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Existence and Multiplicity of Periodic Solutions for Semilinear Duffing Equations at Resonance

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Abstract. In this paper, we consider the semilinear Duffing equation

$$\ddot{x} + n^2 x + g(x) = p(t),$$

where $p(t)$ is a 2π-periodic function and $\lim_{|x|\to\infty} x^{-1} g(x) = 0$. We give some sharp sufficient conditions for the existence and multiplicity of periodic solutions in terms of the time map. Unlike many existing results in the literature, our main results here allow $g(x)$ be unbounded or oscillatory without asymptotic limits.

AMS Subject Classifications: 34C11, 34C28, 58F35

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1. Introduction and Main Results

As one of the simplest but nontrivial conservative system, the Duffing equation

$$\ddot{x} + \bar{g}(x) = p(t), \qquad p(t + 2\pi) \equiv p(t) \tag{1.0}$$

has been widely investigated by many authors and many results have been obtained for the existence and multiplicity of periodic solutions by various methods, such as critical point theory, phase plane technique and continuation methods based on degree theory. We refer to [2, 3, 5, 7, 6, 11] and the references therein.

In the present paper, we consider the Duffing equation of the form

$$\ddot{x} + n^2 x + g(x) = p(t), \tag{1.1}$$

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where $n \in N, p(t+2\pi) = p(t)$ and $\lim_{|x|\to\infty} x^{-1}g(x) = 0$.

Set

$$\bar{p} = \int_0^{2\pi} p(t)e^{-int}dt. \tag{1.2}$$

In a classical paper [7], Lazer and Leach proved that if $p(t) \in C(R/2\pi Z)$, $g(x) \in C(R)$ is bounded and if

$$|\bar{p}| < 2\max\{\liminf_{x\to+\infty} g(x) - \limsup_{x\to-\infty} g(x),\ \liminf_{x\to-\infty} g(x) - \limsup_{x\to+\infty} g(x)\},$$

then the equation (1.1) has at least one 2π-periodic solution. Moreover, if $g(x)$ is not constant and if

$$|\bar{p}| \geq 2\left(\sup_R g(x) - \inf_R g(x)\right),$$

then the equation (1.1) has no 2π-periodic solution.

In particular, if the limits

$$g(+) := \lim_{x\to+\infty} g(x), \qquad g(-) := \lim_{x\to-\infty} g(x)$$

are finite, and

$$g(-) \leq g(x) \leq g(+), \quad \text{or} \quad g(+) \leq g(x) \leq g(-), \qquad \text{for all } \ x \in R.$$

Then the condition

$$|\bar{p}| < 2|g(+) - g(-)|. \tag{1.3}$$

is a necessary and sufficient condition for the existence of 2π-periodic solutions of (1.1).

Set

$$\Phi(\rho) = \int_0^{2\pi} g(\rho\cos\theta)\cos\theta d\theta \tag{1.4}$$

and

$$\phi^+ = \limsup_{\rho\to+\infty} \Phi(\rho), \quad \phi_+ = \liminf_{\rho\to+\infty} \Phi(\rho). \tag{1.5}$$

In a recent paper [6], Krasnosel'skii and Mawhin gave a new formulation of the Lazer-Leach conditions for the existence of 2π-periodic solutions. A slight modification of Krasnosel'skii and Mawhin's result is given as follows

Theorem A. *Assume that $p(t) \in C(\mathrm{R}/2\pi\mathrm{Z})$ and $g(x) \in C(\mathrm{R})$ is bounded. Then the following statements hold true:*

(i) If one of the following four relations is valid:

$$\phi^+ > |\bar{p}| > \phi_+ > 0, \qquad -\phi_+ > |\bar{p}| > -\phi^+ > 0,$$

$$\phi^+ > |\bar{p}| > 0 \geq \phi_+, \qquad -\phi_+ > |\bar{p}| > 0 \geq -\phi^+,$$

then the equation (1.1) has an unbounded sequence of 2π-periodic solutions.

