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Oscillation for an Odd-Order Delay Difference Equation with Several Delays*

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Abstract. In this paper, we obtain some oscillation criteria for the odd-order difference equation with several delays

$$\Delta^m x_n + \sum_{i=1}^{N} p_i(n) x_{n-k_i} = 0, \quad n = 0, 1, 2, \cdots,$$

which include some existing criteria for $m = 1$ as special cases.

AMS Subject Classifications: 39A10

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1. Introduction

In recent years, oscillation of solutions of difference equations has attracted many researchers, see, for example, [1-14] and the references cited therein. Now, numerous results exist for first order delay difference equations. However, results dealing with oscillation of the higher odd-order delay difference equations such as the form

$$\Delta^m x_n + \sum_{i=1}^{N} p_i(n) x_{n-k_i} = 0, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

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are relatively scarce [2-4, 6, 12, 14], where $p_i(n) \geq 0$, $m \geq 1$ is an odd integer, k_i are positive integers, $i = 1, 2, \ldots, N$, Δ denotes the forward difference operator, i.e. $\Delta x_n = x_{n+1} - x_n$, $\Delta^i x_n = \Delta(\Delta^{i-1} x_n), i = 1, 2, \ldots, m$ and $\Delta^0 x_n = x_n$. Furthermore, even in the mentioned literature, only the oscillation of unbounded solutions was involved. Few oscillation criteria for the higher odd-order delay difference equations are founded in the literature. When $m = 1$, Eq.(1.1) reduces to the first-order delay difference equation

$$\Delta x_n + \sum_{i=1}^{N} p_i(n) x_{n-k_i} = 0, \qquad n = 0, 1, 2, \ldots. \tag{1.2}$$

To the best of our knowledge, the best two oscillation criteria are

$$\liminf_{n\to\infty} \sum_{i=1}^{N} \left(\frac{k_i+1}{k_i}\right)^{k_i+1} \sum_{s=n+1}^{n+k_i} p_i(s) > 1 \tag{1.3}$$

and

$$\limsup_{n\to\infty} \sum_{i=1}^{N} \sum_{s=n}^{n+k_i} p_i(s) > 1 \tag{1.4}$$

obtained by Tang and Yu [7] and Tang and Zhang [11], respectively. In this paper, we are concerned with oscillation of Eq.(1.1) and aim to establish some oscillation criteria. In particular, as corollaries of our results in present paper, two explicit oscillation criteria for Eq.(1.1)

$$\liminf_{n\to\infty} \sum_{i=1}^{N} \left(\frac{k_i+1}{k_i}\right)^{k_i+1} \sum_{s=n+1}^{n+k_i} (s-n)^{(m-1)} p_i(s) > (m-1)! \tag{1.5}$$

and

$$\limsup_{n\to\infty} \sum_{i=1}^{N} \sum_{s=n}^{n+k_i} (s-n+1)^{(m-1)} p_i(s) > (m-1)! \tag{1.6}$$

are obtained. Obviously, wehn $m = 1$ conditions (1.5) and (1.6) reduce to (1.3) and (1.4), respectively.

For $t \in (-\infty, \infty)$ and positive integer n, we define $t^{(n)} = \prod_{i=0}^{n-1}(t+i)$ with $t^{(0)} = 1$; As usual, when $n_1 > n_2$, we define $\sum_{i=n_1}^{n_2} x_i = 0$ and $\prod_{i=n_1}^{n_2} x_i = 1$.

2. Main Results

In this section, we first state a lemma taken from [7] and [11], which is useful in the proofs of our main results.

Lemma 2.1.[7,11] *Assume that (1.3) or (1.4) holds. Then the following delay difference inequality corresponding to Eq.(1.2)*

$$\Delta x_n + \sum_{i=1}^{N} p_i(n) x_{n-k_i} \leq 0, \qquad n = 0, 1, 2, \ldots \tag{2.1}$$

has no eventually positive solutions.

