International Jr, of Qualitative Theory of Differential Equations and Applications
Vol. 3 No. 2 (December, 2017)
ISSN: 0973-3590

A New Approach to the Boundary Value Problems of Discrete Hamiltonian Systems*

Jianshe Yu, Zhan Zhou†

School of Mathematics and Information Sciences, Guangzhou University, Guangzhou, Guangdong 510006, P. R. China

Abstract. Consider the discrete Hamiltonian system,

$$\begin{cases} \Delta x_1(n) = -H_{x_2}(n, x_1(n+1), x_2(n)) \\ \Delta x_2(n) = H_{x_1}(n, x_1(n+1), x_2(n)) \end{cases}, \quad n \in \mathbb{Z}(0, m-1),$$

with the boundary value conditions

$$x_1(0) = A, \quad x_2(m) = B,$$

where m is a positive integer, $x_1, x_2 \in \mathbb{R}$, $H(n, x_1, x_2) \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ for each $n \in \mathbb{Z}(0, m-1)$. For the first time, results on the existence of solutions of such a system are established by using the critical point theory.

AMS Subject Classifications: 39A11

Keywords: Boundary value problem; Discrete Hamiltonian systems; Saddle point theorem

1. Introduction

Let $\mathbb{N}, \mathbb{Z}$, and $\mathbb{R}$ be the sets of all natural numbers, integers, and real numbers respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \cdots\}$, $\mathbb{Z}(a, b) = \{a, a+1, \cdots, b\}$ when $a \leq b$.

E-mail addresses: jsyu@gzhu.edu.cn (J. Yu), zzhou@gzhu.edu.cn (Z. Zhou)

*This project is supported by National Natural Science Foundation of China (No. 10625104) and Research Fund for the Doctoral Program of Higher Education of PR China (Grant No. 200610788002).

†Corresponding author

Recently, some scholars have investigated the discrete Hamiltonian system

$$\begin{cases} \Delta x_1(n) = -H_{x_2}(n, x_1(n+1), x_2(n)) \\ \Delta x_2(n) = H_{x_1}(n, x_1(n+1), x_2(n)) \end{cases}, \quad n \in \mathbb{Z}$$

for the disconjugacy, oscillations and asymptotic behavior [3, 5, 7, 8, 11]. There are also few papers which deal with the existence of periodic and subharmonic solutions of discrete Hamiltonian systems [9, 17] by using the critical point theory. As for the existence of periodic solutions to the general difference equations, we refer to [1, 2, 10, 16].

Unlike the discrete case, the corresponding continuous Hamiltonian system

$$\begin{cases} \dfrac{dx}{dt} = -H_y(t, x(t), y(t)) \\ \dfrac{dy}{dt} = H_x(t, x(t), y(t)) \end{cases} \tag{1.1}$$

has been studied extensively by many scholars. In particular, using the critical point theory, Benci, Chang, Mahwin, Rabinowitz and others have obtained some significant results for the existence of periodic and subharmonic solutions of (1.1). We refer to [4, 6, 12, 13, 14, 15] and the references therein for details.

In this paper, we shall consider the discrete Hamiltonian system,

$$\begin{cases} \Delta x_1(n) = -H_{x_2}(n, x_1(n+1), x_2(n)) \\ \Delta x_2(n) = H_{x_1}(n, x_1(n+1), x_2(n)) \end{cases}, \quad n \in \mathbb{Z}(0, m-1), \tag{1.2}$$

with boundary value conditions

$$x_1(0) = A, \quad x_2(m) = B, \tag{1.3}$$

where $m \in \mathbb{N}$, $x_1, x_2 \in \mathbb{R}$, $H(n, x_1, x_2) \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ for each $n \in \mathbb{Z}(0, m-1)$, $\Delta x_i(n) = x_i(n+1) - x_i(n), i = 1, 2$. The main purpose of this paper is to establish the existence of the boundary value problem (BVP for short) of (1.2) and (1.3) by using the critical point theory. The main idea is to set up a suitable variational framework for (1.2)—(1.3) such that the existence of solutions to (1.2) and (1.3) is equivalent to the existence of critical points of the variational functional. This is the first time in the literature that the critical point theory is used to deal with the BVP of discrete Hamiltonian systems.

