x(s)ds$$

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in which D and K are constant matrices, $r > 0$, and $G : R^+ \to R^{n\times n}$ is continuous. For each bounded continuous function $x : R \to R^n$ and $t \in R$, x_t is defined by $x_t(s) = x(t+s)$ for all $s \leq 0$ so that $x_t \in C$.

The existence of periodic solutions of (1.1) has been the subject of extensive investigations for many years. Our interest here centers on fixed point theorems of continuation type which are nonlinear alternatives of Leray-Schauder degree theory. Continuation theorems such as Schaefer's fixed point theorem, without actually calculating degree, require less restrictive growth condition on the functions involved. For the historical background and discussion of applications, we refer the reader to, for example, the work of Burton ([1], [2]). Burton and Kirk [3], Burton and Zhang [4], Hale and Mawhin [6], Hatvani and Krisztin [7], Ma, Wang, and Yu [10], Wu, Xia, and Zhang [13], and Zhang [14]. A common method of applying fixed point theory consists of writing the differential equation as an integral equation which then defines a mapping; if the mapping has a fixed point, then it is a solution of the differential equation. In this paper, we use the direct fixed point mapping technique introduced in Burton and Zhang [4] to construct a homotopy directly from $F(t,\phi)$ and $\int_0^\infty [dE(s)]x(t-s)$. This involves writing the solution as an integral equation and it eliminates many of the problems encountered in writing the differential equation as an integral equation. Then main difficulty is in selecting the constant of integration. Examples will be given to illustrate the method of finding such a constant.

Let R^-, R^+, R denote the intervals $(-\infty, 0]$, $[0,\infty)$, and $(-\infty,\infty)$ respectively. $|\cdot|$ denotes the Euclidean norm on R^n. For an $n\times n$ matrix D, we denote the norm of D by $\|D\| = \sup\{|Dx| : \ |x| \leq 1\}$. Let $(P_T, \|\cdot\|)$ be the Banach space of continuous T-periodic functions $\phi : R \to R^n$ with the supremum norm and define

$$P_T^0 = \{\phi \in P_T : \int_0^T \phi(s)ds = 0\}$$

For $F : R\times C \to R^n$, we say $F(t,\phi)$ is continuous in $\phi \in C$ uniformly with respect to $t \in R$ if for each $\varepsilon > 0$ and $\phi_0 \in C$, there exists $\delta > 0$ such that $[\phi \in C, \|\phi - \phi_0\| < \delta]$ imply

$$|F(t,\phi) - F(t,\phi_0)| < \varepsilon \ \text{ for all } \ t \in R.$$

2. The Main Result

Our result rests on a fixed point theorem of Burton and Kirk [3] which is a combination of the contraction mapping theorem and Schaefer's theorem. The theorem may be viewed as a continuation theorem of Krasnoselskii [8] type. Continuation theorems and their relations to Leray-Schauder degree theory are discussed in Smart [12].

Theorem 2.1. (Burton-Kirk) *Let V be a Banach space, A, $B : V \to V$, B a contraction with contraction number $\alpha < 1$, and A continuous with A mapping bounded sets into compact sets. Then either* *(i*)* $x = \lambda B(x/\lambda) + \lambda Ax$ *has a solution in* V *for* $\lambda = 1$*, or*

(ii)* *the set of all such solutions,* $0 < \lambda < 1$*, is unbounded.*

Some extensions of Theorem 2.1 may be found in Dhage [5], Liu and Li [9]. It is clear that $\lambda B(x/\lambda) = B(x)$ when B is linear.

Theorem 2.2. *Suppose the following conditions hold:*

(i) *There exists a constant* γ*,* $0 < \gamma < 1$*, such that* $\displaystyle\int_0^\infty \|d[E(s)]\| = \gamma$.

(ii) *For each* $\phi \in P_T^0$*, there is a constant* $k_\phi \in R$ *such that* $\int_0^T F(t, \Phi_t)dt = 0$ *where* $\Phi(t) = k_\phi + \int_0^t \phi(s)ds$.