(ii) If one of the following two conditions

$$\phi_+ > |\bar{p}|, \qquad -\phi^+ > |\bar{p}|,$$

holds, then the equation (1.1) has at least one 2π*-periodic solution, and the set of such solutions is bounded.*

Consider the auxiliary equation

$$\ddot{x} + n^2 x + g(x) = 0, \tag{1.6}$$

which is equivalent to the following autonomous Hamiltonian system

$$\dot{x} = -ny, \quad \dot{y} = nx + n^{-1} g(x), \tag{1.7}$$

with the Hamiltonian function

$$H_0(x, y) = \frac{1}{2} n(x^2 + y^2) + n^{-1} \int_0^x g(s) ds.$$

For $h > 0$, we denote by $\tau(h)$ the least positive period of the orbit $\Gamma_h : H_0(x, y) = h$ of the system (1.7). For $h > 0$, set

$$\Gamma(h) = \sqrt{h} \left(\tau(h) - \frac{2\pi}{n} \right). \tag{1.8}$$

There have been several authors [2, 5, 11] investigating the existence and multiplicity of periodic solutions of (1.1) in terms of the asymptotic behavior of the time map $\tau(h)$. In particular, the following theorem has been proved.

Theorem B. *Assume that* $p(t) \in C(\mathrm{R}/2\pi\mathrm{Z})$ *and* $\lim_{|x| \to +\infty} x^{-1} g(x) = 0$*, then the following statements hold true:*

(i) If $g(x)$ *is continuous and*

$$\limsup_{h \to +\infty} \Gamma(h) = +\infty, \quad \liminf_{h \to +\infty} \Gamma(h) = -\infty,$$

then (1.1) has infinitely many 2π*-periodic solutions.*

(ii) If $g(x)$ *is Lipschitz continuous and*

$$\limsup_{h \to +\infty} |\Gamma(h)| = +\infty,$$

then (1.1) has at least one 2π*-periodic solution.*

A natural question is whether or not (1.1) admits a 2π-periodic solution if

$$\limsup_{h \to +\infty} |\Gamma(h)| < +\infty.$$

In the present paper, we study this problem for (1.1) and obtain the following theorem.

Theorem 1. *Suppose that* $p(t) \in C^2(\mathrm{R}/2\pi\mathrm{Z})$, $g(x) \in C^1(\mathrm{R})$ *and*

$$\lim_{|x|\to+\infty} x^{k-1} g^{(k)}(x) = 0, \qquad k = 0, 1. \tag{1.9}$$

Then the following statements hold true:

(i) (1.1) has at least one 2π*-periodic solution if*

$$|\bar{p}| < \sqrt{2} n^{5/2} \limsup_{h\to+\infty} |\Gamma(h)|; \tag{1.10}$$

(ii) (1.1) has an unbounded sequence of 2π*-periodic solutions if*

$$|\bar{p}| < \sqrt{2} n^{5/2} \min\{\limsup_{h\to+\infty} \Gamma(h), -\liminf_{h\to+\infty} \Gamma(h)\}, \tag{1.11}$$

where $\bar{p}$ *and* $\Gamma(h)$ *are given by (1.2) and (1.8), respectively.*

As a corollary of Theorem 1, we have the following

Theorem 2. *Suppose that* $p(t) \in C^2(\mathrm{R}/2\pi\mathrm{Z})$, $g(x) \in C^1(\mathrm{R})$ *and*

$$\lim_{|x|\to+\infty} |x|^{-1/2} g(x) = 0, \qquad \limsup_{|x|\to+\infty} |x|^{1/2} |g'(x)| < +\infty. \tag{1.12}$$

Then the following statements hold true:

(i) (1.1) has at least one 2π*-periodic solution if*

$$|\bar{p}| < \limsup_{\rho\to+\infty} |\Phi(\rho)|\,; \tag{1.13}$$

(ii) (1.1) has an unbounded sequence of 2π*-periodic solutions if*

$$|\bar{p}| < \min\left\{\limsup_{\rho\to+\infty} \Phi(\rho), -\liminf_{\rho\to+\infty} \Phi(\rho)\right\}, \tag{1.14}$$

where $\bar{p}$ *and* $\Phi(\rho)$ *are given by (1.2) and (1.4), respectively.*

Remark 1. In our main results, the quantities

$$\limsup_{h\to+\infty} |\Gamma(h)|, \qquad \limsup_{\rho\to+\infty} |\Phi(\rho)|,$$

may be infinity. In [6], the authors proposed the following question: It would be interesting to prove some analogs of Theorem A for unbounded functions $g(x)$, either for finite or infinite values ϕ^+ and ϕ_+. Our results can be seen as an answer to this question.