Remark 2.1. *In the above condition (1.4), even if $k_i = 0$ for some $i \in \{1, 2, \ldots N\}$, the conclusion of Lemma 2.1 still holds.*

Theorem 2.1. *Every solution of Eq.(1.1) oscillates if the inequality*

$$\Delta y_n + \sum_{i=1}^{N} \sum_{s=n}^{\infty} \frac{(s-n+1)^{(m-2)}}{(m-2)!} p_i(s) y_{s-k_i} \leq 0, \qquad n = 0, 1, 2, \ldots \tag{2.2}$$

has no eventually positive solutions.

Proof. For the sake of contradiction, assume that Eq.(1.1) has an eventually positive solution $\{x_n\}$. Then $\Delta^m x_n \leq 0$ eventually. By Discrete Kneser's Theorem [1], there exist an even integer $l \in \{0, 2, \ldots, m-1\}$ and integer $n_0 > 0$ such that

$$\Delta^i x_n > 0, \qquad i = 0, \ldots, l, \qquad n \geq n_0, \tag{2.3}$$

and

$$(-1)^{l+i} \Delta^i x_n > 0, \qquad i = l+1, \ldots, m-1, \qquad n \geq n_0. \tag{2.4}$$

Summing (1.1) $m-l-1$ times from n to ∞ and using (2.4), we have

$$-\Delta^{l+1} x_n \geq \sum_{i=1}^{N} \sum_{s=n}^{\infty} \frac{(s-n+1)^{(m-l-2)}}{(m-l-2)!} p_i(s) x_{s-k_i}, \qquad n \geq n_0. \tag{2.5}$$

If $l = 0$, then it follows from (2.5) that

$$-\Delta^{l+1} x_n \geq \sum_{i=1}^{N} \sum_{s=n}^{\infty} \frac{(s-n+1)^{(m-2)}}{(m-2)!} p_i(s) \Delta^l x_{s-k_i}, \qquad n \geq n_0. \tag{2.6}$$

If $2 \leq l \leq m-1$, then, in view of the discrete Taylor's Formula [1], we can obtain

$$x_n = \sum_{j=0}^{l-1} \frac{(n+1-n_1-j)^{(j)}}{j!} \Delta^j x_{n_1} + \frac{1}{(l-1)!} \sum_{r=n_1}^{n-l} (n+1-r-l)^{(l-1)} \Delta^l x_r, \qquad n \geq n_1.$$

Again by using (2.3), we have

$$\begin{aligned}
x_{s-k_i} &= \sum_{j=0}^{l-1} \frac{(s+m-n-j-1)^{(j)}}{j!} \Delta^j x_{n+2-m-k_i} \\
&\quad + \frac{1}{(l-1)!} \sum_{r=n+2-m-k_i}^{s-k_i-l} (s+1-k_i-r-l)^{(l-1)} \Delta^l x_r \\
&\geq \frac{\Delta^l x_{s-k_i}}{(l-1)!} \sum_{r=n+2-m-k_i}^{s-k_i-l} (s+1-k_i-r-l)^{(l-1)} \\
&= \frac{1}{l!} (s+m-n-l-1)^{(l)} \Delta^l x_{s-k_i}, \qquad s \geq n \geq n_0 + m + \sum_{i=1}^{N} k_i.
\end{aligned}$$

Substituting this into (2.5) and using the fact that $l!(m-l-2)! \le (m-2)!$, we can also conclude (2.6). Set $y_n = \Delta^l x_n$. Then $y_n > 0$ for $n \ge n_0$ and

$$\Delta y_n + \sum_{i=1}^{N} \sum_{s=n}^{\infty} \frac{(s-n+1)^{(m-2)}}{(m-2)!} p_i(s) y_{s-k_i} \le 0, \qquad n \ge n_0,$$

which shows that inequality (2.2) has an eventually positive solution $\{y_n\}$. This contradiction completes the proof. □

From Theorem 2.1, it is easy to conclude the following corollary.

Corollary 2.1. *Every solution of Eq.(1.1) oscillates if the inequality*

$$\Delta y_n + \sum_{i=1}^{N} \sum_{j=0}^{k_i} \frac{(k_i+1-j)^{(m-2)}}{(m-2)!} p_i(n+k_i-j) y_{n-j} \le 0, \qquad n = 0, 1, 2, \dots \tag{2.7}$$

has no eventually positive solutions.

