International Jr, of Qualitative Theory of Differential Equations and Applications
Vol. 2 No. 1 (June, 2016)
ISSN: 0973-3590

Exponential Stability of a Class of Neural Networks with Time-Varying Delay*

Jingbing Wu, Lihong Huang†

College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, P. R. China

Abstract. Global exponential stability for a class of cellular neural networks (CNNs) with time-varying delay is considered. By using the method of Lyapunov Krasovskii functional and linear matrix inequality (LMI) technique, some sufficient conditions for global exponential stability of CNNs are obtained. The conditions presented here are related to the size of delay. An example is given to illustrate the feasibility of our results.

AMS Subject Classifications: 34K20; 92B20

Keywords: Cellular neural network; Exponential stability; Delay; LMI; Lyapunov Krasovskii

1. Introduction

Cellular Neural Network (CNN) was introduced by Chua and Yang [1]. Applications of CNN in physical systems include connected component detection, hole filling, optimization, associative memories, pattern recognition, and signal processing [2]. However, in order to deal with moving images, one must introduce the time delay in signal transmission among the cells. This leads to the model of CNN with delay (DCNN) [3]. It is well known that time delay may cause instability, divergent oscillation in many systems. In recent years, the stability of DCNN has become an important topic of theoretical studies. As a result, many sufficient conditions ensuring

E-mail addresses: wujingbing820411@163.com (J. Wu), lhhuang@hnu.cn (L. Huang)

*Research supported by National Natural Science Foundation of China (10771055) and the Doctor Program Foundation of Chinese Ministry of Education (20050532023).

†Corresponding author

global asymptotic stability and global exponential stability for CNN with constant or time-varying delays have been derived. Some of them can be found in, for example, [4-10]. In this note, we study the exponential stability of DCNN. Some criteria on exponential stability are presented by employing a more general type of Lyapunov-Krasovskii functional and linear matrix inequality (LMI). These conditions in our criteria are dependent on the size of delay. One example is presented to show the applicability of our results.

2. Notations

For convenience of expressions, throughout this paper, we will use the following notations:
B^{T}: transpose of matrix B;
B^{-1}: inverse of matrix B;
$\lambda_m(B)$: minimal eigenvalue of matrix B;
$\lambda_M(B)$: maximal eigenvalue of matrix B;
$\mathrm{diag}\{b_i > 0\}$: diagonal matrix with the positive diagonal elements $b_i, i = 1, 2, \cdots n$;
$B > 0$ (resp. $B < 0$): B is a positive (resp., negative) definite and symmetric matrix;
x_t: segment of $x(s)$ on $[t - \tau(t), t]$;
$\| x_t \|$: $\sup\limits_{t-\tau(t)\le s\le t} \| x(s) \|$.

3. Preliminaries

Consider the following DCNN:

$$\dot{u}_i(t) = -u_i(t) + \sum_{j=1}^{n} a_{ij} g_j(u_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(u_j(t - \tau(t))) + I_i \quad ,$$

or equivalently

$$\dot{u}(t) = -u(t) + Ag(u(t)) + Bg(u(t - \tau(t))) + I, \tag{1}$$

where $u(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^{\mathrm{T}}$, $u(t - \tau(t)) = [u_1(t - \tau(t)), u_2(t - \tau(t)), \cdots, u_n(t-\tau(t))]^{\mathrm{T}}$, $n \ge 2$ is the number of neurons in the network; $g(u(t)) = [g_1(u_1(t)), \cdots, g_n(u_n(t))]^{\mathrm{T}} \in R^{\mathrm{n}}$ denotes the activation function of the neurons; $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ are known constant matrices, A is referred to as the feedback matrix, B represents the delayed feedback matrix, $I = (I_1, I_2, \cdots, I_n)^{\mathrm{T}}$ is an external bias vector, $\tau(t)$ is nonnegative, bounded, and differentiable with $0 \le \tau(t) \le \tau_M$ and $\tau'(t) \le d < 1$ where

$$\tau_M = \sup_{t\epsilon R} \tau(t) \qquad \text{and} \qquad d = \sup_{t\epsilon R} \tau'(t).$$

Lemma 1. ([13], Theorem 2.3) *Suppose that there exist non-negative constants p_j and q_j such that $| g_j(x) | \le p_j | x | + q_j$ for all $x \in R^1$ and $j = 1, 2, \cdots, n$. Let*

$\bar{\kappa}_{ij} = (| a_{ij} | + | b_{ij} |)p_j$ *and* $\bar{K} = (\bar{\kappa}_{ij})_{n\times n}$. *If* $\rho(\bar{K}) < 1$, *then system (1) has at least one equilibrium.*

The following assumptions will be employed in our main results.