Remark 2. If the asymptotic limits $g(+)$ and $g(-)$ are finite, then by the dominated convergence theorem, (1.13), and hence (1.10), reduce to (1.3), therefore, the inequalities (1.10) and (1.13) are sharp.

The rest of this paper is organized as follows. In Section 2, some technical lemmas will be established and those lemmas will be employed in the proof of our main theorems. The proofs of theorem 1 and theorem 2 will be given in Section 3 and Section 4, respectively.

2. Preliminaries

By introducing a new variable y as $y = -n^{-1}\dot{x}$, (1.1) is changed into the following planar Hamiltonian system

$$\dot{x} = -ny, \quad \dot{y} = nx + \frac{1}{n}g(x) - \frac{1}{n}p(t) \tag{2.1}$$

with the Hamiltonian function

$$H(x, y, t) = \frac{1}{2}n(x^2 + y^2) + \frac{1}{n}G(x) - \frac{1}{n}xp(t)$$

and (1.6) is changed into the Hamiltonian system (1.7) with the Hamiltonian function

$$H_0(x, y) = \frac{1}{2}n(x^2 + y^2) + \frac{1}{n}G(x),$$

where $G(x) = \int_0^x g(s)ds$.

Under the standard symplectic transformation $(r, \theta) \mapsto (x, y)$ with $r > 0$ and $\theta \pmod{2\pi}$, given by

$$x = \sqrt{2r}\cos\theta, \quad y = \sqrt{2r}\sin\theta, \tag{2.2}$$

the systems (2.1) and (1.7) are transformed into the following two Hamiltonian systems

$$\dot{r} = -\frac{\partial}{\partial\theta}h(r, \theta, t), \quad \dot{\theta} = \frac{\partial}{\partial r}h(r, \theta, t) \tag{2.3}$$

and

$$\dot{r} = -\frac{\partial}{\partial\theta}h_0(r, \theta), \quad \dot{\theta} = \frac{\partial}{\partial r}h_0(r, \theta), \tag{2.4}$$

respectively, where

$$h(r, \theta, t) = H(x, y, t) = nr + \frac{1}{n}G(\sqrt{2r}\cos\theta) - \frac{1}{n}\sqrt{2r}p(t)\cos\theta, \tag{2.5}$$

and

$$h_0(r, \theta) = H_0(x, y) = nr + \frac{1}{n}G(\sqrt{2r}\cos\theta). \tag{2.6}$$

From (1.9) and (2.6), we see that

$$\begin{aligned} \lim_{r\to+\infty} \tfrac{h_0}{r} &= \lim_{r\to+\infty} \left[n + \tfrac{1}{nr}G(\sqrt{2r}\cos\theta)\right] \\ &= \lim_{r\to+\infty} \left[n + \tfrac{1}{n\sqrt{2r}}g(\sqrt{2r}\cos\theta)\cos\theta\right] \\ &= n \end{aligned}$$

and

$$\frac{\partial h_0}{\partial r} = n + \frac{1}{n\sqrt{2r}}g(\sqrt{2r}\cos\theta)\cos\theta > 0,$$

for $r \gg 1$. By the implicit function theorem, we know that there exists a function $R_1 = R_1(h, \theta)$ of class C^2 with $|R_1(h, \theta)| \le \epsilon(h)h$ such that

$$r_0(h, \theta) = n^{-1}h + R_1(h, \theta)$$

solve the equation

$$h = h(r_0, \theta) = nr_0 + \frac{1}{n}G(\sqrt{2r_0}\cos\theta).$$

In a similar way, we may show that there is a function $R_2 = R_2(h, t, \theta)$ of class C^2 with $|R_2(h, t, \theta)| \le \epsilon(h)h$ such that

$$r(h, t, \theta) = n^{-1}h + R_2(h, t, \theta)$$

satisfies

$$h = h(r, t, \theta) = nr + \frac{1}{n}G(\sqrt{2r}\cos\theta) - \frac{1}{n}\sqrt{2r}p(t)\cos\theta.$$