For BVP (1.2) and (1.3), define a functional F on $\mathbb{R}^{2m}$ by

$$\begin{aligned} F(x) \;=\; & \sum_{n=1}^{m-1} (\Delta x_1(n), x_2(n)) + \sum_{n=0}^{m-1} H(n, x_1(n+1), x_2(n)) \\ & + (x_1(1) - A)x_2(0) - Bx_1(m), \end{aligned} \tag{1.4}$$

where

$$x = (x_1(1), x_1(2), \cdots, x_1(m), x_2(0), x_2(1), \cdots, x_2(m-1))^T \in \mathbb{R}^{2m}.$$

After a simple computation, we can prove the following result.

Lemma 1.1. *There is a one to one correspondence between the critical points of the functional F and the solutions of BVP (1.2) and (1.3). More precisely, $x = (x_1(1), x_1(2), \cdots, x_1(m), x_2(0), x_2(1), \cdots, x_2(m-1))^T$ is a critical point of F if and only if $\{x_1(n), x_2(n)\}_{n=0}^{m}$ with $x_1(0) = A$ and $x_2(m) = B$ is a solution of BVP (1.2) and (1.3).*

From Lemma 1.1, the existence of solutions to BVP (1.2) and (1.3) is transferred to the existence of critical points of the functional F.

Rewrite F as

$$F(x) = \frac{1}{2}(Dx, x) + \sum_{n=0}^{m-1} H(n, x_1(n+1), x_2(n)) - (\eta, x), \tag{1.5}$$

where $\eta = (\eta_1, \eta_2, \cdots, \eta_{2m})^T$ with $\eta_i = 0$ for $i = 1, \cdots, m-1, m+2, \cdots, 2m$ and $\eta_m = B, \eta_{m+1} = A$,

$$D = \begin{pmatrix} 0 & Q \\ Q^T & 0 \end{pmatrix}_{2m\times 2m} \quad \text{with} \quad Q = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{m\times m}.$$

It is easy to see that λ is an eigenvalue of D if and only if λ^2 is an eigenvalue of Q^TQ. Since

$$Q^TQ = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}_{m\times m}$$

is positive definite, Q^TQ has m positive eigenvalues denoted by $\lambda_1^2, \lambda_2^2, \cdots, \lambda_m^2$, where $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$. Then the eigenvalues of D are$\lambda_{\pm 1}, \lambda_{\pm 2}, \cdots, \lambda_{\pm m}$, where $\lambda_{-j} = -\lambda_j$ for $j = 1, \ldots, m$.

Let X_1, X_2 denote the eigenspaces associated with all negative eigenvalues and all positive eigenvalues of D, respectively. Then

$$\mathbb{R}^{2m} = X_1 \oplus X_2.$$

For any $x = (x_1, x_2, \cdots, x_{2m})^T \in \mathbb{R}^{2m}$ and $r > 1$, define

$$\|x\|_r = \left(\sum_{i=1}^{2m} x_i^r\right)^{\frac{1}{r}}.$$

Then $\|\cdot\|_r$ is a norm in $\mathbb{R}^{2m}$. Clearly, the usual Euclidean norm $|x| = \|x\|_2$. Since $\mathbb{R}^{2m}$ is a finite dimensional space, $\|\cdot\|_2$ and $\|\cdot\|_r$ are equivalent, i.e. there exist positive constants C_{1r} and C_{2r} such that

$$C_{1r}\|x\|_r \leq \|x\|_2 \leq C_{2r}\|x\|_r, \forall x \in \mathbb{R}^{2m}. \tag{1.6}$$

Moreover, we have

$$-\lambda_m\|u\|_2^2 \leq (Du, u) \leq -\lambda_1\|u\|_2^2, \text{ for } u \in X_1, \tag{1.7}$$

$$\lambda_1\|v\|_2^2 \leq (Dv, v) \leq \lambda_m\|v\|_2^2, \text{ for } v \in X_2. \tag{1.8}$$

Here, $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^{2m}$.