(H_1) There exists a non-negative constant l such that

$$0 \le \frac{g_i(x) - g_i(y)}{x - y} \le l, \ \forall x \ne y.$$

(H_2) $\rho(K) < 1$, where $\rho(K)$ denotes the spectral radius of matrix $K = (\kappa_{ij})_{n\times n}$ and $\kappa_{ij} = l(| a_{ij} | + | b_{ij} |)$.

Lemma 2. *Suppose that* (H_1) *and* (H_2) *are satisfied, then system (1) has at least one equilibrium.*
Proof. In view of (H_1), we have

$$0 \le \frac{g_i(x) - g_i(0)}{x} \le l \quad \text{for} \quad x \ne 0.$$

It follows that,

$$| g_i(x) - g_i(0) | \le l \mid x \mid .$$

Therefore,

$$\begin{aligned} | g_i(x) | &= | (g_i(x) - g_i(0)) + g_i(0) | \\ &\le | g_i(x) - g_i(0) | + | g_i(0) | \\ &\le l \mid x \mid + | g_i(0) |, \end{aligned}$$

which, together with (H_2) and Lemma 1, implies that system (1) has at least one equilibrium. □

Definition 1. *An equilibrium* $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^{\mathrm{T}}$ *of (1) is said to be globally exponentially stable, if there exist* $k > 0$ *and* $\gamma(k) > 0$ *such that for any solution* $u(t)$ *of (1), we have,*

$$\| u(t) - u^* \| \le \gamma(k) e^{-kt} \sup_{-\tau(0)\le s\le 0} \| u(s) - u^* \|, \ t \ge 0,$$

where k *is convergence rate of exponential stability.*

We point out that if a u^* is global exponential stable, then it must be unique.

4. Global exponential stability analysis

In this section, we always assume that (H_1) and (H_2) hold so that there is an equilibribirum $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^{\mathrm{T}}$ for (1). In order to study the global exponential stability of this equilibrium, we let $x(t) = u(t) - u^*$. Then model (1) is transformed to the following:

$$\dot{x}(t) = -x(t) + Af(x(t)) + Bf(x(t - \tau(t))), \tag{2}$$

where $f_i(x_i) = g_i(x_i + u_i^*) - g_i(u_i^*)$ with $f_i(0) = 0$ for $i = 1, 2, \cdots, n$. Note that under the assumption (H_1) the functions $f_i(x_i)$ satisfies the condition $0 \le \frac{f_i(x_i)}{x_i} \le l$ for $x_i \ne 0$, $i = 1, 2, \cdots, n$.

Theorem 1. *The equilibrium u^* is globally exponentially stable, if there exist positive definite matrices P and Q, a positive diagonal matrix D, and positive constants β and k, such that the following conditions are satisfied:*

$$\Omega_1(\beta, k) = P - 2kP - 2k\beta l D - \frac{e^{2k\tau_M}}{1-d} PBQ^{-1}B^{\mathrm{T}}P > 0, \tag{3}$$

$$\Omega_2(\beta, k) = \frac{2\beta D}{l} - \beta(DA + A^{\mathrm{T}}D) - A^{\mathrm{T}}PA - 2Q - \frac{\beta^2 e^{2k\tau_M}}{1-d} DBQ^{-1}B^{\mathrm{T}}D \ge 0. \tag{4}$$

Proof. To prove the theorem, it suffices to show that the trivial equilibrium $x = 0$ of system (2) is globally exponentially stable.

Set the following positive definite Lyapunov functional:

$$\begin{aligned} V(x(t)) &= e^{2kt}x^{\mathrm{T}}(t)Px(t) + 2\beta e^{2kt}\sum_{i=1}^{n} d_i \int_0^{x_i(t)} f_i(s)\,\mathrm{d}s \\ &\quad +2\int_{t-\tau(t)}^{t} e^{2ks} f^{\mathrm{T}}(x(s))Qf(x(s))\,\mathrm{d}s, \end{aligned}$$

where $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, $D = \mathrm{diag}\{d_i\}$, $d_i > 0, i = 1, 2, \cdots n$, and β and k are positive constants. Calculate the derivatives of $V(x(t))$ along the solutions of (2), we have

