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A Survey on the Oscillation of Delay and Difference Equations with Variable Delay

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Abstract. Consider the first-order linear delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$, $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$, $\lim_{t\to\infty} \tau(t) = \infty$, and the (discrete analogue) difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, ..., \tag*{(1)$'$}$$

where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and $\tau(n)$ is a nondecreasing sequence of integers such that $\tau(n) \leq n - 1$ for all $n \geq 0$ and $\lim_{n\to\infty} \tau(n) = \infty$. Optimal conditions for the oscillation of all solutions to the above equations are presented.

AMS Subject Classifications: 39A11, 39A12

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1. Introduction

The problem of establishing sufficient conditions for the oscillation of all solutions to the differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1}$$

where the functions $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$ (here $\mathbb{R}^+ = [0, \infty)$), $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$, has been the subject of many investigations.

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See, for example, [11, 15, 17, 21–26, 28, 29–32, 33-42, 44, 47-52, 54, 55, 59, 60, 66, 73–80, 82-84, 90] and the references cited therein.

By a solution of Eq.(1) we understand a continuously differentiable function defined on $[\tau(T_0), \infty)$ for some $T_0 \geq t_0$ and such that Eq.(1) is satisfied for $t \geq T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros, and otherwise it is called *nonoscillatory.*

The oscillation theory of the (discrete analogue) delay difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, ..., \tag{1$'$}$$

where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and $\tau(n)$ is a nondecreasing sequence of integers such that $\tau(n) \leq n-1$ for all $n \geq 0$ and $\lim_{n\to\infty} \tau(n) = \infty$, has also attracted growing attention in the last decades, especially in the case where the delay $n - \tau(n)$ is a constant, that is, in the special case of the difference equation,

$$\Delta x(n) + p(n)x(n-k) = 0, \quad n = 0, 1, 2, ... \tag{1$''$}$$

where k is a positive integer. The reader is referred to [5–10, 12, 13, 16, 18-20, 43, 46, 53, 56, 57, 61, 62, 63-65, 67-72, 81, 85-89] and the references cited therein.

By a solution of Eq.$(1)'$ we mean a sequence $x(n)$ which is defined for $n \geq -k$ and which satisfies $(1)'$ for $n \geq 0$. A solution $x(n)$ of Eq.$(1)'$ is said to be *oscillatory* if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative, and otherwise the solution is said to be *nonoscillatory.* (Analogously for Eq.$(1)''$.)

In this paper our main purpose is to present the state of the art on the oscillation of all solutions to Eq.(1) especially in the case where

$$0 < \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds \leq \frac{1}{e} \quad \text{and} \quad \limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)ds < 1,$$

and (the discrete analogues) for Eq.$(1)'$ when

$$\liminf_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) \leq \frac{1}{e} \quad \text{and} \quad \limsup_{n\to\infty} \sum_{i=\tau(n)}^{n} p(i) < 1.$$

2. Oscillation Criteria for Eq. (1)

In this section we study the delay equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0. \tag{1}$$

where the functions $p, \tau \in C([t_0, \infty), \mathbb{R}^+)$, $\tau(t)$ is nondecreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

The first systematic study for the oscillation of all solutions to Eq.(1) was made by Myshkis. In 1950 [58] he proved that every solution of Eq.(1) oscillates if

$$\limsup_{t\to\infty}[t-\tau(t)]<\infty \quad \text{and} \quad \liminf_{t\to\infty}[t-\tau(t)]\liminf_{t\to\infty}p(t)>\frac{1}{e}. \qquad (C_1)$$

In 1972, Ladas, Lakshmikantham and Papadakis [44] proved that the same conclusion holds if

$$A:=\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds>1. \qquad (C_2)$$

In 1979, Ladas [42] established integral conditions for the oscillation of Eq.(1) with constant delay. Tomaras [77-79] extended this result to Eq.(1) with variable delay. For related results see Ladde [49-51]. The following most general result is due to Koplatadze and Canturija [37].

If

$$\mathfrak{a}:=\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds>\frac{1}{e}, \qquad (C_3)$$

then all solutions of Eq.(1) oscillate; If

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds<\frac{1}{e}, \qquad (N_1)$$

then Eq.(1) has a nonoscillatory solution.