(iii) $\Gamma : P_T^0 \to P_T$ *defined by* $\Gamma(\phi)(t) = \Phi(t)$ *in (ii) is continuous and for each* $\alpha > 0$*, there exists a constant* $L_\alpha > 0$ *such that* $|k_\phi| \le L_\alpha$ *whenever* $\|\phi\| \le \alpha$.

(iv) $F : R \times C \to R^n$ *is continuous and maps bounded sets into bounded sets.*

(v) *There exists a constant* $B_1 > 0$ *such that* $\|x\| < B_1$ *whenever* $x = x(t)$ *is a* T*-periodic solution of*

$$\frac{d}{dt}\Big(x(t) - \int_0^\infty [dE(s)]x(t-s)\Big) = \lambda F(t, x_t), \quad \lambda \in (0, 1]. \tag{2.1}$$

Then (1.1) has a T*-periodic solution.*

Proof. First observe that $\int_0^t \phi(s)ds$ is T-periodic for each $\phi \in P_T^0$. For $B_1 > 0$ in (v), by (iv) there exists $L > 0$ such that $|F(t, \psi)| \le L$ whenever $\psi \in C$ and $\|\psi\| \le B_1$. We now define

$$(B\phi)(t) = \int_0^\infty [dE(s)]\phi(t-s) \quad \text{and} \quad (A\phi)(t) = F(t, \Phi_t).$$

It is clear that $B : P_T^0 \to P_T^0$ is a contraction. Since F is continuous, we see that $F(t, \Phi_t)$ is continuous in t. By (ii), we also find $(A\phi) \in P_T^0$. Thus, $A : P_T^0 \to P_T^0$ is well-defined and continuous since Γ in (iii) is continuous and $F(t, \phi)$ is continuous in ϕ uniformly with respect to t. □

We now show that $A : P_T^0 \to P_T^0$ is compact. Consider $\{\Gamma(\phi) : \phi \in P_T^0, \|\phi\| \le M\}$ for each $M > 0$,. This set is uniformly bounded by (iii) and equi-continuous by the definition of Φ. Thus, Γ is compact by Ascoli-Arzela's theorem. This implies that A is compact since F satisfies the continuity condition in (iv).

If there exists $\phi \in P_T^0$ such that

$$\begin{aligned} \phi(t) &= (B\phi)(t) + \lambda(A\phi)(t) \\ &= \int_0^\infty [dE(s)]\phi(t-s) + \lambda F(t, \Phi_t), \end{aligned} \tag{2.2}$$

then

$$\frac{d}{dt}\Big(\Phi(t) - \int_0^\infty [dE(s)]\Phi(t-s)\Big) = \lambda F(t, \Phi_t). \tag{2.3}$$

Thus Φ is a solution of (2.1). For $\lambda \in (0, 1]$, we have $\|\Phi\| \le B_1$ by (v), and so $|F(t, \Phi_t)| \le L$. It follows from (2.2) that

$$|\phi(t)| \le \|\phi\| \int_0^\infty \|[dE(s)]\| + |F(t, \Phi_t)| \le \gamma\|\phi\| + L$$

and hence $\|\phi\| \le L/(1-\gamma) =: B_2$. This implies that $\|\phi\| \le B_2$ whenever ϕ is a fixed point of $B + \lambda A$. By Theorem 2.1, there exists $\phi \in P_T^0$ such that (2.2) holds for $\lambda = 1$, and we assert from (2.3) that Φ is a T-periodic solution of (1.1). The proof is complete.

Finally in this section, we consider a linear form of equation (1.1)

$$\frac{d}{dt}\Big(x(t) - \int_0^\infty [dE(s)]x(t-s)\Big) = L(t, x_t) + p(t) \tag{2.4}$$

where $L : R \times C \to R^n$ is continuous, linear in ϕ, T-periodic in t with $|L(t, \phi)| \le K\|\phi\|$ for some constant $K > 0$, and $p \in P_T$. In this case, conditions (ii)-(iv) can be verified directly.