$$\begin{aligned} \dot{V}(x(t)) &= 2ke^{2kt}x^{\mathrm{T}}(t)Px(t) + 2e^{2kt}x^{\mathrm{T}}(t)P\dot{x}(t) + 4k\beta e^{2kt}\sum_{i=1}^{n} d_i \int_0^{x_i(t)} f_i(s)\,\mathrm{d}s \\ &\quad +2\beta e^{2kt} f^{\mathrm{T}}(x(t))D\dot{x}(t) + 2e^{2kt}f^{\mathrm{T}}(x(t))Qf(x(t)) \\ &\quad -2e^{2k(t-\tau(t))}(1-\dot{\tau}(t))f^{\mathrm{T}}(x(t-\tau(t)))Qf(x(t-\tau(t))). \end{aligned}$$

Since

$$\int_0^{x_i(t)} f_i(s)\,\mathrm{d}s \le \frac{l}{2}x_i^2(t),$$

we have

$$\sum_{i=1}^{n} d_i \int_0^{x_i(t)} f_i(s)\,\mathrm{d}s \le \sum_{i=1}^{n} \frac{l}{2} d_i x_i^2(t) = \frac{l}{2}x^{\mathrm{T}}(t)Dx(t).$$

Thus, we can write

$$\begin{aligned}\dot{V}(x(t)) &\leq e^{2kt}\{2kx^{\mathrm{T}}(t)Px(t)+2x^{\mathrm{T}}(t)P(-x(t)+Af(x(t))+Bf(x(t-\tau(t))))\\&\quad+2k\beta lx^{\mathrm{T}}(t)Dx(t)+2\beta f^{\mathrm{T}}(x(t))D(-x(t)+Af(x(t))\\&\quad+Bf(x(t-\tau(t))))+2f^{\mathrm{T}}(x(t))Qf(x(t))\\&\quad-2(1-d)e^{-2k\tau(t)}f^{T}(x(t-\tau(t)))Qf(x(t-\tau(t)))\}\\&\leq e^{2kt}\{2kx^{\mathrm{T}}(t)Px(t)-2x^{\mathrm{T}}(t)Px(t)+2x^{\mathrm{T}}(t)PAf(x(t))\\&\quad+2x^{\mathrm{T}}(t)PBf(x(t-\tau(t)))+2k\beta lx^{\mathrm{T}}(t)Dx(t)-2\beta f^{\mathrm{T}}(x(t))Dx(t)\\&\quad+2\beta f^{\mathrm{T}}(x(t))DAf(x(t))+2\beta f^{\mathrm{T}}(x(t))DBf(x(t-\tau(t)))\\&\quad+2f^{\mathrm{T}}(x(t))Qf(x(t))-(1-d)e^{-2k\tau_M}f^{\mathrm{T}}(x(t-\tau(t)))Qf(x(t-\tau(t)))\\&\quad-(1-d)e^{-2k\tau_M}f^{\mathrm{T}}(x(t-\tau(t)))Qf(x(t-\tau(t)))\}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}2x^{\mathrm{T}}(t)PAf(x(t)) &= -[P^{1/2}x(t)-P^{1/2}Af(x(t))]^{\mathrm{T}}[P^{1/2}x(t)-P^{1/2}Af(x(t))]\\&\quad+x^{\mathrm{T}}(t)Px(t)+f^{\mathrm{T}}(x(t))A^{\mathrm{T}}PAf(x(t))\\&\leq x^{\mathrm{T}}(t)Px(t)+f^{\mathrm{T}}(x(t))A^{\mathrm{T}}PAf(x(t)),\end{aligned}\tag{5}$$

$$\begin{aligned}&-(1-d)e^{-2k\tau_M}f^{\mathrm{T}}(x(t-\tau(t)))Qf(x(t-\tau(t)))+2x^{\mathrm{T}}(t)PBf(x(t-\tau(t)))\\&=-[\sqrt{1-d}e^{-k\tau_M}Q^{1/2}f(x(t-\tau(t)))-\frac{1}{\sqrt{1-d}}e^{k\tau_M}Q^{-1/2}B^{\mathrm{T}}Px(t)]^{\mathrm{T}}\\&\quad[\sqrt{1-d}e^{-k\tau_M}Q^{1/2}f(x(t-\tau(t)))-\frac{1}{\sqrt{1-d}}e^{k\tau_M}Q^{-1/2}B^{\mathrm{T}}Px(t)]\\&\quad+\frac{1}{1-d}e^{2k\tau_M}x^{\mathrm{T}}(t)PBQ^{-1}B^{\mathrm{T}}Px(t)\\&\leq\frac{1}{1-d}e^{2k\tau_M}x^{\mathrm{T}}(t)PBQ^{-1}B^{\mathrm{T}}Px(t),\end{aligned}\tag{6}$$