It is obvious that there is a gap between the conditions (C_2) and (C_3) when the limit $\lim\limits_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds$ does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors.

In 1988, Erbe and Zhang [26] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$ for possible nonoscillatory solutions $x(t)$ of Eq.(1). Their result says that all the solutions of Eq.(1) are oscillatory, if $0<\mathfrak{a}\leq\frac{1}{e}$ and

$$A>1-\frac{\mathfrak{a}^2}{4}. \qquad (C_4)$$

Since then several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$.

In 1991, Jian [35] derived the condition

$$A>1-\frac{\mathfrak{a}^2}{2(1-\mathfrak{a})}, \qquad (C_5)$$

while in 1992, Yu and Wang [83] and Yu, Wang, Zhang and Qian [84] obtained the condition

$$A>1-\frac{1-\mathfrak{a}-\sqrt{1-2\mathfrak{a}-\mathfrak{a}^2}}{2}. \qquad (C_6)$$

In 1990, Elbert and Stavroulakis [23] and in 1991 Kwong [41], using different techniques, improved (C_4), in the case where $0 < \mathfrak{a} \leq \frac{1}{e}$, to the conditions

$$A > 1 - (1 - \frac{1}{\sqrt{\lambda_1}})^2 \tag{C_7}$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1}, \tag{C_8}$$

respectively, where λ_1 is the smaller real root of the equation $\lambda = e^{\mathfrak{a}\lambda}$.

In 1994, Koplatadze and Kvinikadze [38] improved (C_6), while in 1998, Philos and Sficas [59] and in 1999, Zhou and Yu [90] and Jaroš and Stavroulakis [34] derived the conditions

$$A > 1 - \frac{\mathfrak{a}^2}{2(1-\mathfrak{a})} - \frac{\mathfrak{a}^2}{2}\lambda_1, \tag{C_9}$$

$$A > 1 - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2} - (1 - \frac{1}{\sqrt{\lambda_1}})^2, \tag{C_{10}}$$

and

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \mathfrak{a} - \sqrt{1 - 2\mathfrak{a} - \mathfrak{a}^2}}{2}, \tag{C_{11}}$$

respectively.

Consider Eq.(1) and assume that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$ such that $p(\tau(t))\tau'(t) \geq \theta p(t)$ eventually for all t. Under this additional condition, in 2000, Kon, Sficas and Stavroulakis [36] and in 2003, Sficas and Stavroulakis [60] established the conditions

$$A > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - \mathfrak{a} - \sqrt{(1-\mathfrak{a})^2 - 4\Theta}}{2} \tag{2.1}$$

and

$$A > \frac{\ln \lambda_1}{\lambda_1} - \frac{1 + \sqrt{1 + 2\theta - 2\theta\lambda_1 M}}{\theta\lambda_1} \tag{2.2}$$

respectively, where $\Theta = \frac{e^{\lambda_1\theta\mathfrak{a}} - \lambda_1\theta\mathfrak{a} - 1}{(\lambda_1\theta)^2}$ and $M = \frac{1-\mathfrak{a}-\sqrt{(1-\mathfrak{a})^2-4\Theta}}{2}$.

Remark 2.1. ([36], [60]) Observe that when $\theta = 1$, then $\Theta = \frac{\lambda_1 - \lambda_1\mathfrak{a} - 1}{\lambda_1{}^2}$, and (2.1) reduces to

$$A > 2\mathfrak{a} + \frac{2}{\lambda_1} - 1, \tag{C_{12}}$$

while in this case it follows that $M = 1 - \mathfrak{a} - \frac{1}{\lambda_1}$ and (2.2) reduces to

$$A > \frac{\ln \lambda_1 - 1 + \sqrt{5 - 2\lambda_1 + 2\mathfrak{a}\lambda_1}}{\lambda_1}. \tag{C_{13}}$$