Now we recall some basic conceptions and lemmas in the critical point theory.

Let X be a real Banach space, $I \in C^1(X, \mathbb{R})$, i.e. I is a continuously Fréchet differentiable functional defined on X. I is said to satisfy the Palais-Smale condition (P-S condition), if any sequence $\{u_n\} \subset X$ for which $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0 (n \to \infty)$ possesses a convergent subsequence in X.

Let B_r denote the open ball in X about 0 of radius r and let ∂B_r denote its boundary.

Lemma 1.2. *(Saddle Point Theorem) (see [15]): Let X be a real Banach space, $X = X_1 \oplus X_2$, where $X_1 \neq \{0\}$ and is finite dimensional. Suppose $I \in C^1(X, \mathbb{R})$ satisfies the P-S condition and*

($\mathbf{I}_1$) there exist constants σ, $\rho > 0$ such that $I|_{\partial B_\rho \cap X_1} \leq \sigma$, and

($\mathbf{I}_2$) there is $e \in B_\rho \cap X_1$ and a constant $\omega > \sigma$ such that $I|_{e+X_2} \geq \omega$.

Then I possesses a critical value $c \geq \omega$ and

$$c = \inf_{h\in\Gamma} \max_{u \in B_\rho \cap X_1} I(h(u)),$$

where $\Gamma = \{h \in C(\bar{B}_\rho \cap X_1, X) : h|_{\partial B_\rho \cap X_1} = id\}$.

Let

$$H(n, x_1, x_2) = H(n, z), \quad n \in \mathbb{Z}(0, m-1),$$

where $z = (x_1, x_2)^T \in \mathbb{R}^2$. The rest of this paper is organized as follows. In section 2, we consider the case where H is subquadratic. Then in scetion 3, we discuss the case where H is superquadratic. Finally, in section 4, we consider the case where $H_z(n, z)$ is Lipschitzian.

2. The Subquadratic Case

Theorem 2.1. *Suppose $H(n, z)$ satisfies the following two conditions.*

($\mathbf{H}_1$) There exist constants $R_1 > 0$ and $\alpha \in (1, 2)$ such that for any $(n, z) \in \mathbb{Z}(0, m-1) \times \mathbb{R}^2$ with $|z| \geq R_1$,

$$0 < z \cdot H_z(n, z) \leq \alpha H(n, z). \tag{2.1}$$

($\mathbf{H}_2$) *There exist constants* $a_1 > 0, a_2 \geq 0$ *and* $\gamma \in (1, \alpha]$ *such that*

$$H(n, z) \geq a_1|z|^\gamma - a_2, \quad \forall (n, z) \in \mathbb{Z}(0, m-1) \times \mathbb{R}^2. \tag{2.2}$$

Then the BVP (1.2) with (1.3) has at least one solution.

Remark 2.1 Integrating inequality (2.1) gives us

$$H(n, z) \leq a_3|z|^\alpha + a_4 \tag{2.3}$$

for some positive constants a_3 and a_4. It follows from 2.3 that

$$\lim_{|z|\to\infty} \frac{H(n, z)}{z^2} = 0.$$

Such $H(n, z)$ is called subquadratic at infinity.

Proof of Theorem 2.1. First, we show that F satisfies the P-S condition.