$$\begin{aligned}&-(1-d)e^{-2k\tau_M}f^{\mathrm{T}}(x(t-\tau(t)))Qf(x(t-\tau(t)))+2\beta f^{\mathrm{T}}(x(t))DBf(x(t-\tau(t)))\\&=-[\sqrt{1-d}e^{-k\tau_M}Q^{1/2}f(x(t-\tau(t)))-\frac{\beta}{\sqrt{1-d}}e^{k\tau_M}Q^{-1/2}B^{\mathrm{T}}Df(x(t))]^{\mathrm{T}}\\&\quad[\sqrt{1-d}e^{-k\tau_M}Q^{1/2}f(x(t-\tau(t)))-\frac{\beta}{\sqrt{1-d}}e^{k\tau_M}Q^{-1/2}B^{\mathrm{T}}Df(x(t))]\\&\quad+\frac{\beta^2}{1-d}e^{2k\tau_M}f^{\mathrm{T}}(x(t))DBQ^{-1}B^{\mathrm{T}}Df(x(t))\\&\leq\frac{\beta^2}{1-d}e^{2k\tau_M}f^{\mathrm{T}}(x(t))DBQ^{-1}B^{\mathrm{T}}Df(x(t)),\end{aligned}\tag{7}$$

$$-2\beta f^{\mathrm{T}}(x(t))Dx(t)\leq-\frac{2\beta}{l}f^{\mathrm{T}}(x(t))Df(x(t)).\tag{8}$$

From the inequalities (5)-(8), it follows that

$$\begin{aligned}\dot{V}(x(t)) &\leq -e^{2kt}x^{\mathrm{T}}(t)(P-2kP-2k\beta lD-\frac{1}{1-d}e^{2k\tau_M}PBQ^{-1}B^{\mathrm{T}}P)x(t)\\ &\quad -e^{2kt}f^{\mathrm{T}}(x(t))(\frac{2\beta D}{l}-\beta(DA+A^{\mathrm{T}}D)-A^{\mathrm{T}}PA-2Q\\ &\quad -\frac{\beta^2}{1-d}e^{2k\tau_M}DBQ^{-1}B^{\mathrm{T}}D)f(x(t))\\ &= -e^{2kt}x^{\mathrm{T}}(t)\Omega_1(\beta,k)x(t)-e^{2kt}f^{\mathrm{T}}(x(t))\Omega_2(\beta,k)f(x(t)). \qquad (9)\end{aligned}$$

Since $\Omega_2(\beta,k)\geq 0$, (9) can be written as

$$V(x(t))\leq V(x(0)).$$

Noting that

$$\begin{aligned}V(x(0)) &= x^{\mathrm{T}}(0)Px(0)+2\beta\sum_{i=1}^{n}d_i\int_0^{x_i(0)}f_i(s)\,\mathrm{d}s\\ &\quad +2\int_{-\tau(0)}^{0}e^{2ks}f^{\mathrm{T}}(x(s))Qf(x(s))\,\mathrm{d}s\\ &\leq \lambda_M(P)\parallel\phi\parallel^2+\beta ld_M\parallel\phi\parallel^2+2\lambda_M(Q)l^2\parallel\phi\parallel^2\int_{-\tau(0)}^{0}e^{2ks}\,\mathrm{d}s\\ &= \{\lambda_M(P)+\beta ld_M+2\lambda_M(Q)l^2\frac{1-e^{-2k\tau(0)}}{2k}\}\parallel\phi\parallel^2,\end{aligned}$$

where $d_M=\max(d_i)$, and $\parallel\phi\parallel=\sup\limits_{-\tau(0)\leq s\leq 0}\parallel x(s)\parallel$, we also have

$$V(x(t))\geq e^{2kt}\lambda_m(P)\parallel x(t)\parallel^2,$$

which implies that

$$e^{2kt}\lambda_m(P)\parallel x(t)\parallel^2\leq\{\lambda_M(P)+\beta ld_M+2\lambda_M(Q)l^2\frac{1-e^{-2k\tau(0)}}{2k}\}\parallel\phi\parallel^2.$$

Therefore, we obtain

$$\parallel x(t)\parallel\leq\sqrt{\frac{\lambda_M(P)+\beta ld_M+2\lambda_M(Q)l^2\frac{1-e^{-2k\tau(0)}}{2k}}{\lambda_m(P)}}\parallel\phi\parallel e^{-kt}.$$