Theorem 2.3. *Suppose that (i) holds and there is an $n \times n$ matrix $L(t, \cdot)$ such that for every $k \in R^n$ there is the relation $L(t, \cdot)k = L(t, k)$. If the linear function $\int_0^T L(t, \cdot)dt$ is invertible and*

(v^*) there exists a constant $B^* > 0$ such that $\|x\| < B^*$ whenever $x = x(t)$ is a T-periodic solution of

$$\frac{d}{dt}\Big(x(t) - \int_0^\infty [dE(s)]x(t-s)\Big) = \lambda[L(t, x_t) + p(t)],\ \lambda \in (0, 1]. \tag{2.5}$$

Then (2.4) has a T-periodic solution.

Proof. Define $F(t, x_t) = L(t, x_t) + p(t)$. In view of Theorem 2.2, we need to verify conditions (ii)-(iv). Let $\phi \in P_T^0$ and $k \in R^n$. Consider

$$\int_0^T L\Big(t, (k + \int_0^t \phi(s)ds)_t\Big)dt + \int_0^T p(s)ds = 0.$$

Since L is linear with respect to the second argument, we have

$$\int_0^T L(t, k)dt + \int_0^T L\Big(t, (\int_0^t \phi(s)ds)_t\Big)dt + \int_0^T p(s)ds = 0.$$

Thus,

$$k = \Big(\int_0^T L(t, \cdot)dt\Big)^{-1}\Big[-\int_0^T L\Big(t, (\int_0^t \phi(s)ds)_t\Big)dt - \int_0^T p(s)ds\Big].$$

We designate that unique constant as k_ϕ. It is clear that $\Gamma : P_T^0 \to P_T$ defined by $\Gamma(\phi) = \Phi$ with $\Phi(t) = k_\phi + \int_0^t \phi(s)ds$ is continuous and $\int_0^T F(t, \Phi_t)dt = 0$. Observe that

$$\Big|L\Big(t, (\int_0^t \phi(s)ds)_t\Big)\Big| \leq KT\|\phi\|$$

and

$$|k_\phi| \leq \Big\|\Big(\int_0^T L(t, \cdot)dt\Big)^{-1}\Big\|(KT\alpha + \|p\|)T =: L_\alpha$$

for $\|\phi\| \leq \alpha$. Thus, (ii) and (iii) are satisfied. It is also clear that (iv) holds. This completes the proof. $\square$

Remark 2.1. If $L(t, x_t) = A(t)x(t) + \int_{-\infty}^t B(t, s)x(s)ds + \sum_{k=1}^\infty A_k(t)x(t-h_k)$, where $A(t), B(t, s)$, and $A_k(t)$ are $n \times n$ matrices and $h_k > 0$, then $\int_0^T L(t, \cdot)dt$ is invertible if and only if the matrix

$$\int_0^T \Big(A(t) + \int_{-\infty}^t B(t, s)ds + \sum_{k=1}^\infty A_k(t)\Big)dt$$

has an inverse.

3. Examples

In this section, we give examples to illustrate how to apply Theorem 2.2 and Theorem 2.3 to some linear and nonlinear equations. Our emphasis will be on proving the existence of k_ϕ described in Theorem 2.2 and the use of Liapunov functions to derive a priori bounds on periodic solutions. The examples are shown in simple forms for illustrative purposes and they can be easily generalized.

Example 3.1. *Consider the scalar equation*

$$\frac{d}{dt}\Big(x(t) - \int_{-\infty}^t C(t-s)x(s)ds\Big) = b(t)x(t) + p(t) \tag{3.1}$$

where $b, p \in P_T$ *and* $C \in L^1(R^+)$ *with* $\int_0^\infty |C(u)|du = \gamma < 1$. *If*

$$(2-\gamma)|b(t)| - \gamma\|b\| \geq \alpha \tag{3.2}$$

for some constant $\alpha > 0$, *then equation (3.1) has a* T*-periodic solution.*

Proof. In view of Theorem 2.3, we need to show that there exists $B^* > 0$ such that $\|x\| < B^*$ whenever $x = x(t)$ is a T-periodic solution of

$$\frac{d}{dt}\Big(x(t) - \int_{-\infty}^t C(t-s)x(s)ds\Big) = \lambda[b(t)x(t) + p(t)],\ \lambda \in (0, 1] \tag{3.3}$$