By employing Lemma 2.1 to inequality (2.7) directly, we have

Theorem 2.2. *Assume that*

$$\liminf_{n\to\infty} \sum_{i=1}^{N} \sum_{j=1}^{k_i} (k_i+1-j)^{(m-2)} \left(\frac{j+1}{j}\right)^{j+1} \sum_{s=n+1}^{n+j} p_i(s+k_i-j) > (m-2)!, \tag{2.8}$$

or

$$\limsup_{n\to\infty} \sum_{i=1}^{N} \sum_{j=0}^{k_i} (k_i+1-j)^{(m-2)} \sum_{s=n}^{n+j} p_i(s+k_i-j) > (m-2)!. \tag{2.9}$$

Then every solution of Eq.(1.1) oscillates.

Note that

$$\begin{aligned}
&\sum_{i=1}^{N} \sum_{j=1}^{k_i} \frac{(k_i+1-j)^{(m-2)}}{(m-2)!} \left(\frac{j+1}{j}\right)^{j+1} \sum_{s=n+1}^{n+j} p_i(s+k_i-j) \\
\ge\ & \sum_{i=1}^{N} \left(\frac{k_i+1}{k_i}\right)^{k_i+1} \sum_{j=1}^{k_i} \frac{(k_i+1-j)^{(m-2)}}{(m-2)!} \sum_{s=n+1}^{n+j} p_i(s+k_i-j) \\
=\ & \sum_{i=1}^{N} \left(\frac{k_i+1}{k_i}\right)^{k_i+1} \sum_{s=n+1}^{n+k_i} p_i(s) \sum_{j=1}^{s-n} \frac{j^{(m-2)}}{(m-2)!} \\
=\ & \sum_{i=1}^{N} \left(\frac{k_i+1}{k_i}\right)^{k_i+1} \sum_{s=n+1}^{n+k_i} \frac{(s-n)^{(m-1)}}{(m-1)!} p_i(s),
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=1}^{N}\sum_{j=0}^{k_i}\frac{(k_i+1-j)^{(m-2)}}{(m-2)!}\sum_{s=n}^{n+j}p_i(s+k_i-j)\\
=&\sum_{s=n}^{n+k_i}p_i(s)\sum_{j=1}^{s-n+1}\frac{j^{(m-2)}}{(m-2)!}\\
=&\sum_{i=1}^{N}\sum_{s=n}^{n+k_i}\frac{(s-n+1)^{(m-1)}}{(m-1)!}p_i(s).
\end{aligned}$$

Hence, from Theorem 2.2, we have

Corollary 2.2. *Assume that (1.5) or (1.6) holds. Then every solution of Eq.(1.1) oscillates.*

To further improve condition (1.6), we need the following lemma.

Lemma 2.2. *Assume that*

$$\liminf_{n\to\infty} p_i(n)=p_i, \qquad i=1,2,\ldots,N, \tag{2.10}$$

and

$$\liminf_{n\to\infty}\sum_{i=1}^{N}\sum_{s=n+1}^{n+k_i}\frac{(s-n)^{(m-1)}}{(m-1)!}p_i(s)=d. \tag{2.11}$$

Let $\{y_n\}$ be an eventually positive solution of (2.7). Then

$$\liminf_{n\to\infty}\frac{y_{n+1}}{y_n}\geq\frac{1}{2}\left[1+d-\sum_{i=1}^{N}\frac{p_ik_i^{(m-1)}}{(m-1)!}-\sqrt{\left(1+d-\sum_{i=1}^{N}\frac{p_ik_i^{(m-1)}}{(m-1)!}\right)^2-4d}\,\right]. \tag{2.12}$$