Clearly, $F \in C^1(\mathbb{R}^{2m}, \mathbb{R})$. Let $x^{(k)} \in \mathbb{R}^{2m}, k \in \mathbb{Z}(1)$ be such that $\{F(x^{(k)})\}$ is bounded and $F'(x^{(k)}) \to 0$ as $k \to \infty$. Then there exists a constant $M_1 > 0$ and $k_0 \in \mathbb{Z}(1)$ such that

$$|F(x^{(k)})| \leq M_1 \text{ for } k \in \mathbb{Z}(1), \quad |(F'(x^{(k)}), x)| \leq \|x\|_2 \text{ for } k \in \mathbb{Z}(k_0),\ x \in \mathbb{R}^{2m}.$$

Since

$$(F'(x^{(k)}), x^{(k)}) = (Dx^{(k)}, x^{(k)}) + \sum_{n=0}^{m-1} H_z(n, Lx^{(k)}(n+1)) \cdot Lx^{(k)}(n+1) - (\eta, x^{(k)}),$$

where $Lx(n) \in \mathbb{R}^2$ is defined as $(x_1(n), x_2(n-1))^T$, we see that, for $k \in \mathbb{Z}(k_0)$,

$$\begin{aligned} & M_1 + \frac{1}{2}\|x^{(k)}\|_2 \\ \geq\ & F(x^{(k)}) - \frac{1}{2}(F'(x^{(k)}), x^{(k)}) \\ =\ & \sum_{n=0}^{m-1} \left[H(n, Lx^{(k)}(n+1)) - \frac{1}{2} Lx^{(k)}(n+1) \cdot H_z(n, Lx^{(k)}(n+1)) \right] - \frac{1}{2}(\eta, x^{(k)}) \end{aligned}$$

For any $k \in \mathbb{Z}(k_0)$, denote

$$S_1^k = \{n \in \mathbb{Z}(1, m) |\ |Lx^{(k)}(n)| \geq R_1\}, \quad S_2^k = \{n \in \mathbb{Z}(1, m) |\ |Lx^{(k)}(n)| < R_1\}.$$

Then $S_1^k \cup S_2^k = \mathbb{Z}(1, m)$, and

$$\begin{aligned} & M_1 + \frac{1}{2}\|x^{(k)}\|_2 \\ \geq\ & \sum_{n=1}^{m} H(n-1, Lx^{(k)}(n)) - \frac{1}{2} \sum_{n \in S_1^k} Lx^{(k)}(n) \cdot H_z(n-1, Lx^{(k)}(n)) \\ & - \frac{1}{2} \sum_{n \in S_2^k} Lx^{(k)}(n) \cdot H_z(n-1, Lx^{(k)}(n)) - \frac{1}{2}(\eta, x^{(k)}). \end{aligned}$$

In view of (2.1), we have

$$\begin{aligned}
& M_1 + \frac{1}{2}\|x^{(k)}\|_2 \\
\geq\ & \sum_{n=1}^{m} H(n-1, Lx^{(k)}(n)) - \frac{\alpha}{2}\sum_{n\in S_1^k} H(n-1, Lx^{(k)}(n)) \\
& -\frac{1}{2}\sum_{n\in S_2^k} Lx^{(k)}(n)\cdot H_z(n-1, Lx^{(k)}(n)) - \frac{1}{2}(\eta, x^{(k)}) \\
=\ & (1-\frac{\alpha}{2})\sum_{n=1}^{pm} H(n-1, Lx^{(k)}(n)) - \frac{1}{2}(\eta, x^{(k)}) \\
& +\frac{1}{2}\sum_{n\in S_2^k}[\alpha H(n-1, Lx^{(k)}(n)) - Lx^{(k)}(n)\cdot H_z(n-1, Lx^{(k)}(n))].
\end{aligned}$$

Since $\alpha H(n-1,z) - z\cdot H_z(n-1,z)$ is continuous with respect to $z\in\mathbb{R}^2$ for each $n\in\mathbb{Z}(1,m)$, there exists a constant $M_2>0$ such that

$$|\alpha H(n-1,z) - z\cdot H_z(n-1,z)| \leq M_2, \quad \forall z\in\mathbb{R}^2 \text{ and } |z|\leq R_1,\ n\in\mathbb{Z}(1,m).$$