Let $x = x(t)$ be a T-periodic solution of (3.3) and define

$$V(t) = \Big(x(t) - \int_{-\infty}^{t} C(t-s)x(s)ds\Big)^2.$$

Without loss of generality, we may assume $b(t) < 0$. Then for $t \geq 0$, we have

$$\begin{aligned}
V'(t) &= 2\Big(x(t) - \int_{-\infty}^{t} C(t-s)x(s)ds\Big)\lambda(b(t)x(t) + p(t)) \\
&= 2\lambda b(t)x^2(t) - 2\lambda b(t)x(t)\int_{-\infty}^{t} C(t-s)x(s)ds \\
&\quad +2\lambda p(t)\Big(x(t) - \int_{-\infty}^{t} C(t-s)x(s)ds\Big) \\
&\leq -2\lambda|b(t)|x^2(t) + \lambda|b(t)|\int_{-\infty}^{t} |C(t-s)|(x^2(t) + x^2(s))ds \\
&\quad +2\lambda|p(t)|\Big(|x(t)| + \int_{-\infty}^{t} |C(t-s)||x(s)|ds\Big) \\
&\leq -2\lambda|b(t)|x^2(t) + \gamma\lambda|b(t)|x^2(t) + \lambda\|b\|\int_{-\infty}^{t} |C(t-s)|x^2(s)ds \\
&\quad +2\lambda\|p\|\Big(|x(t)| + \int_{-\infty}^{t} |C(t-s)||x(s)|ds\Big).
\end{aligned}$$

Integrate from 0 to T to obtain

$$\begin{aligned}
0 &= V(T) - V(0) \\
&\leq -\lambda(2-\gamma)\int_0^T |b(t)|x^2(t)dt + \lambda\|b\|\int_{-\infty}^{0} |C(-s)|\int_0^T x^2(t+s)dtds \\
&\quad +2\lambda\|p\|\int_0^T |x(t)|dt + 2\lambda\|p\|\int_{-\infty}^{0} |C(-s)|\int_0^T |x(t+s)|dtds \\
&= -\lambda\int_0^T [(2-\gamma)|b(t)| - \gamma\|b\|]x^2(t)dt + 2\lambda\|p\|(1+\gamma)\int_0^T |x(t)|dt \\
&\leq -\alpha\lambda\int_0^T x^2(t)dt + 2\lambda\|p\|(1+\gamma)\int_0^T |x(t)|dt. \qquad (3.4)
\end{aligned}$$

Here we have used the equality $\int_0^T x^2(t+s)dt = \int_0^T x^2(t)dt$ for any $s \in R$ since x is T-periodic. Thus, it follows from (3.4) that

$$\alpha\int_0^T x^2(t)dt \leq 2\|p\|(1+\gamma)\int_0^T |x(t)|dt \leq (\alpha/2)\int_0^T x^2(t)dt + \beta^*$$

for some constant $\beta^* > 0$. To obtain the last term above, we have applied the Cauchy inequality $2ab \leq \beta a^2 + b^2/\beta$ with $a = |x(t)|$, $b = \|p\|(1+\gamma)$, and $\beta = \alpha/2$, and hence

$\beta^* = (2/\alpha)T\|p\|^2(1+\gamma)^2$. This implies that there exists a constant $K_1 > 0$ such that

$$\int_0^T x^2(t)dt \le K_1 \text{ and } \int_0^T \Big(x(t) - \int_{-\infty}^t C(t-s)x(s)ds\Big)^2 dt \le K_1. \tag{3.5}$$

Using (3.3), (3.5), and the boundedness of $p(t)$, we also obtain

$$\int_0^T \left|\frac{d}{dt}\Big[x(t) - \int_{-\infty}^t C(t-s)x(s)ds\Big]^2\right| dt \le K_2$$

for some constant $K_2 > 0$. By Sobolev's inequality, there exists a constant $K_3 > 0$ such that

$$\sup_{0\le t\le T}\left|x(t) - \int_{-\infty}^t C(t-s)x(s)ds\right| \le K_3.$$

Thus, $\|x\| \le K_3/(1-\gamma)$. Therefore, (v*) of Theorems 2.3 is satisfied. We conclude that (3.1) has a T-periodic solution. The proof is complete. □