Proof. Let $\theta\in(0,1)$. Choose a positive integer n_0 such that

$$y_{n-k}>0, \qquad \sum_{i=1}^{N}\sum_{s=n+1}^{n+k_i}\frac{(s-n)^{(m-1)}}{(m-1)!}p_i(s)\geq\theta d, \qquad n\geq n_0$$

and

$$p_i(n)\geq\theta p_i, \qquad i=1,2,\ldots,N, \qquad n\geq n_0,$$

here and in the sequel, $k=\max\{k_1,k_2,\ldots,k_N\}$. Summing (2.7) from $n+1$ to ∞, we have

$$\begin{aligned} y_{n+1} &\geq \sum_{i=1}^{N}\sum_{j=1}^{k_i}\frac{(k_i+1-j)^{(m-2)}}{(m-2)!}\sum_{s=n+1}^{\infty}p_i(s+k_i-j)y_{s-j} \\ &\geq y_n\sum_{i=1}^{N}\sum_{j=1}^{k_i}\frac{(k_i+1-j)^{(m-2)}}{(m-2)!}\sum_{s=n+1}^{n+j}p_i(s+k_i-j) \\ &= y_n\sum_{i=1}^{N}\sum_{s=n+1}^{n+k_i}\frac{(s-n)^{(m-1)}}{(m-1)!}p_i(s) \\ &\geq \theta d y_n, \qquad n\geq n_0+k. \end{aligned}$$

It follows that

$$\frac{y_{n+1}}{y_n}\geq \theta d := d_1, \qquad n\geq n_0+k.$$

Again from (2.7) and using the above, we have

$$\begin{aligned} y_{n+1} &\geq \sum_{i=1}^{N}\sum_{j=1}^{k_i}\frac{(k_i+1-j)^{(m-2)}}{(m-2)!}\sum_{s=n+1}^{\infty}p_i(s+k_i-j)y_{s-j} \\ &= \sum_{i=1}^{N}\sum_{j=1}^{k_i}\frac{(k_i+1-j)^{(m-2)}}{(m-2)!}\left[\sum_{s=n+1}^{n+j}p_i(s+k_i-j)y_{s-j}\right. \\ &\qquad \left. +\sum_{s=n+j+1}^{\infty}p_i(s+k_i-j)y_{s-j}\right] \\ &\geq y_n\sum_{i=1}^{N}\sum_{j=1}^{k_i}\frac{(k_i+1-j)^{(m-2)}}{(m-2)!}\sum_{s=n+1}^{n+j}p_i(s+k_i-j) \\ &\quad +y_{n+1}\sum_{i=1}^{N}\sum_{j=1}^{k_i}\frac{(k_i+1-j)^{(m-2)}}{(m-2)!}\sum_{s=n+j+1}^{\infty}p_i(s+k_i-j)d_1^{s-n-j-1} \\ &= y_n\sum_{i=1}^{N}\sum_{s=n+1}^{n+k_i}\frac{(s-n)^{(m-1)}}{(m-1)!}p_i(s) \\ &\quad +y_{n+1}\sum_{i=1}^{N}\sum_{j=1}^{k_i}\frac{(k_i+1-j)^{(m-2)}}{(m-2)!}\sum_{s=0}^{\infty}p_i(n+s+k_i+1)d_1^{s} \\ &\geq d_1y_n+y_{n+1}\sum_{i=1}^{N}\frac{\theta p_i}{1-d_1}\sum_{j=1}^{k_i}\frac{(k_i+1-j)^{(m-2)}}{(m-2)!} \\ &= d_1y_n+\frac{y_{n+1}}{1-d_1}\sum_{i=1}^{N}\frac{\theta p_i k_i^{(m-1)}}{(m-1)!}, \qquad n\geq n_0+2k, \end{aligned}$$

which implies

$$\frac{y_{n+1}}{y_n} \geq d_1 \left[1 - \frac{1}{1-d_1} \sum_{i=1}^{N} \frac{\theta p_i k_i^{(m-1)}}{(m-1)!}\right]^{-1} := d_2, \qquad n \geq n_0 + 2k.$$

Following this iterative procedure, we have

$$\frac{y_{n+1}}{y_n} \geq d_1 \left[1 - \frac{1}{1-d_j} \sum_{i=1}^{N} \frac{\theta p_i k_i^{(m-1)}}{(m-1)!}\right]^{-1} := d_{j+1}, \qquad n \geq n_0 + (j+1)k.$$