Thus,

$$M_1 + \frac{1}{2}\|x^{(k)}\|_2 \geq (1-\frac{\alpha}{2})\sum_{n=1}^{m} H(n-1, Lx^{(k)}(n)) - \frac{1}{2}\|\eta\|_2\|x^{(k)}\|_2 - \frac{1}{2}mM_2.$$

By (2.2) and (1.6), we get

$$\begin{aligned}
& M_1 + \frac{1}{2}\|x^{(k)}\|_2 \\
\geq\ & (1-\frac{\alpha}{2})a_1\sum_{n=1}^{m}|Lx^{(k)}(n)|^\gamma - (1-\frac{\alpha}{2})a_2 m - \frac{1}{2}\|\eta\|_2\|x^{(k)}\|_2 - \frac{1}{2}mM_2 \\
\geq\ & (1-\frac{\alpha}{2})a_1\left(\frac{1}{C_{2\gamma}}\right)^\gamma \|x^{(k)}\|_2^\gamma - \frac{1}{2}\|\eta\|_2\|x^{(k)}\|_2 - M_3,
\end{aligned}$$

where $M_3 = (1-\frac{\alpha}{2})a_2 m + \frac{1}{2}mM_2$ and

$$\|Lx\|_2^2 = \sum_{n=1}^{m}|Lx(n)|^2 = \sum_{n=1}^{m}(|x_1(n)|^2 + |x_2(n-1)|^2) = \|x\|_2^2.$$

Therefore,

$$(1-\frac{\alpha}{2})a_1\left(\frac{1}{C_{2\gamma}}\right)^\gamma \|x^{(k)}\|_2^\gamma - \frac{1}{2}(1+\|\eta\|_2)\|x^{(k)}\|_2 \leq M_1 + M_3.$$

Because $\gamma\in(1,2)$, we see that $\{\|x^{(k)}\|_2\}$ is bounded. Since $\mathbb{R}^{2m}$ is finite dimensional, $\{x^{(k)}\}$ has a subsequence which is convergent in $\mathbb{R}^{2m}$. Therefore, F satisfies the P-S condition.

Now we prove F satisfies $(\mathbf{I}_1)$ and $(\mathbf{I}_2)$. To this end, let $v \in X_2$. Then

$$\begin{aligned} F(v) &= \frac{1}{2}(Dv, v) + \sum_{n=1}^{m} H(n-1, Lv(n)) - (\eta, v) \\ &\geq \frac{1}{2}\lambda_1 \|v\|_2^2 + \sum_{n=1}^{m}(a_1|Lv(n)|^\gamma - a_2) - \|\eta\|_2\|v\|_2 \\ &\geq -a_2 m - \frac{\|\eta\|_2^2}{2\lambda_1}. \end{aligned}$$

Let $\omega = -a_2 m - \frac{\|\eta\|_2^2}{2\lambda_1}$, $e = 0$, $\sigma = \omega - 1$. Then F satisfies $(\mathbf{I}_2)$. By (1.7), for $u \in X_1$, it follows from (2.3) that

$$\begin{aligned} F(u) &= \frac{1}{2}(Du, u) + \sum_{n=1}^{m} H(n-1, Lu(n)) - (\eta, u) \\ &\leq -\frac{1}{2}\lambda_1 \|u\|_2^2 + a_3 \sum_{n=1}^{m} |Lu(n)|^\alpha + a_4 m + \|\eta\|_2\|u\|_2 \\ &\leq -\frac{1}{2}\lambda_1 \|u\|_2^2 + a_3 \left(\frac{1}{C_{1\alpha}}\right)^\alpha \|u\|_2^\alpha + \|\eta\|_2\|u\|_2 + a_4 m. \end{aligned}$$

Since $1 < \alpha < 2$, there exists a sufficiently large constant $\rho > 0$ such that

$$F(u) \leq \sigma, \qquad \forall u \in X_1 \text{ with } \|u\| = \rho.$$

Thus $(\mathbf{I}_1)$ holds. By Lemma 1.2, there exists at least one critical point of F. This completes the proof. □