In the case where $\mathfrak{a} = \frac{1}{e}$, then $\lambda_1 = e$, and (C_{13}) leads to

$$A > \frac{\sqrt{7-2e}}{e} \approx 0.459987065.$$

It is to be noted that as $\mathfrak{a} \to 0$, then all the previous conditions $(C_4) - (C_{12})$ reduce to the condition (C_2), i.e. $A > 1$. However, the condition (C_{13}) leads to

$$A > \sqrt{3} - 1 \approx 0.732,$$

which is an essential improvement. Moreover (C_{13}) improves all the above conditions when $0 < \mathfrak{a} \leq \frac{1}{e}$ as well. Note that the value of the lower bound on A can not be less than $\frac{1}{e} \approx 0.367879441$. Thus the aim is to establish a condition which leads to a value *as close as possible to* $\frac{1}{e}$. For illustrative purpose, we give the values of the lower bound on A under these conditions when $\mathfrak{a} = \frac{1}{e}$.

(C_4):	0.966166179
(C_5):	0.892951367
(C_6):	0.863457014
(C_7):	0.845181878
(C_8):	0.735758882
(C_9):	0.709011646
(C_{10}):	0.708638892
(C_{11}):	0.599215896
(C_{12}):	0.471517764
(C_{13}):	0.459987065

We see that the condition (C_{13}) essentially improves all the known results in the literature.

Example 2.1. ([60]) Consider the delay differential equation

$$x'(t) + px\left(t - q\sin^2\sqrt{t} - \frac{1}{pe}\right) = 0,$$

where $p > 0$, $q > 0$ and $pq = 0.46 - \frac{1}{e}$. Then

$$\mathfrak{a} = \liminf_{t\to\infty} \int_{\tau(t)}^{t} pds = \liminf_{t\to\infty} p(q\sin^2\sqrt{t} + \frac{1}{pe}) = \frac{1}{e}$$

and

$$A = \limsup_{t\to\infty} \int_{\tau(t)}^{t} pds = \limsup_{t\to\infty} p(q\sin^2\sqrt{t} + \frac{1}{pe}) = pq + \frac{1}{e} = 0.46.$$

Thus, according to Remark 2.1, all solutions of this equation oscillate. Observe that none of the conditions (C_4)-(C_{12}) apply to this equation.

Following this historical (and chronological) review we also mention that in the case where

$$\int_{\tau(t)}^{t} p(s)ds \geq \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e}$$

this problem has been studied in 1995, by Elbert and Stavroulakis [24], by Kozakiewicz [39], Li [54, 55] and in 1996, by Domshlak and Stavroulakis [22].

3. Oscillation Criteria for Eq. $(1)'$

In this section we study the difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \quad n = 0, 1, 2, ..., \tag{1$'$}$$

where $\Delta x(n) = x(n+1) - x(n)$, $p(n)$ is a sequence of nonnegative real numbers and $\tau(n)$ is a nondecreasing sequence of integers such that $\tau(n) \leq n-1$ for all $n \geq 0$ and $\lim_{n\to\infty} \tau(n) = \infty$.

In the special case where the delay $n - \tau(n)$ is a constant, the delay difference equation $(1)'$ becomes

$$\Delta x(n) + p(n)x(n-k) = 0, \quad n = 0, 1, 2, ... \tag{1$''$}$$

where k is a positive integer.

In 1981, Domshlak [12]was the first who studied this problem in the case where $k = 1$.Then, in 1989, Erbe and Zhang [27] established that all solutions of Eq.$(1)''$ are oscillatory if

$$\liminf_{n\to\infty} p(n) > \frac{k^k}{(k+1)^{k+1}} \tag{3.1}$$

or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p(i) > 1. \tag{$C_2$$''$}$$

In the same year, 1989, Ladas, Philos and Sficas [46] proved that a sufficient condition for all solutions of Eq.$(1)''$ to be oscillatory is that

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > \left(\frac{k}{k+1}\right)^{k+1} \tag{$C_3$$''$}$$

Therefore they improved the condition (3.1) by replacing the $p(n)$ of (3.1) by the arithmetic mean of $p(n-k), ..., p(n-1)$ in $(C_3)''$.

Concerning the constant $\frac{k^k}{(k+1)^{k+1}}$ in (3.1) it should be emphasized that, as it is shown in [27], if

$$\sup p(n) < \frac{k^k}{(k+1)^{k+1}}$$

then Eq.(1)$''$ has a nonoscillatory solution.