Notice that

$$rd\theta - hdt = -(hdt - rd\theta), \qquad rd\theta - h_0dt = -(h_0dt - rd\theta),$$

we know that

$$\frac{dh}{d\theta} = -\frac{\partial}{\partial t}r(h, t, \theta), \quad \frac{dt}{d\theta} = \frac{\partial}{\partial h}r(h, t, \theta) \tag{2.7}$$

and

$$\frac{dh}{d\theta} = -\frac{\partial}{\partial t}r_0(h, \theta) = 0, \quad \frac{dt}{d\theta} = \frac{\partial}{\partial h}r_0(h, \theta) \tag{2.8}$$

are two Hamiltonian systems with Hamiltonian functions $r = r(h, t, \theta)$ and $r_0 = r_0(h, \theta)$, respectively. Now the action, angle and time variables are h, t and θ respectively. This trick has been used by several authors (see [8, 9, 10]).

It follows from (2.6) and (2.8) that

$$\begin{aligned} \tau(h) &= \int_0^{2\pi} \tfrac{\partial}{\partial h}r_0(h, \theta)d\theta \\ &= \int_0^{2\pi} \frac{d\theta}{\frac{\partial}{\partial r}h_0(r_0, \theta)} \\ &= \int_0^{2\pi} \frac{d\theta}{n + n^{-1}(2r_0)^{-1/2}g(\sqrt{2r_0}\cos\theta)\cos\theta}, \end{aligned} \tag{2.9}$$

where $r_0 = r_0(h, \theta)$.

Lemma 2.1. *If (1.9) holds, then*

$$r(h, t, \theta) = r_0(h, \theta) + \sqrt{2}n^{-5/2}h^{1/2}p(t)\cos\theta + R(h, t, \theta), \tag{2.10}$$

with

$$\left|\frac{\partial^{k+m}}{\partial h^k \partial t^m}R(h, t, \theta)\right| \le \varepsilon(h) \cdot h^{-k+1/2}, \tag{2.11}$$

for $k+m \le 1$, where $\varepsilon(h) \to 0$ as $h \to +\infty$.
Proof. The proof is similar to that of Lemma 2.2 in [10] and is omitted. □

Lemma 2.2. *If (1.9) holds, there is a canonical transformation*

$$\Psi : h = \rho, \quad t = \tau + T(\rho, \theta)$$

with $T(\rho, \theta + 2\pi) = T(\rho, \theta)$ such that the transformed system of (2.7) is of the form

$$\frac{d\rho}{d\theta} = -\frac{\partial}{\partial \tau}\tilde{r}(\rho, \tau, \theta), \qquad \frac{d\tau}{d\theta} = \frac{\partial}{\partial \rho}\tilde{r}(\rho, \tau, \theta), \tag{2.12}$$

where

$$\tilde{r}(\rho, \tau, \theta) = J(\rho) + \sqrt{2}n^{-5/2}\rho^{1/2}p(\tau)\cos\theta + \tilde{R}(\rho, \tau, \theta),$$
$$J(\rho) = \frac{1}{2\pi}\int_0^{2\pi} r_0(\rho, \theta)d\theta.$$

For the new perturbation $\tilde{R}$, we have

$$\left| \frac{\partial^{k+m}}{\partial \rho^k \partial \tau^m}\tilde{R}(\rho, \tau, \theta) \right| \le \varepsilon(\rho) \cdot \rho^{-k+1/2} \tag{2.13}$$

for $k+m \le 1$, where $\varepsilon(\rho) \to 0$ as $\rho \to +\infty$.
Proof. The proof is similar to that of Lemma 2.3 in [10] and is omitted. □

Let $\theta = n\vartheta$, then the system (2.12) is transformed into the form

$$\frac{d\rho}{d\vartheta} = -\frac{\partial}{\partial \tau}\tilde{H}(\rho, \tau, \vartheta), \qquad \frac{d\tau}{d\vartheta} = \frac{\partial}{\partial \rho}\tilde{H}(\rho, \tau, \vartheta), \tag{2.14}$$

where

$$\tilde{H}(\rho, \tau, \vartheta) = n\tilde{r}(\rho, \tau, n\vartheta) = nJ(\rho) + \sqrt{2}n^{-3/2}\rho^{1/2}p(\tau)\cos n\vartheta + n\tilde{R}(\rho, \tau, n\vartheta).$$

As n is a positive integer, the function $\tilde{H}(\rho, \tau, \vartheta)$ is 2π-periodic in ϑ.