It is easy to see that $0 \leq d_1 \leq d_2 \leq \cdots \leq 1$. Therefore, the limit $\lim_{j\to\infty} d_j = d_* = d_*(\theta)$ exists and

$$d_* \left(1 - \frac{1}{1-d_*} \sum_{i=1}^{N} \frac{\theta p_i k_i^{(m-1)}}{(m-1)!}\right) = \theta d. \tag{2.13}$$

Furthermore,

$$\liminf_{n\to\infty} \frac{y_{n+1}}{y_n} \geq d_*(\theta).$$

It is easy to see that

$$\mu_* = \frac{1}{2}\left[1 + d - \sum_{i=1}^{N} \frac{p_i k_i^{(m-1)}}{(m-1)!} - \sqrt{\left(1 + d - \sum_{i=1}^{N} \frac{p_i k_i^{(m-1)}}{(m-1)!}\right)^2 - 4d}\,\right]$$

is the smaller of two roots of the equation

$$\mu \left(1 - \frac{1}{1-\mu} \sum_{i=1}^{N} \frac{p_i k_i^{(m-1)}}{(m-1)!}\right) = d. \tag{2.14}$$

Then from (2.13) and (2.14), it is easy to show that $\liminf_{\theta\to 1^-} d_*(\theta) \geq \mu_*$. Hence, $\liminf n \to \infty (y_{n+1}/y_n) \geq \mu_*$ and so (2.12) holds and the proof is complete. □

Theorem 2.3. *Assume that (2.10) and (2.11) hold, and that*

$$\limsup_{n\to\infty} \sum_{i=1}^{N} \sum_{s=n}^{n+k_i} (s-n+1)^{(m-1)} p_i(s) > (m-1)! - \sum_{i=1}^{N} p_i (k_i+1)^{(m-1)}$$
$$\times \frac{1 + d - \sum_{i=1}^{N} \frac{p_i k_i^{(m-1)}}{(m-1)!} - \sqrt{\left(1 + d - \sum_{i=1}^{N} \frac{p_i k_i^{(m-1)}}{(m-1)!}\right)^2 - 4d}}{1 + \sum_{i=1}^{N} \frac{p_i k_i^{(m-1)}}{(m-1)!} - d - \sqrt{\left(1 + d - \sum_{i=1}^{N} \frac{p_i k_i^{(m-1)}}{(m-1)!}\right)^2 - 4d}}. \tag{2.15}$$

Then every solution of Eq.(1.1) oscillates.

Proof. By Corollary 2.1, we only need to prove that inequality (2.7) has no eventually positive solutions. For the sake of contradiction, assume that (2.7) has an eventually positive solution $\{y_n\}$. Then there exists a positive integer n_0 such that

$$y_{n-k} > 0 \quad \text{and} \quad y_{n+1} - y_n \leq 0, \qquad n \geq n_0.$$

Summing (2.7) from n to ∞, we have

$$y_n \geq \sum_{i=1}^{N}\sum_{j=0}^{k_i} \frac{(k_i+1-j)^{(m-2)}}{(m-2)!}\left[\sum_{s=n}^{n+j} p_i(s+k_i-j)y_{s-j} + \sum_{s=1}^{\infty} p_i(n+s+k_i)y_{n+s}\right], \qquad n \geq n_0. \tag{2.16}$$

Let

$$\mu_* = \frac{1}{2}\left[1 + d - \sum_{i=1}^{N}\frac{p_i k_i^{(m-1)}}{(m-1)!} - \sqrt{\left(1 + d - \sum_{i=1}^{N}\frac{p_i k_i^{(m-1)}}{(m-1)!}\right)^2 - 4d}\,\right].$$