This implies that the origin of (2) is exponentially stable with convergence rate $k>0$, the proof is complete. □

Theorem 2. *The equilibrium u^* is globally exponentially stable, if there exist a positive definite matrix P, positive diagonal matrices D and Q, and positive constants*

β and k, such that the following conditions are satisfied:

$$\Omega_3(\beta,k) = P - 2kP - 2Q - 2k\beta lD - \frac{l^2e^{2k\tau_M}}{1-d}PBQ^{-1}B^{\mathrm{T}}P > 0, \quad (10)$$

$$\Omega_4(\beta,k) = \frac{2\beta D}{l} - \beta(DA + A^{\mathrm{T}}D) - A^{\mathrm{T}}PA - \frac{l^2\beta^2e^{2k\tau_M}}{1-d}DBQ^{-1}B^{\mathrm{T}}D \geq 0. \quad (11)$$

Proof. To obtain the result, it suffices to show that the origin is a equilibrium of system (2) and it is globally exponentially stable.

Consider the following Lyapunov functional defined by

$$V(x(t)) = e^{2kt}x^{\mathrm{T}}(t)Px(t) + 2\beta e^{2kt}\sum_{i=1}^{n} d_i\int_0^{x_i(t)} f_i(s)\,\mathrm{d}s + 2\int_{t-\tau(t)}^{t} e^{2ks}x^{\mathrm{T}}(s)Qx(s)\,\mathrm{d}s,$$

where $P = P^{\mathrm{T}} > 0$, $D = \mathrm{diag}\{d_i\}$, $d_i > 0, i = 1,2,\cdots n$, Q is a positive diagonal matrix, and β and k are positive constants. Calculate the time derivative of the functional along the trajectories of system (2), we obtain

$$\begin{aligned}\dot{V}(x(t)) \;=\; & 2ke^{2kt}x^{\mathrm{T}}(t)Px(t) + 2e^{2kt}x^{\mathrm{T}}(t)P\dot{x}(t) + 4k\beta e^{2kt}\sum_{i=1}^{n} d_i\int_0^{x_i(t)} f_i(s)\,\mathrm{d}s \\ & +2\beta e^{2kt}f^{\mathrm{T}}(x(t))D\dot{x}(t) + 2e^{2kt}x^{\mathrm{T}}(t)Qx(t) \\ & -2e^{2k(t-\tau(t))}(1-\dot{\tau}(t))x^{\mathrm{T}}(t-\tau(t))Qx(t-\tau(t)).\end{aligned}$$

From $0 \leq \frac{f_i(x_i)}{x_i} \leq l$ for $x_i \neq 0$ and $i = 1,2,\cdots n$, it follows that

$$\begin{aligned}\dot{V}(x(t)) \;\leq\; & e^{2kt}\{2kx^{\mathrm{T}}(t)Px(t) - 2x^{\mathrm{T}}(t)Px(t) + 2x^{\mathrm{T}}(t)PAf(x(t)) \\ & +2x^{\mathrm{T}}(t)PBf(x(t-\tau(t))) + 2k\beta lx^{\mathrm{T}}(t)Dx(t) - 2\beta f^{\mathrm{T}}(x(t))Dx(t) \\ & +2\beta f^{\mathrm{T}}(x(t))DAf(x(t)) + 2\beta f^{\mathrm{T}}(x(t))DBf(x(t-\tau(t))) \\ & +2x^{\mathrm{T}}(t)Qx(t) - \frac{1-d}{l^2}e^{-2k\tau_M}f^{\mathrm{T}}(x(t-\tau(t)))Qf(x(t-\tau(t))) \\ & -\frac{1-d}{l^2}e^{-2k\tau_M}f^{\mathrm{T}}(x(t-\tau(t)))Qf(x(t-\tau(t)))\}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned} & - \frac{1-d}{l^2}e^{-2k\tau_M}f^{\mathrm{T}}(x(t-\tau(t)))Qf(x(t-\tau(t))) + 2x^{\mathrm{T}}(t)PBf(x(t-\tau(t))) \\ = \; & -[\frac{\sqrt{1-d}}{l}e^{-k\tau_M}Q^{1/2}f(x(t-\tau(t))) - \frac{l}{\sqrt{1-d}}e^{k\tau_M}Q^{-1/2}B^{\mathrm{T}}Px(t)]^{\mathrm{T}} \\ & [\frac{\sqrt{1-d}}{l}e^{-k\tau_M}Q^{1/2}f(x(t-\tau(t))) - \frac{l}{\sqrt{1-d}}e^{k\tau_M}Q^{-1/2}B^{\mathrm{T}}Px(t)] \\ & +\frac{l^2}{1-d}e^{2k\tau_M}x^{\mathrm{T}}(t)PBQ^{-1}B^{\mathrm{T}}Px(t) \\ \leq \; & \frac{l^2}{1-d}e^{2k\tau_M}x^{\mathrm{T}}(t)PBQ^{-1}B^{\mathrm{T}}Px(t), \end{aligned} \quad (12)$$