Lemma 2.3. *Assume that (1.9) holds, then*

$$\lim_{h \to +\infty} h^{k-1/2}\Gamma^{(k)}(h) = 0, \quad k = 0, 1.$$

Proof. By (2.9), it is easy to check that

$$h^{-1/2}\Gamma(h) = -\frac{1}{n}\int_0^{2\pi} \frac{(2r_0)^{-1/2}g(\sqrt{2r_0}\cos\theta)\cos\theta}{n^2 + (2r_0)^{-1/2}g(\sqrt{2r_0}\cos\theta)\cos\theta}d\theta,$$

and

$$\begin{aligned} h^{1/2}\Gamma'(h) &= \frac{1}{2}h^{-1/2}\Gamma(h) \\ &\quad + \frac{1}{n}\int_0^{2\pi} \frac{h(2r_0)^{-1}[(2r_0)^{-1/2}g(\sqrt{2r_0}\cos\theta)\cos\theta - g'(\sqrt{2r_0}\cos\theta)\cos^2\theta]}{[n + n^{-1}(2r_0)^{-1/2}g(\sqrt{2r_0}\cos\theta)\cos\theta]^3}d\theta, \end{aligned}$$

where $r_0 = r_0(h, \theta)$. Now the conclusion follows from (1.9) and the dominated convergence theorem and the proof is complete. □

3. Proof of Theorem 1

At first, we give an expression for the Poincaré map of the system (2.14).

In order to calculate the Poincaré map, we introduce a new variable v and a small positive parameter δ by the formula

$$\rho = \delta^{-2} v, \qquad v \in [a, b], \tag{3.1}$$

where $b > a > 0$ are independent of δ.

In the new action and angle variables (v, τ), the system (2.14) can be written in the form

$$\frac{dv}{d\vartheta} = -\frac{\partial}{\partial \tau}\hat{H}(v, \tau, \vartheta, \delta), \qquad \frac{d\tau}{d\vartheta} = \frac{\partial}{\partial v}\hat{H}(v, \tau, \vartheta, \delta), \tag{3.2}$$

where

$$\begin{aligned} \hat{H}(v, \tau, \vartheta, \delta) &= \delta^2 \tilde{H}(\delta^{-2} v, \tau, \vartheta) \\ &= \delta^2 n J(\delta^{-2} v) + \sqrt{2}\delta n^{-3/2} v^{1/2} p(\tau) \cos n\vartheta + \delta^2 n \tilde{R}(\delta^{-2} v, \tau, n\vartheta). \end{aligned}$$

Let

$$\hat{R}(v, \tau, \vartheta, \delta) = \delta^2 n \tilde{R}(\delta^{-2} v, \tau, n\vartheta).$$

By virtue of Lemma 2.2, it is easy to show that

$$\delta^{-1} \cdot \left| \frac{\partial^{k+m}}{\partial v^k \partial \tau^m} \hat{R}(v, \tau, \vartheta, \delta) \right| \le n\varepsilon(\delta^{-2} v) v^{-k+1/2} \to 0 \quad \text{as} \quad \delta \to 0^+ \tag{3.3}$$

for $k + m \le 1$.

Since

$$\tau(\delta^{-2} v) = \frac{2\pi}{n} + \delta v^{-1/2} \Gamma(\delta^{-2} v),$$

we may rewrite the system (3.2) explicitly

$$\begin{cases} \frac{dv}{d\vartheta} = -\sqrt{2}\delta n^{-3/2} v^{1/2} p'(\tau) \cos n\vartheta - \frac{\partial \hat{R}}{\partial \tau}, \\ \frac{d\tau}{d\vartheta} = 1 + \frac{1}{2\pi}\delta n v^{-1/2} \Gamma(\delta^{-2} v) + \frac{\sqrt{2}}{2}\delta n^{-3/2} v^{-1/2} p(\tau) \cos n\vartheta + \frac{\partial \hat{R}}{\partial v}. \end{cases} \tag{3.4}$$