In 1990, Ladas [43] conjectured that Eq.(1)$''$ has a nonoscillatory solution if

$$\sum_{i=n-k}^{n-1} p(i) < \left(\frac{k}{k+1}\right)^{k+1}$$

holds eventually. However, a counterexample to this conjecture was given in 1994, by Yu, Zhang and Wang [86].

It is interesting to establish sufficient oscillation conditions for the equation (1)$''$ in the case where neither $(C_2)''$ nor $(C_3)''$ is satisfied.

In 1995, the following oscillation criterion was established by Stavroulakis [63]:

If $0 < \alpha_0 \leq \left(\frac{k}{k+1}\right)^{k+1}$, where

$$\alpha_0 = \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p(i)$$

then the condition

$$\limsup_{n\to\infty} p(n) > 1 - \frac{\alpha_0^2}{4} \tag{3.2}$$

implies that all solutions of Eq.(1)$''$ oscillate. In 2004, the same author [64] improved the condition (3.2) to the following

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\alpha_0^2}{4} \tag{C_4}''$$

or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \alpha_0^k, \tag{3.3}$$

while in 2006, Chatzarakis and Stavroulakis [5], established the condition

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p(i) > 1 - \frac{\alpha_0^2}{2(2-\alpha_0)}. \tag{3.4}$$

Also, Chen and Yu [6] obtained the following oscillation condition

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p(i) > 1 - \frac{1-\alpha_0-\sqrt{1-2\alpha_0-\alpha_0^2}}{2}. \tag{C_6}''$$

Remark 3.1. Observe that the conditions $(C_2)''$, $(C_3)''$, $(C_4)''$ and $(C_6)''$ are the discrete analogues of the conditions (C_2), (C_3), (C_4) and (C_6) respectively for Eq.(1)$''$.

In the case of Eq.(1)$'$ with a general delay argument $\tau(n)$, from Chatzarakis, Koplatadze and Stavroulakis [2], it follows the following

Theorem 3.1. ([2]) *If*

$$\limsup_{n\to\infty} \sum_{i=\tau(n)}^{n} p(i) > 1 \qquad (C_2)'$$

then all solutions of Eq. (1)$'$ *oscillate.*

This result generalizes the oscillation criterion $(C_2)''$. Also Chatzarakis, Koplatadze and Stavroulakis [3] extended the oscillation criterion $(C_3)''$ to the general case of Eq. (1)$'$. More precisely, the following theorem has been established in [3].

Theorem 3.2. ([3]) *Assume that*

$$\limsup_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) < +\infty \qquad (3.5)$$

and

$$\alpha := \liminf_{n\to\infty} \sum_{i=\tau(n)}^{n-1} p(i) > \frac{1}{e}. \qquad (C_3)'$$

Then all solutions of Eq.(1)$'$ *oscillate.*

Remark 3.2. It is to be pointed out that the conditions $(C_2)'$ and $(C_3)'$ are the discrete analogues of the conditions (C_2) and (C_3) and also the analogues of the conditions $(C_2)''$ and $(C_3)''$ for Eq.(1)$'$ in the case of a general delay argument $\tau(n)$.

Remark 3.3. ([3]). The condition $(C_3)'$ is optimal for Eq.(1)$'$ under the assumption that $\lim_{n\to+\infty}(n-\tau(n)) = \infty$, since in this case the set of natural numbers increases infinitely in the interval $[\tau(n), n-1]$ for $n \to \infty$.

Now, we are going to present an example to show that the condition $(C_3)'$ is optimal, in the sense that it cannot be replaced by the non-strong inequality.

Example 3.1. ([3]) Consider Eq.(1)$'$, where

$$\tau(n) = [\beta n],\ p(n) = \left(n^{-\lambda} - (n+1)^{-\lambda}\right)([\beta n])^{\lambda},\ \beta \in (0,1),\ \lambda = -\ln^{-1}\beta \qquad (3.6)$$

and $[\beta n]$ denotes the integer part of βn.