$$
\begin{aligned}
&- \frac{1-d}{l^2}e^{-2k\tau_M}f^{\mathrm{T}}(x(t-\tau(t)))Qf(x(t-\tau(t))) + 2\beta f^{\mathrm{T}}(x(t))DBf(x(t-\tau(t)))\\
&= -[\frac{\sqrt{1-d}}{l}e^{-k\tau_M}Q^{1/2}f(x(t-\tau(t))) - \frac{l\beta}{\sqrt{1-d}}e^{k\tau_M}Q^{-1/2}B^{\mathrm{T}}Df(x(t))]^{\mathrm{T}}\\
&\quad [\frac{\sqrt{1-d}}{l}e^{-k\tau_M}Q^{1/2}f(x(t-\tau(t))) - \frac{l\beta}{\sqrt{1-d}}e^{k\tau_M}Q^{-1/2}B^{\mathrm{T}}Df(x(t))]\\
&\quad + \frac{l^2\beta^2}{1-d}e^{2k\tau_M}f^{\mathrm{T}}(x(t))DBQ^{-1}B^{\mathrm{T}}Df(x(t))\\
&\leq \frac{l^2\beta^2}{1-d}e^{2k\tau_M}f^{\mathrm{T}}(x(t))DBQ^{-1}B^{\mathrm{T}}Df(x(t)). \qquad (13)
\end{aligned}
$$

Base on the inequalities (5), (8), (12) and (13), we obtain that

$$
\begin{aligned}
\dot{V}(x(t)) &\leq -e^{2kt}x^{\mathrm{T}}(t)(P - 2kP - 2Q - 2k\beta lD - \frac{l^2}{1-d}e^{2k\tau_M}PBQ^{-1}B^{\mathrm{T}}P)x(t)\\
&\quad -e^{2kt}f^{\mathrm{T}}(x(t))(\frac{2\beta D}{l} - \beta(DA + A^{\mathrm{T}}D)\\
&\quad -A^{\mathrm{T}}PA - \frac{l^2\beta^2}{1-d}e^{2k\tau_M}DBQ^{-1}B^{\mathrm{T}}D)f(x(t))\\
&= -e^{2kt}x^{\mathrm{T}}(t)\Omega_3(\beta,k)x(t) - e^{2kt}f^{\mathrm{T}}(x(t))\Omega_4(\beta,k)f(x(t))\\
&\leq -e^{2kt}x^{\mathrm{T}}(t)\Omega_3(\beta,k)x(t).
\end{aligned}
$$

The remaining part of the proof is similar to that of Theorem 1.

It is easy to show that

$$\| x(t) \| \leq \sqrt{\frac{\lambda_M(P) + \beta l d_M + 2\lambda_M(Q)\frac{1-e^{-2k\tau(0)}}{2k}}{\lambda_m(P)}} \| \phi \| e^{-kt},$$

which means that the origin of system (2) is exponentially stable. □

If we take $\beta = 1$ in Theorem 1, then the conditions (3) and (4) become into the following (14) and (15) respectively:

$$\Omega_1(1,k) = P - 2kP - 2klD - \frac{e^{2k\tau_M}}{1-d}PBQ^{-1}B^{\mathrm{T}}P > 0, \qquad (14)$$

$$\Omega_2(1,k) = \frac{2D}{l} - (DA + A^{\mathrm{T}}D) - A^{\mathrm{T}}PA - 2Q - \frac{e^{2k\tau_M}}{1-d}DBQ^{-1}B^{\mathrm{T}}D \geq 0. \qquad (15)$$

We now rewrite (14) into

$$\Omega_1(1,k) = P - \frac{PBQ^{-1}B^{\mathrm{T}}P}{1-d} - 2kP - 2klD - \frac{e^{2k\tau_M}-1}{1-d}PBQ^{-1}B^{\mathrm{T}}P > 0. \qquad (16)$$