Denote by $(v(\vartheta, v_0, \tau_0), \tau(\vartheta, v_0, \tau_0))$ the solution of (3.4) with the initial condition

$$(v(0, v_0, \tau_0), \tau(0, v_0, \tau_0)) = (v_0, \tau_0).$$

From (3.3), we know that for $\delta \ll 1$, the solution $(v(\vartheta, v_0, \tau_0), \tau(\vartheta, v_0, \tau_0))$ exists in $[0, 4\pi]$ for any $(v_0, \tau_0) \in [a, b] \times [0, 2\pi]$. Moreover,

$$0 < \frac{1}{2}a \le v(\vartheta, v_0, \tau_0) \le 2b, \qquad \forall\ \vartheta \in [0, 4\pi].$$

Assume that the solution $(v(\vartheta, v_0, \tau_0), \tau(\vartheta, v_0, \tau_0))$ has the following expression

$$v(\vartheta, v_0, \tau_0) = v_0 + \delta F_2(\vartheta, v_0, \tau_0), \qquad \tau(\vartheta, v_0, \tau_0) = \tau_0 + \vartheta + \delta F_1(\vartheta, v_0, \tau_0). \tag{3.5}$$

Then the Poincaré map of (3.4), denoted by P, has the form

$$P(v_0,\tau_0) = (v_0 + \delta F_2(2\pi, v_0, \tau_0), \tau_0 + 2\pi + \delta F_1(2\pi, v_0, \tau_0)).$$

From the above discussions, we know that if $\delta \ll 1$, this map is well defined in the region $[a,b]\times[0,2\pi]$.

Since $(v(\vartheta, v_0, \tau_0), \tau(\vartheta, v_0, \tau_0))$ is the solution of (3.4), we have

$$\begin{cases} \frac{dF_1}{d\vartheta} = \left[\frac{1}{2\pi} n\Gamma(\delta^{-2}(v_0+\delta F_2)) + \frac{\sqrt{2}}{2} n^{-3/2} p(\tau)\cos n\vartheta\right](v_0+\delta F_2)^{-1/2} + \delta^{-1}\frac{\partial \hat{R}}{\partial v}, \\ \frac{dF_2}{d\vartheta} = -\sqrt{2} n^{-3/2}(v_0+\delta F_2)^{1/2} p'(\tau)\cos n\vartheta - \delta^{-1}\frac{\partial \hat{R}}{\partial \tau}. \end{cases} \tag{3.6}$$

As in the proof in [1], we can show from (3.6) and (3.3) that

$$|F_1(\vartheta, v_0, \tau_0)|, \quad |F_2(\vartheta, v_0, \tau_0)| \le C, \tag{3.7}$$

uniformly in ϑ.

By virtue of Lemma 2.3, it follows from (3.3),(3.6) and (3.7) that

$$\begin{aligned} F_1(2\pi, v_0, \tau_0) &= \int_0^{2\pi}\left[\frac{1}{2\pi} n\Gamma(\delta^{-2}(v_0+\delta F_2)) + \frac{\sqrt{2}}{2} n^{-3/2} p(\tau(\vartheta))\cos n\vartheta\right] \\ &\qquad \times (v_0+\delta F_2)^{-1/2} d\vartheta + o(1) \\ &= n v_0^{-1/2}\Gamma(\delta^{-2}v_0) + \frac{\sqrt{2}}{2} n^{-3/2} v_0^{-1/2}\int_0^{2\pi} p(\tau(\vartheta))\cos n\vartheta d\vartheta + o(1), \\ F_2(2\pi, v_0, \tau_0) &= -\sqrt{2} n^{-3/2}\int_0^{2\pi}(v_0+\delta F_2)^{1/2} p'(\tau(\vartheta))\cos n\vartheta d\vartheta + o(1) \\ &= -\sqrt{2} n^{-3/2} v_0^{1/2}\int_0^{2\pi} p'(\tau(\vartheta))\cos n\vartheta d\vartheta + o(1). \end{aligned}$$

In the following, we compute the integrals in the above formulas. Let

$$p_n^c = \int_0^{2\pi} p(\vartheta)\cos n\vartheta d\vartheta, \qquad p_n^s = \int_0^{2\pi} p(\vartheta)\sin n\vartheta d\vartheta.$$