It is obvious that

$$n^{1+\lambda}\left(n^{-\lambda} - (n+1)^{-\lambda}\right) \to \lambda \quad \textit{for} \quad n \to \infty.$$

Therefore

$$n\left(n^{-\lambda} - (n+1)^{-\lambda}\right)([\beta n])^{\lambda} \to \frac{\lambda}{e} \quad \textit{for} \quad n \to \infty. \qquad (3.7)$$

Hence, in view of (3.6) and (3.7), we have

$$\begin{aligned}\liminf_{n\to\infty}\sum_{i=\tau(n)}^{n-1}p(i) &= \frac{\lambda}{e}\ \liminf_{n\to\infty}\sum_{i=[\beta n]}^{n-1}\frac{e}{\lambda}i\left(i^{-\lambda}-(i+1)^{-\lambda}\right)([\beta i])^{\lambda}\cdot\frac{1}{i}\\ &= \frac{\lambda}{e}\ \liminf_{n\to\infty}\sum_{i=[\beta n]}^{n-1}\frac{1}{i}=\frac{\lambda}{e}\ \ln\frac{1}{\beta}=\frac{1}{e}\end{aligned}$$

or

$$\liminf_{n\to\infty}\sum_{i=\tau(n)}^{n-1}p(i)=\frac{1}{e}. \tag{3.8}$$

Observe that all the conditions of Theorem 3.2 are satisfied except the condition $(C_3)'$. In this case it is not guaranteed that all solutions of Eq.$(1)'$ oscillate. Indeed, it is easy to see that the function $u=n^{-\lambda}$ is a positive solution of Eq.$(1)'$.

As it has been mentioned above, it is an interesting problem to find new sufficient conditions for the oscillation of all solutions of the delay difference equation $(1)'$, in the case where neither $(C_2)'$ nor $(C_3)'$ is satisfied.

In 2007, Chatzarakis, Koplatadze and Stavroulakis [2] investigated for the first time this question for the difference equation $(1)'$ in the case of a general delay argument $\tau(n)$ and derived the following theorem.

Theorem 3.3. ([2]) *Assume that* $0<\alpha\le\frac{1}{e}$. *Then we have:*

(I) *If*

$$\limsup_{n\to\infty}\sum_{j=\tau(n)}^{n}p(j)>1-\left(1-\sqrt{1-\alpha}\right)^2 \tag{3.9}$$

then all solutions of Eq.$(1)'$ *oscillate.*

(II) *If in addition,*

$$p(n)\ge 1-\sqrt{1-\alpha}\ \textit{for all large } n, \tag{3.10}$$

and

$$\limsup_{n\to\infty}\sum_{j=\tau(n)}^{n}p(j)>1-\alpha\frac{1-\sqrt{1-\alpha}}{\sqrt{1-\alpha}} \tag{3.11}$$

then all solutions of Eq.$(1)'$ *oscillate.*

Recently the above result was improved in [4] as follows.

Theorem 3.4. ([4]) (I) *If* $0<\alpha\le\frac{1}{e}$ *and*

$$\limsup_{n\to\infty}\sum_{j=\tau(n)}^{n}p(j)>1-\frac{1}{2}\left(1-\alpha-\sqrt{1-2\alpha}\right) \tag{3.12}$$

then all solutions of Eq.$(1)'$ *oscillate.*

(II) *If* $0 < \alpha \leq 6 - 4\sqrt{2}$ *and in addition,*

$$p(n) \geq \frac{\alpha}{2} \text{ for all large } n, \tag{3.13}$$

and

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{4}\left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}\right) \tag{3.14}$$

then all solutions of Eq.(1)$'$ *are oscillatory.*

Remark 3.4. Observe the following:

(i) When $0 < \alpha \leq \frac{1}{e}$, it is easy to verify that

$$\frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha}\right) > \left(1 - \sqrt{1 - \alpha}\right)^2,$$

and therefore the inequality (3.12) improves the inequality (3.9).

(ii) When $0 < \alpha \leq 6 - 4\sqrt{2}$, because

$$1 - \sqrt{1 - \alpha} > \frac{\alpha}{2},$$

we see that the assumption (3.13) is weaker than the assumption (3.10), and moreover, we can show that

$$\frac{1}{4}\left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}\right) > \alpha\frac{1 - \sqrt{1 - \alpha}}{\sqrt{1 - \alpha}}$$

and so the inequality (3.14) is an improvement of the inequality (3.11).