3. The Superquadratic Case

Theorem 3.1. *Assume the following condition,*

$(\mathbf{H}_3)$ *There exist some constants $R_2 > 0, \beta > 2$ such that for any $(n, z) \in \mathbb{Z}(0, m-1) \times \mathbb{R}^2, |z| \geq R_2$,*

$$(z, H_z(n, z)) \geq \beta H(n, z) > 0, \tag{3.1}$$

holds. Then BVP (1.2) with (1.3) possesses at least one solution.

Remark 3.1. It follows from (3.1) easily that there exist positive constants a_5 and a_6 such that

$$H(n, z) \geq a_5|z|^\beta - a_6, \quad \forall (n, z) \in \mathbb{Z}(0, m-1) \times \mathbb{R}^2. \tag{3.2}$$

Therefore the assumption $(\mathbf{H}_3)$ implies that $H(t, z)$ grows superquadratically at infinity.

In order to prove Theorem 3.1, we need the following lemma.

Lemma 3.1. *Suppose that $H(n,z)$ satisfies* ($\mathbf{H}_3$)*. Then $F(x)$ defined in (1.4) is bounded from below in $\mathbb{R}^{2m}$.*

Proof. For any $x \in \mathbb{R}^{2m}$, let $x = u + v \in X_1 \oplus X_2$. Then, in view of (1.5),

$$\begin{aligned} & F(x) \\ = \;& \tfrac{1}{2}[(Du,u)+(Dv,v)] + \sum_{n=1}^{m} H(n-1, x_1(n), x_2(n-1)) - (\eta, x) \\ \geq \;& -\tfrac{1}{2}\lambda_m \|u\|_2^2 + \tfrac{1}{2}\lambda_1\|v\|_2^2 + a_5 \sum_{n=1}^{m} |Lx(n)|^\beta - a_6 m - \|\eta\|_2\|x\|_2 \\ \geq \;& -\tfrac{1}{2}\lambda_m \|u\|_2^2 + a_5 \sum_{n=1}^{m} |Lx(n)|^\beta - \|\eta\|_2\|x\|_2 - a_6 m \\ \geq \;& -\tfrac{1}{2}\lambda_m \|x\|_2^2 + a_5 \left(\tfrac{1}{C_{2\beta}}\right)^\beta \|x\|_2^\beta - \|\eta\|_2\|x\|_2 - a_6 m. \end{aligned} \tag{3.3}$$

Since $\beta > 2$, it is clear that there exists a positive constant M_4 such that

$$F(x) \geq -M_4, \quad \forall x \in \mathbb{R}^{2m}.$$

The proof is complete. □

Proof of Theorem 3.1. By Lemma 3.1, $F(x)$ is bounded from below on $\mathbb{R}^{2m}$. Let

$$c_0 = \inf_{x \in \mathbb{R}^{2m}} F(x).$$

According to (3.3), we know that $F(x)$ is coercive. Then, there must be a point $\bar{x} \in \mathbb{R}^{2m}$ such that $F(\bar{x}) = c_0$ and thus $\bar{x} \in \mathbb{R}^{2m}$ is a critical point of F. The proof is complete. □

4. The Lipschitz Case

In this section, we suppose that $H_z(n,z)$ satisfies the Lipschitz condition.

($\mathbf{H}_4$) $H(n,z)$ is Lipschitizian in z, namely, there exists a positive constant L such that

$$|H_z(n,z_1) - H_z(n,z_2)| \leq L|z_1 - z_2| \tag{4.1}$$

holds for $n \in \mathbb{Z}(0, m-1),\ z_1, z_2 \in \mathbb{R}^2$.