Example 3.2. *Consider the nonlinear scalar equation*

$$\frac{d}{dt}\Big(x(t) - ax(t-r)\Big) = b(t)x^3(t) + \int_{-\infty}^t C(t-s)x^3(s)ds + p(t) \tag{3.6}$$

where $a \in R$, $r > 0$, $b \in P_T$, $C \in L^1(R^+)$, *and* $p \in P_T^0$. *If*

$$|a|\|b\| + (1+|a|)\int_0^\infty |C(u)|du < |b(t)| \tag{3.7}$$

for $t \in [0,T]$, *then (3.6) has a* T*-periodic solution.*

Proof. We verify that all conditions of Theorem 2.2 hold. Let $\int_0^\infty [dE(s)]x(t-s) = ax(t-r)$ and

$$F(t,\phi) = b(t)\phi^3(0) + \int_{-\infty}^0 C(-s)\phi^3(s)ds + p(t).$$

It is clear that $F : R \times C \to R^n$ is continuous, T-periodic in t, and

$$F(t,x_t) = b(t)x^3(t) + \int_{-\infty}^0 C(-s)x^3(t+s)ds + p(t)$$

for each $x \in P_T$. Condition (i) of Theorem 2.2 is readily satisfied since $|a| < 1$. We now show that for $\phi \in P_T^0$, there exists $k_\phi \in R$ such that $\int_0^T F(t,\Phi_t)dt = 0$ with $\Phi(t) = k_\phi + \int_0^t \phi(s)ds$. Notice that $\int_0^t \phi(s)ds$ is T-periodic for $\phi \in P_T^0$ so that for each $s \in R$, we have

$$\int_0^T \Big(k_\phi + \int_0^{t+s}\phi(u)du\Big)^3 dt = \int_0^T \Big(k_\phi + \int_0^t \phi(u)du\Big)^3 dt = \int_0^T \Phi^3(t)dt.$$

Since $\int_0^T p(s)ds = 0$, we have

$$\begin{aligned}\int_0^T F(t,\Phi_t)dt &= \int_0^T b(t)\Phi^3(t)dt + \int_0^T\int_{-\infty}^0 C(-s)\Big(k_\phi + \int_0^{t+s}\phi(u)du\Big)^3 dsdt\\ &= \int_0^T \Big(b(t) + \int_0^\infty C(u)du\Big)\Phi^3(t)dt =: \int_0^T \theta(t)\Phi^3(t)dt.\end{aligned}$$

For $k \in R$, we define

$$Q(k) = \int_0^T \theta(t)\Big(k + \int_0^t \phi(s)ds\Big)^3 dt.$$

Without loss of generality, we may assume that $b(t) > 0$. By (3.7), there exists a constant $\theta^* > 0$ such that $\theta(t) \geq \theta^*$. Since $Q'(k) = 3\int_0^T \theta(t)\Big(k + \int_0^t \phi(s)ds\Big)^2 dt \geq 0$ and $\lim_{k\to\pm\infty} Q(k) = \pm\infty$, there exists a unique $k_\phi \in R$ such that $Q(k_\phi) = 0$. Therefore, $\int_0^T F(t,\Phi_t)dt = 0$ and (ii) holds. This also implies that $|k_\phi| \leq T\|\phi\|$.

We now show that $\Gamma : P_T^0 \to P_T$ defined by $\Gamma(\phi)(t) = \Phi(t)$ is continuous. Let $\{\phi_n\}$ be a sequence in P_T^0 and $\phi_n \to \phi \in P_T^0$ as $n \to \infty$. We first show that $k_{\phi_n} \to k_\phi$ as $n \to \infty$. By way of contradiction, if $k_{\phi_n} \not\to k_\phi$, then there exists a subsequence, say $\{k_{\phi_n}\}$ again, and $\mu > 0$ with $|k_{\phi_n} - k_\phi| \geq \mu$. By the definition of k_{ϕ_n} and k_ϕ, we have