(iii) When $0 < \alpha \leq \frac{1}{e}$, it is easy to see that

$$\frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right) > \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha}\right)$$

and therefore, in the case of Eq.(1)$''$, the condition $(C_6)''$ is weaker than the condition (3.12).

Observe, however, that when $0 < \alpha \leq 6 - 4\sqrt{2}$, it is easy to show that

$$\frac{1}{4}\left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}\right) > \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right),$$

and therefore in this case and when (3.13) holds, inequality (3.14) improves the inequality $(C_6)''$ and especially, when $\alpha = 6 - 4\sqrt{2} \simeq 0.3431457$, the lower bound in $(C_6)''$ is 0.8929094 while in (3.14) is 0.7573593.

We illustrate by the following example.

Example 3.2. ([4]) Consider the equation

$$\Delta x(n) + p(n)x(n-2) = 0,$$

where

$$p(3n) = \frac{1474}{10000}, \quad p(3n+1) = \frac{1488}{10000}, \quad p(3n+2) = \frac{6715}{10000}, \qquad n = 0, 1, 2, \ldots.$$

Here $k = 2$ and it is easy to see that

$$\alpha_0 = \liminf_{n\to\infty} \sum_{j=n-2}^{n-1} p(j) = \frac{1474}{10000} + \frac{1488}{10000} = 0.2962 < \left(\frac{2}{3}\right)^3 \simeq 0.2962963,$$

and

$$\limsup_{n\to\infty} \sum_{j=n-2}^{n} p(j) = \frac{1474}{10000} + \frac{1488}{10000} + \frac{6715}{10000} = 0.9677.$$

Observe that

$$0.9677 > 1 - \frac{1}{2}\left(1 - \alpha_0 - \sqrt{1 - 2\alpha_0}\right) \simeq 0.967317794,$$

that is, condition (3.12) of Theorem 3.4 is satisfied and therefore all solutions oscillate. Also, condition $(C_6)''$ is satisfied. Observe, however, that

$$0.9677 < 1,$$

$$\alpha_0 = 0.2962 < \left(\frac{2}{3}\right)^3 \simeq 0.2962963,$$

$$0.9677 < 1 - \left(1 - \sqrt{1 - \alpha_0}\right)^2 \simeq 0.974055774,$$

and therefore none of the conditions $(C_2)''$, $(C_3)''$ and (3.9) is satisfied.

If, on the other hand, in the above equation

$$p(3n) = p(3n+1) = \frac{1481}{10000}, \quad p(3n+2) = \frac{6138}{10000}, \qquad n = 0, 1, 2, \ldots,$$

it is easy to see that

$$\alpha_0 = \liminf_{n\to\infty} \sum_{j=n-2}^{n-1} p(j) = \frac{1481}{10000} + \frac{1481}{10000} = 0.2962 < \left(\frac{2}{3}\right)^3 \simeq 0.2962963,$$

and

$$\limsup_{n\to\infty} \sum_{j=n-2}^{n} p(j) = \frac{1481}{10000} + \frac{1481}{10000} + \frac{6138}{10000} = 0.91.$$

Furthermore, it is clear that

$$p(n) \geq \frac{\alpha_0}{2} \quad \text{for all large } n.$$

In this case

$$0.91 > 1 - \frac{1}{4}\left(2 - 3\alpha_0 - \sqrt{4 - 12\alpha_0 + \alpha_0^2}\right) \simeq 0.904724375,$$

that is, condition (3.14) of Theorem 3.4 is satisfied and therefore all solutions oscillate. Observe, however, that

$$0.91 < 1,$$

$$\alpha_0 = 0.2962 < \left(\frac{2}{3}\right)^3 \simeq 0.2962963,$$

$$0.91 < 1 - \left(1 - \sqrt{1 - \alpha_0}\right)^2 \simeq 0.974055774,$$

$$0.91 < 1 - \frac{1}{2}\left(1 - \alpha_0 - \sqrt{1 - 2\alpha_0 - \alpha_0^2}\right) \simeq 0.930883291,$$

and therefore none of the conditions $(C_2)''$, $(C_3)''$, (3.9) and $(C_6)''$ is satisfied.

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