Then, from (3.5) and (3.7), we see that

$$\begin{aligned} \int_0^{2\pi} p(\tau(\vartheta))\cos n\vartheta d\vartheta &= \int_0^{2\pi} p(\tau_0+\vartheta)\cos n\vartheta d\vartheta + o(1) \\ &= \int_0^{2\pi} p(\vartheta)\cos n(\vartheta-\tau_0) d\vartheta + o(1) \\ &= p_n^c\cos n\tau_0 + p_n^s\sin n\tau_0 + o(1), \end{aligned}$$

$$\begin{aligned}\int_0^{2\pi} p'(\tau(\vartheta))\cos n\vartheta d\vartheta &= \int_0^{2\pi} p'(\tau_0+\vartheta)\cos n\vartheta d\vartheta + o(1)\\ &= \int_0^{2\pi} p'(\vartheta)\cos n(\vartheta-\tau_0)d\vartheta + o(1)\\ &= n\int_0^{2\pi} p(\vartheta)\sin n(\vartheta-\tau_0)d\vartheta + o(1)\\ &= n(p_n^s\cos n\tau_0 - p_n^c \sin n\tau_0) + o(1).\end{aligned}$$

Now we get an expression of the Poincaré map P as

$$P:\begin{cases}\tau_1 = \tau_0 + 2\pi + \delta\ell_1(v_0,\tau_0) + o(\delta),\\ v_1 = v_0 - \delta\ell_2(v_0,\tau_0) + o(\delta),\end{cases} \tag{3.8}$$

where

$$\begin{cases}\ell_1(v_0,\tau_0) = nv_0^{-1/2}\Gamma(\delta^{-2}v_0) + \frac{\sqrt{2}}{2}n^{-3/2}v_0^{-1/2}(p_n^c\cos n\tau_0 + p_n^s \sin n\tau_0),\\ \ell_2(v_0,\tau_0) = \sqrt{2}n^{-1/2}v_0^{1/2}(p_n^s\cos n\tau_0 - p_n^c\sin n\tau_0).\end{cases} \tag{3.9}$$

Since

$$|p_n^c\cos n\tau_0 + p_n^s\sin n\tau_0| \le \left|\int_0^{2\pi} p(t)e^{-int}dt\right|,$$

now the statements of Theorem 1 are easy consequence of (3.8), (3.9) and the following two fixed point theorems and the details will be omitted.

Fixed Point Theorem. (T. R. Ding [3]) *Let $B\subset \mathrm{R}^2$ be a compact domain with star-shaped boundary about the origin O, and $T: B\to \mathrm{R}^2$ be a continuous mapping. If for any $p\in\partial B$ and $\lambda\ge 1$, $\overline{OT(p)}\ne\lambda\overline{Op}$, then there exists at least one fixed point $p_0\in B$ for T.*

Twist Theorem. (W. Y. Ding [4]) *Let $T:\mathrm{R}^2\to\mathrm{R}^2$ be a area-preserving homeomorphism and D_1, D_2 be two open domains bounded by Γ_1 and Γ_2 star-shaped curves about the origin O, respectively, such that $O\in D_1\subset\bar{D}_1\subset D_2$. If the polar coordinate expression of T,*

$$\bar{r} = f(r,\theta),\qquad \bar{\theta} = \theta + g(r,\theta),$$

satisfies the twist condition $g(r,\theta)>0$ on Γ_1 and $g(r,\theta)<0$ on Γ_2, then T has at least two fixed points in $B = D_2\setminus\bar{D}_1$.

4. Proof of Theorem 2

In this section, we denote by $\varepsilon(h)$ an universal function satisfying $\lim_{h\to+\infty}\varepsilon(h) = 0$ not concerning it's quantity.