Then from (2.16) and using Lemma 2.2, we have

$$\begin{aligned}
1 &\geq \limsup_{n\to\infty} \sum_{i=1}^{N}\sum_{j=0}^{k_i} \frac{(k_i+1-j)^{(m-2)}}{(m-2)!}\left[\sum_{s=n}^{n+j} p_i(s+k_i-j) + \sum_{s=1}^{\infty} p_i(n+s+k_i)\frac{y_{n+s}}{y_n}\right] \\
&\geq \limsup_{n\to\infty} \sum_{i=1}^{N}\sum_{j=0}^{k_i} \frac{(k_i+1-j)^{(m-2)}}{(m-2)!}\sum_{s=n}^{n+j} p_i(s+k_i-j) \\
&\quad + \sum_{i=1}^{N}\sum_{j=0}^{k_i} \frac{(k_i+1-j)^{(m-2)}}{(m-2)!}\sum_{s=1}^{\infty} \liminf_{n\to\infty} p_i(n+s+k_i) \liminf_{n\to\infty}\frac{y_{n+s}}{y_n} \\
&\geq \limsup_{n\to\infty} \sum_{i=1}^{N}\sum_{j=0}^{k_i} \frac{(k_i+1-j)^{(m-2)}}{(m-2)!}\sum_{s=n}^{n+j} p_i(s+k_i-j) \\
&\quad + \sum_{i=1}^{N}\sum_{j=0}^{k_i} \frac{(k_i+1-j)^{(m-2)}}{(m-2)!} p_i \sum_{s=1}^{\infty} \mu_*^s \\
&= \frac{1}{(m-1)!}\left[\limsup_{n\to\infty}\sum_{i=1}^{N}\sum_{s=n}^{n+k_i}(s-n+1)^{(m-1)}p_i(s) + \frac{\mu_*}{1-\mu_*}\sum_{i=1}^{N} p_i(k_i+1)^{(m-1)}\right].
\end{aligned}$$

That is

$$\limsup_{n\to\infty}\sum_{i=1}^{N}\sum_{s=n}^{n+k_i}(s-n+1)^{(m-1)}p_i(s) \leq (m-1)! - \frac{\mu_*}{1-\mu_*}\sum_{i=1}^{N} p_i(k_i+1)^{(m-1)},$$

which contradicts (2.15), and so the proof is complete. □

References

[1] R. P. Agarwal, Difference Equations and Inequalities, New York, Marcel Dekker, 1999.

[2] R. P. Agarwal, Said R. Grace, Donal O'reqan, Oscillatory Theory for Difference Equations and Functional Differential Equations, Kluwer Academic Publishers, Dordrecht, 2000.

[3] R. P. Agarwal, E. Thandapani and P. J. Y. Wong, Oscillation for higher-order neutral difference equations, Appl. Math. Lett., 10:1 (1997) 71-78.

[4] G. Grzecorczyk and J. Werbowski, Oscillation of solutions of higher-order linear difference equations, Computers Math. Appl., 42 (2001) 711-717.

[5] B. Li, Discrete oscillations, J. Difference Equations Applic., 2 (1996) 389-399.

[6] B. Szmanda, Oscillation of solutions of higher order nonlinear difference equations, Bull. Inst. Math. Acad. Sinica, 25 (1997) 71-78.

[7] X. H. Tang and J. S. Yu, Oscillation of delay difference equations, Computers Math. Applic., 37:7 (1999) 11-20.

[8] X. H. Tang and J. S. Yu, Oscillations of delay difference equations in a critical state, Applied Math. Letters, 13 (2000) 9-15.

[9] X. H. Tang and J. S. Yu, A Further result on the oscillation of delay difference equations, Computers Math. Applic., 38:11-12 (1999) 229-237.

[10] X. H. Tang and J. S. Yu, Oscillation of delay difference equations, Hokkaido Mathematical J., 29 (2000) 213-228.

[11] X. H. Tang and R. Y. Zhang, New oscillation criteria for delay difference equations, Computers Math. Applic., 42 (2001) 1319-1330.

[12] A. Wyrwinska, Oscillation criteria of higher-order linear difference equations, Bull. Inst. Math. Acad. Sinica, 22 (1994) 259-266.

[13] J. S. Yu, B. G. Zhang and X. Z. Qian, Oscillation of delay difference equations with oscillating coefficients, J. Math. Anal. Appl., 177 (1993) 432-444.

[14] G. Zhang and Y. Gao, Oscillatory Theory for Difference Equations (in Chinese), Higher Education Press, Beijing, 2001.