$$\begin{aligned}0 &= \int_0^T \theta(t)\Big[\Phi_n^3(t) - \Phi^3(t)\Big]dt\\ &= \int_0^T \theta(t)\Big(\Phi_n(t) - \Phi(t)\Big)\Big[\Phi_n^2(t) + \Phi_n(t)\Phi(t) + \Phi^2(t)\Big]dt \qquad (3.8)\end{aligned}$$

where $\Phi_n(t) = k_{\phi_n} + \int_0^t \phi_n(s)ds$. Since $\phi_n \to \phi$ as $n \to \infty$, there exists a constant $Q_1 > 0$ such that $\|\phi_n\| \leq Q_1$ for all $n = 1, 2, \cdots$. Thus, $|k_{\phi_n}| \leq TQ_1$ and so there exists a subsequence $\{k_{\phi_{n_j}}\}$ of $\{k_{\phi_n}\}$ such that $k_{\phi_{n_j}} \to k_*$ as $j \to \infty$. Replace n by n_j in (3.8), let $j \to \infty$, and apply Lebesgue's convergence theorem to obtain

$$0 = \int_0^T \theta(t)(k_* - k_\phi)\Big[\Phi_*^2(t) + \Phi_*(t)\Phi(t) + \Phi^2(t)\Big]dt$$

where $\Phi_*(t) = k_* + \int_0^t \phi(s)ds$. Thus,

$$\begin{aligned}0 &\geq \frac{1}{2}|k_* - k_\phi|\theta^* \int_0^T \Big[\Phi_*^2(t) + \Phi^2(t)\Big]dt\\ &= \frac{1}{2}|k_* - k_\phi|\theta^* \int_0^T \Big[\,(k_* - k_\phi + \Phi(t))^2 + \Phi^2(t)\Big]dt\\ &\geq \frac{1}{2}|k_* - k_\phi|\theta^* \int_0^T \frac{1}{2}(k_* - k_\phi)^2 dt = \frac{1}{4}T\theta^*|k_* - k_\phi|^3 \geq \frac{1}{4}T\theta^*\mu^3 > 0,\end{aligned}$$

a contradiction. Thus, $k_{\phi_n} \to k_\phi$ as $n \to \infty$. Moreover,

$$\Big|\int_0^t \phi_n(s)ds - \int_0^t \phi(s)ds\Big| \le T\|\phi_n - \phi\| \to 0$$

as $n \to \infty$. This shows that $\|\Gamma(\phi_n) - \Gamma(\phi)\| = \|\Phi_n - \Phi\| \to 0$ as $n \to \infty$. Thus, $\Gamma : P_T^0 \to P_T$ is continuous. Also, $|k_\phi| \le T\alpha$ if $\|\phi\| \le \alpha$, and therefore, (iii) holds.

It is also clear that $F : R \times C \to R^n$ maps bounded sets into bounded sets and $F(t, \phi)$ is continuous in ϕ uniformly with respect to t, and so (iv) holds. We now show that there exists $B_1 > 0$ such that $\|x\| < B_1$ whenever $x = x(t)$ is a T-periodic solution of

$$\frac{d}{dt}\Big(x(t) - ax(t-r)\Big) = \lambda\Big[b(t)x^3(t) + \int_{-\infty}^t C(t-s)x^3(s)ds + p(t)\Big],\ \lambda \in (0,1]. \quad (3.9)$$

Let $x = x(t)$ be a T-periodic solution of (3.9) and define $V(t) = (x(t) - ax(t-r))^2$. Then for $t \ge 0$, we have

$$\begin{aligned} V'(t) &= 2(x(t) - ax(t-r))\Big[\lambda b(t)x^3(t) + \lambda\int_{-\infty}^t C(t-s)x^3(s)ds + \lambda p(t)\Big] \\ &= 2\lambda b(t)x^4(t) - 2\lambda ab(t)x(t-r)x^3(t) + 2\lambda x(t)\int_{-\infty}^t C(t-s)x^3(s)ds \\ &\quad -2\lambda ax(t-r)\int_{-\infty}^t C(t-s)x^3(s)ds + 2\lambda(x(t) - ax(t-r))p(t). \end{aligned}$$