By ($\mathbf{H}_4$), there exists a positive constant a_7 such that

$$|H_z(n,z)| \leq L|z| + a_7, \quad \forall (n,z) \in \mathbb{Z}(0,m-1) \times \mathbb{R}^2. \tag{4.2}$$

Theorem 4.1. *Assume that $L < \lambda_1$ and (4.2) holds. Then the BVP (1.2) with (1.3) has at least one solution.*

Proof. In view of (4.2), there exists a constant $a_8 > 0$ such that

$$|H(n,z)| \leq \frac{1}{2}L|z|^2 + a_7|z| + a_8, \quad \forall (n,z) \in \mathbb{Z}(0,m-1) \times \mathbb{R}^2. \tag{4.3}$$

We first show that F satisfies the P-S condition. In fact, suppose that $\{x^{(k)}\}$ is a sequence in $\mathbb{R}^{2m}$ such that for any $k \in \mathbb{Z}(1), |F(x^{(k)})| \leq M_5$ for some positive constant M_5 and $F'(x^{(k)}) \to 0$ as $k \to \infty$. Then for sufficiently large k, $\|F'(x^{(k)})\| \leq 1$. Since

$$(F'(x^{(k)}), x) = (Dx^{(k)}, x) + \sum_{n=1}^{m} H_z(n-1, Lx^{(k)}(n)) \cdot Lx(n) - (\eta, x), \forall x \in \mathbb{R}^{2m}.$$

Then for sufficiently large k,

$$\begin{aligned} \|Dx^{(k)}\|_2 &\leq \left(\sum_{n=1}^{m} |H_z(n-1, Lx^{(k)}(n))|^2 \right)^{\frac{1}{2}} + \|\eta\|_2 + 1 \\ &\leq \left(\sum_{n=1}^{m} (L|Lx^{(k)}(n)| + a_7)^2 \right)^{\frac{1}{2}} + \|\eta\|_2 + 1 \\ &\leq L\|x^{(k)}\|_2 + (a_7\sqrt{m} + \|\eta\|_2 + 1). \end{aligned} \tag{4.4}$$

On the other hand, let $x^{(k)} = u^{(k)} + v^{(k)} \in X_1 \oplus X_2$. Then

$$Dx^{(k)} = Du^{(k)} + Dv^{(k)},$$

which implies that

$$\|Dx^{(k)}\|^2 = \|Du^{(k)}\|^2 + \|Dv^{(k)}\|^2 \geq \lambda_1^2(\|u^{(k)}\|_2^2 + \|v^{(k)}\|_2^2) = \lambda_1^2\|x^{(k)}\|_2^2.$$

From (4.4), we get

$$\lambda_1\|x^{(k)}\|_2 \leq L\|x^{(k)}\|_2 + (a_7\sqrt{m} + \|\eta\|_2 + 1).$$

Noticing the fact that $L < \lambda_1$, we have

$$\|x^{(k)}\|_2 \leq \frac{a_7\sqrt{m} + \|\eta\|_2 + 1}{\lambda_1 - L}.$$

So, $\{x^{(k)}\}$ is bounded.

Now we check that the conditions $(\mathbf{I}_1)$ and $(\mathbf{I}_2)$ in Lemma 1.2. For any $v \in X_2$, according to (4.3), we have

$$\begin{aligned} F(v) &= \frac{1}{2}(Dv, v) + \sum_{n=1}^{m} H(n-1, Lv(n)) - (\eta, v) \\ &\geq \frac{1}{2}\lambda_1\|v\|_2^2 - \sum_{n=1}^{m}(\frac{1}{2}L|Lv(n)|^2 + a_7|Lv(n)| + a_8) - \|\eta\|_2\|v\|_2 \\ &\geq \frac{1}{2}(\lambda_1 - L)\|v\|_2^2 - (a_7\sqrt{m} + \|\eta\|_2)\|v\|_2 - a_8 m, \end{aligned}$$

which implies that

$$F(v) \geq \omega$$

holds for some negative constant ω. Let $e = 0$, then, $(\mathbf{I}_2)$ holds.