Since $k > 0$, it is obvious that (16) implies

$$P - \frac{PBQ^{-1}B^{\mathrm{T}}P}{1-d} > 0, \qquad (17)$$

which means that (14) implies (17). Similarly, from (15), it can be observed that $\Omega_2(1,k) \geq 0$ implies the condition:

$$\frac{2D}{l} - (DA + A^{\mathrm{T}}D) - A^{\mathrm{T}}PA - 2Q - \frac{1}{1-d}DBQ^{-1}B^{\mathrm{T}}D > 0. \tag{18}$$

On the other hand, note that $k > 0$ and

$$\lim_{k\to 0^+} \Omega_1(1,k) = P - \frac{PBQ^{-1}B^{\mathrm{T}}P}{1-d}$$

and

$$\lim_{k\to 0^+} \Omega_2(1,k) = \frac{2D}{l} - (DA + A^{\mathrm{T}}D) - A^{\mathrm{T}}PA - 2Q - \frac{1}{1-d}DBQ^{-1}B^{\mathrm{T}}D,$$

it is easy to see that when (17) holds, (14) also holds for $0 < k \ll 1$ (i.e. $k > 0$ is sufficiently small), and that when (18) holds, (15) also holds for $0 < k \ll 1$.

Summarizing the above, in view of Theorem 1, we have the following.

Theorem 3. *The equilibrium u^* is globally exponentially stable, if there exist positive definite matrices P and Q, a positive diagonal matrix D, such that the following conditions are satisfied:*

$$\begin{pmatrix} A_1 & 0_{2n} \\ 0_{2n} & A_2 \end{pmatrix} > 0,$$

where

$$A_1 = \begin{pmatrix} P & \frac{-PB}{\sqrt{1-d}} \\ \frac{-B^{\mathrm{T}}P}{\sqrt{1-d}} & Q \end{pmatrix}, \qquad A_2 = \begin{pmatrix} M & \frac{-DB}{\sqrt{1-d}} \\ \frac{-B^{\mathrm{T}}D}{\sqrt{1-d}} & Q \end{pmatrix},$$

and $M = \frac{2D}{l} - DA - A^{\mathrm{T}}D - A^{\mathrm{T}}PA - 2Q$.

When take $\beta = 1$ in Theorem 2, the conditions (10) and (11) become into the following (19) and (20) respectively:

$$\Omega_3(1,k) = P - 2kP - 2Q - 2klD - \frac{l^2 e^{2k\tau_M}}{1-d}PBQ^{-1}B^{\mathrm{T}}P > 0, \tag{19}$$

$$\Omega_4(1,k) = \frac{2D}{l} - (DA + A^{\mathrm{T}}D) - A^{\mathrm{T}}PA - \frac{l^2 e^{2k\tau_M}}{1-d}DBQ^{-1}B^{\mathrm{T}}D \geq 0. \tag{20}$$

By an argument similar to above, in view of Theorem 2, we can obtain the following.

Theorem 4. *The equilibrium u^* is globally exponentially stable, if there exist a positive definite matrix P, positive diagonal matrices D and Q, such that the following conditions are satisfied:*

$$\begin{pmatrix} B_1 & 0_{2n} \\ 0_{2n} & B_2 \end{pmatrix} > 0,$$

where

$$B_1 = \begin{pmatrix} P-2Q & \frac{-lPB}{\sqrt{1-d}} \\ \frac{-lB^{\mathrm{T}}P}{\sqrt{1-d}} & Q \end{pmatrix}, \qquad B_2 = \begin{pmatrix} N & \frac{-lDB}{\sqrt{1-d}} \\ \frac{-lB^{\mathrm{T}}D}{\sqrt{1-d}} & Q \end{pmatrix},$$

and $N = \frac{2D}{l} - DA - A^{\mathrm{T}}D - A^{\mathrm{T}}PA$.

Remark 1. Theorem 5 and Theorem 6 in [12] are the special case where $l = 1$ and $\tau(t) \equiv \tau($ non-negative constant) of our Theorem 3 and Theorem 4 respectively.

5. Example

Consider a two-neuron DCNN:

$$\begin{cases} \dot{x_1}(t) = -x_1(t) - \frac{1}{8}[g_1(x_1(t)) + g_2(x_2(t)) - g_1(x_1(t-\tau(t))) + g_2(x_2(t-\tau(t)))] + I_1, \\ \dot{x_2}(t) = -x_2(t) - \frac{1}{8}[g_1(x_1(t)) - g_2(x_2(t)) - g_1(x_1(t-\tau(t))) - g_2(x_2(t-\tau(t)))] + I_2, \end{cases} \tag{21}$$

where I_1 and I_2 are constants, $\tau(t) = \frac{\sin^2(t)}{2}$, and the activation function is described by a PWL function $g_i(x) = 0.5(\mid x+1 \mid - \mid x-1 \mid)(i = 1, 2)$.