Firstly, we notice that $R_1(h,\theta) = r_0(h,\theta) - n^{-1}h$ satisfies

$$R_1(h,\theta) = -n^{-2}G(\sqrt{2(n^{-1}h+R_1)}\cos\theta),\quad |R_1(h,\theta)|\le\varepsilon(h)h,$$

it follows that

$$\Delta \cdot R_1(h,\theta) = -n^{-2}G(\sqrt{2n^{-1}h}\cos\theta), \tag{4.1}$$

$$\Delta = 1 + n^{-2}[2(n^{-1}h+\mu R_1)]^{-1/2}g(\sqrt{2(n^{-1}h+\mu R_1)\cos\theta}\cos\theta, \quad \mu\in[0,1].$$

Since $|R_1(h,\theta)| \le \varepsilon(h)h$, we have

$$\Delta = 1+\varepsilon(h). \tag{4.2}$$

From (4.1), (4.2) and the rule of L'Hospital, it follows that

$$\begin{aligned}
\lim_{h\to+\infty}\tfrac{R_1(h,\theta)}{h^{3/4}} &= \lim_{h\to+\infty}\Delta\tfrac{R_1(h,\theta)}{h^{3/4}}\\
&= -n^{-2}\lim_{h\to+\infty}\tfrac{G(\sqrt{2n^{-1}h}\cos\theta)}{h^{3/4}}\\
&= -n^{-2}\lim_{h\to+\infty}\tfrac{4n^{-1}g(\sqrt{2n^{-1}h}\cos\theta)\cos\theta}{3\sqrt{2n^{-1}h}\cdot h^{-1/4}}\\
&= -\tfrac{2\sqrt{2}}{3}n^{-5/2}\lim_{h\to+\infty}h^{-1/4}g(\sqrt{2n^{-1}h}\cos\theta)\cos\theta\\
&= 0.
\end{aligned}$$

Hence, we have

$$|R_1(h,\theta)| \le \varepsilon(h)h^{3/4}. \tag{4.3}$$

It follows from (1.12), (2.9) and (4.3) that

$$\begin{aligned}
h^{-1/2}\Gamma(h) &= -\tfrac{1}{n^3}\int_0^{2\pi}\tfrac{(2r_0)^{-1/2}g(\sqrt{2r_0}\cos\theta)\cos\theta}{1+n^{-2}(2r_0)^{-1/2}g(\sqrt{2r_0}\cos\theta)\cos\theta}d\theta\\
&= -\tfrac{1}{n^3}\int_0^{2\pi}(2r_0)^{-1/2}g(\sqrt{2r_0}\cos\theta)\cos\theta d\theta+\varepsilon(h)h^{-1/2}\\
&= -\tfrac{1}{n^3}\int_0^{2\pi}(2n^{-1}h)^{-1/2}g(\sqrt{2n^{-1}h}\cos\theta)\cos\theta d\theta\\
&\quad -\tfrac{1}{n^3}\int_0^{2\pi}\{-[2(n^{-1}h+\mu R_1)]^{-3/2}g(\sqrt{2(n^{-1}h+\mu R_1)}\cos\theta)\cos\theta\\
&\quad +[2(n^{-1}h+\mu R_1)]^{-1}g'(\sqrt{2(n^{-1}h+\mu R_1)}\cos\theta)\cos^2\theta\}\cdot R_1 d\theta\\
&\quad +\varepsilon(h)h^{-1/2}\\
&= -\tfrac{1}{n^3}\int_0^{2\pi}(2n^{-1}h)^{-1/2}g(\sqrt{2n^{-1}h}\cos\theta)\cos\theta d\theta\\
&\quad -\tfrac{1}{n^3}\int_0^{2\pi}\{-[2(n^{-1}h+\mu R_1)]^{-1/4}g(\sqrt{2(n^{-1}h+\mu R_1)}\cos\theta)\cos\theta\\
&\quad +[2(n^{-1}h+\mu R_1)]^{1/4}g'(\sqrt{2(n^{-1}h+\mu R_1)}\cos\theta)\cos^2\theta\}\\
&\qquad \times[2(n^{-1}h+\mu R_1)]^{-5/4}\cdot R_1 d\theta+\varepsilon(h)h^{-1/2}\\
&= -\tfrac{1}{n^3}\int_0^{2\pi}(2n^{-1}h)^{-1/2}g(\sqrt{2n^{-1}h}\cos\theta)\cos\theta d\theta+\varepsilon(h)h^{-1/2},
\end{aligned}$$

where $\mu \in [0,1]$, and hence,

$$\Gamma(h) = \frac{\sqrt{2}}{2} n^{-5/2} \int_0^{2\pi} g(\sqrt{2n^{-1}h}\cos\theta)\cos\theta d\theta + \varepsilon(h),$$

Therefore, the conclusions of Theorem 2 follow from Theorem 1 and the proof is complete.

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