We may still assume $b(t) > 0$ so that

$$\begin{aligned} V'(t) \ \ge\ & 2\lambda|b(t)|x^4(t) - 2\lambda|a||b(t)|\Big[\frac{3}{4}x^4(t) + \frac{1}{4}x^4(t-r)\Big] \\ & -2\lambda\int_{-\infty}^t |C(t-s)|\Big[\frac{3}{4}x^4(s) + \frac{1}{4}x^4(t)\Big]ds \\ & -2\lambda|a|\int_{-\infty}^t |C(t-s)|\Big[\frac{3}{4}x^4(s) + \frac{1}{4}x^4(t-r)\Big]ds - 2\lambda|x(t) - ax(t-r)|\|p\|. \end{aligned}$$

Integrate from 0 to T to obtain

$$\begin{aligned} 0 &= V(T) - V(0) \\ &\ge 2\lambda\int_0^T |b(t)|x^4(t)dt - 2\lambda|a|\int_0^T |b(t)|\Big[\frac{3}{4}x^4(t) + \frac{1}{4}x^4(t-r)\Big]dt \\ &\quad -2\lambda\int_0^T\int_{-\infty}^0 |C(-s)|\Big[\frac{3}{4}x^4(t+s) + \frac{1}{4}x^4(t)\Big]dsdt \\ &\quad -2\lambda|a|\int_0^T\int_{-\infty}^0 |C(-s)|\Big[\frac{3}{4}x^4(t+s) + \frac{1}{4}x^4(t-r)\Big]dsdt \end{aligned}$$

$$
\begin{aligned}
& -2\lambda \int_0^T |x(t) - ax(t-r)| \|p\| dt \\
= \quad & 2\lambda \int_0^T \Big[|b(t)| - |a| \|b\| - (1+|a|) \int_0^\infty |C(u)| du \Big] x^4(t) dt \\
& -2\lambda \int_0^T |x(t) - ax(t-r)| \|p\| dt. \qquad (3.10)
\end{aligned}
$$

Here we have use the equality $\int_0^T x^4(t+s)dt = \int_0^T x^4(t)dt$ for any $s \in R$ since x is T-periodic. By (3.7), there exists $\beta > 0$ such that $|b(t)| - |a|\|b\| - (1+|a|)\int_0^\infty |C(u)|du \geq \beta$. Thus, it follows from (3.10) that

$$
\begin{aligned}
\beta \int_0^T x^4(t) dt \quad & \leq \quad \|p\| \int_0^T |x(t) - ax(t-r)| dt \\
& \leq \quad \|p\| \left[\int_0^T |x(t)| dt + \int_0^T |x(t-r)| dt \right] \\
& = \quad 2\|p\| \int_0^T |x(t)| dt \leq (\beta/2) \int_0^T x^4(t) dt + M_1
\end{aligned}
$$

for some constant $M_1 > 0$. Here we have applied the Cauchy inequality twice

$$2ab \leq a^2 + b^2 \leq \theta a^4 + \frac{1}{4\theta} + b^2$$

with $a = |x(t)|$, $b = \|p\|$, $\theta = \beta/2$, and hence $M_1 = T\left(\frac{1}{2\beta} + \|p\|^2\right)$. This implies that there exists a constant $B_2 > 0$ such that

$$\int_0^T x^4(t) dt \leq B_2 \ \text{ and } \ \int_0^T |x(t) - ax(t-r)|^2 dt \leq B_2. \qquad (3.11)$$

Using (3.6), (3.11), and the boundedness of $p(t)$, we also obtain

$$\int_0^T \left| \frac{d}{dt}[x(t) - ax(t-r)]^2 \right| dt \leq B_3$$

for some constants $B_3 > 0$. By Sobolev's inequality, there exists a constant $B_4 > 0$ such that $|x(t) - ax(t-r)| \leq B_4$. This in turn implies that $\|x\| \leq B_4/(1-|a|)$. Therefore, condition (v) holds. Since all conditions of Theorem 2.2 are satisfied, we conclude that (3.6) has a T-periodic solution. The proof is complete. □

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