For such a system, we have

$$\tau_M = 0.5, \qquad d = \frac{1}{2}, \qquad A = B = \begin{pmatrix} -\frac{1}{8} & \frac{1}{8} \\ -\frac{1}{8} & -\frac{1}{8} \end{pmatrix}.$$

Clearly, $g_i(x)$ satisfy the condition (H_1) with l=1. By some simple calculations, we obtain

$$K = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \qquad \rho(K) = 0.5 < 1,$$

which means (H_2) holds. In Theorem 1, take $\beta = 1$, $P = D = 3I$ and $Q = I$, where I denotes the 2×2 identity matrix, we have

$$\begin{aligned} \Omega_1(1,k) &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 6k & 0 \\ 0 & 6k \end{pmatrix} - \begin{pmatrix} 6k & 0 \\ 0 & 6k \end{pmatrix} - \begin{pmatrix} \frac{9e^k}{16} & 0 \\ 0 & \frac{9e^k}{16} \end{pmatrix} \\ &= \begin{pmatrix} 3 - 12k - \frac{9e^k}{16} & 0 \\ 0 & 3 - 12k - \frac{9e^k}{16} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} \Omega_2(1,k) &= \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} - \begin{pmatrix} -\frac{3}{4} & 0 \\ 0 & -\frac{3}{4} \end{pmatrix} - \begin{pmatrix} \frac{3}{32} & 0 \\ 0 & \frac{3}{32} \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} \frac{9e^k}{16} & 0 \\ 0 & \frac{9e^k}{16} \end{pmatrix} \\ &= \begin{pmatrix} \frac{149}{32} - \frac{9e^k}{16} & 0 \\ 0 & \frac{149}{32} - \frac{9e^k}{16} \end{pmatrix}. \end{aligned}$$

It is clear that sufficiently small k can ensure that $\Omega_1(1,k) > 0$ and $\Omega_2(1,k) > 0$. Therefore, it follows from Theorem 1 that system (21) possesses exactly one equilibrium, and it is globally exponentially stable. □

References

[1] Chua, L. O., Yang L., Cellular neural networks: theory, IEEE Trans. Circ. Syst., 35 (1988) 1257-1272.

[2] Chua, L. O., Roska T., Cellular neural networks and visual computing, Cambridge, UK, Cambridge University Press, 2002.

[3] Roska, T., Chua, L. O., Cellular neural networks with nonlinear and delay-type template, Int. J. Circuit Theory Appl., 20 (1992) 469-481.

[4] Zhang, Q., Ma, R., Wang, C. and Xu, J., On the global stability of delayed neural networks, IEEE Trans. Autom. Contr., 48 (2003) 794-797.

[5] Arik, S., An improved global stability result for delayed cellular neural networks, IEEE Trans. Circuits Syst. I, 49 (2002) 1211-1214.

[6] Cao, J., A set of stability criteria for delayed cellular neural networks, IEEE Trans. Circuits Syst. I, 48 (2001) 494-8.

[7] van den Driessche, P., Zou, X., Global attractivity in delayed Hopfield neural network models, SIMA J. Appl. Math., 58 (1998) 1878-1890.

[8] Zhang, J., Globally exponential stability of neural networks with variable delays, IEEE Trans. Circuits Syst. I, 50 (2003) 288-290.

[9] Liao, X., Chen, G., Sanchez, E. N., Delay-dependent exponential stability analysis of delayed neural networks: an LMI approach, Neural Networks, 15 (2002) 855-566.

[10] Lu, H., On stability of nonlinear continue-time neural networks with delays, Neural Networks, 13 (2000) 1135-1143.

[11] Cao, J., Global asymptotic stability of neural networks with transmission delays, Int. J. Syst. Sci., 31 (2000) 1313-1316.

[12] Singh, V., On global exponential stability of delayed cellular neural networks, Chaos, Solitons and Fractals, 33 (2007) 188-193.

[13] Huang, L. H., Huang, C. H. and Liu, B. W., Dynamics of a class of cellular neural networks with time-varying delays, Physics Letters A, 345 (2005) 330-344.