For any $u \in X_1$, according to (1.7) and (4.3), we have

$$\begin{aligned} F(u) &= \frac{1}{2}(Du, u) + \sum_{n=1}^{m} H(n-1, Lu(n)) - (\eta, u) \\ &\leq -\frac{1}{2}\lambda_1 \|u\|_2^2 + \sum_{n=1}^{m}(\frac{1}{2}L|Lu(n)|^2 + a_7|Lu(n)| + a_8) + \|\eta\|_2\|u\|_2 \\ &\leq \frac{1}{2}(-\lambda_1 + L)\|u\|_2^2 + (a_7\sqrt{m} + \|\eta\|_2)\|u\|_2 + a_8 m. \end{aligned}$$

This implies $F(u) \to -\infty$ as $\|u\|_2 \to \infty$. Let $\sigma = \omega - 1$. Then there exists a sufficiently large $\rho > 0$ such that

$$F(u) \leq \sigma, \quad \forall u \in X_1, \text{ with } \|u\|_2 = \rho. \tag{4.5}$$

Thus $(\mathbf{I}_1)$ is satisfied. Now, the result follows directly from Lemma 1.2 and hence the proof of Theorem 4.1 is complete. □

The following corollary is obvious.

Corollary 4.1. *Assume that $L < \lambda_1$ and $(\mathbf{H}_4)$ holds. Then the BVP (1.2) with (1.3) has at least one solution.*

References

[1] R. P. Agarwal, Difference Equations and Inequalities: Theory, Methods and Applications, Second Edition, Marcel Dekker, New York, 2000.

[2] R. P. Agarwal and W. Zhang, Periodic solutions of difference equations with general periodicity, Comput. Math. Appl., 42 (2001) 719-727.

[3] C. D. Ahlbrandt and A. C. Peterson, Discrete Hamiltonian Systems, Kluwer Academic Publishers, Dordrecht, 1996.

[4] V. Benci and P. H. Rabinowitz, Critical point theorems for indefinite functionals, Invent. Math., 52 (1979) 241-273.

[5] M. Bohner, Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions, J. Math. Anal. Appl., 199 (1996) 804-826.

[6] K. C. Chang, Critical Point Theory and its Applications, Science and Technical Press, Shanghai, China, 1980.

[7] S. Chen, Disconjugacy, disfocality, and oscillation of second order difference equations, J. Differential Equations, 107 (1994) 383-394.

[8] L. H. Erbe and P. Yan, Disconjugacy for linear Hamiltonian difference systems, J. Math. Anal. Appl., 167 (1992) 355-367.

[9] Z. Guo and J. Yu, Periodic and subharmonic solutions for superquadratic discrete Hamiltonian systems, Nonl. Anal., 55 (2003) 969-983.

[10] Z. Guo and J. Yu, The existence of periodic and subharmonic solutions for second-order superlinear difference equations, Science in China, Mathematics, 46 (2003) 506-515.

[11] P. Hartman, Difference equations: disconjugacy, principal solutions, Green's functions, complete monotonicity, Trans. Amer. Math. Soc., 246 (1978) 1-30.

[12] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, Springer-Verlag, New York Inc., 1989.

[13] P. H. Rabinowitz, Periodic solutions of Hamiltonian systems, Comm. Pure Appl. Math., 31 (1978) 157-184.

[14] P. H. Rabinowitz, On subharmonic solutions of Hamiltonian systems, Comm. Pure Appl. Math., 33 (1980) 609-633.

[15] P. H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. Math., vol 65, Amer. Math. Soc., Providence, RI, 1986.

[16] Z. Zhou and J. Wu, Attractive periodic orbits in nonlinear delayed discrete-time neural networks with delayed feedback, J. Difference Equa. Appl., 8 (2002) 467-483.

[17] Z. Zhou, J. Yu and Z. Guo, The existence of periodic and subharmonic solutions to subquadratic discrete Hamiltonian system, ANZIAM J., 47 (2005) 89-102.