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Linearized Oscillation of Odd Order Nonlinear Neutral Delay Differential Equations (II)

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Abstract. In this paper, we proved the odd order nonlinear neutral delay differential equation

$$[x(t) - p(t)g(x(t-\tau))]^{(n)} + q(t)h(x(t-\sigma)) = 0$$

has the same oscillatory character as its linearized equation

$$[x(t) - p_0x(t-\tau)]^{(n)} + q_0x(t-\sigma) = 0$$

under some rather relaxed conditions on $g(u)$ and $h(u)$, where $1 < p_0 = \lim_{t\to\infty} p(t)$, $q_0 = \lim_{t\to\infty} q(t)$.

AMS Subject Classifications: 34K15

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1. Introduction

In the past 15 years, the linearized oscillation theory for nonlinear neutral delay differential equations has been extensively developed, for example see [1-14]. Linearization is an important method dealt with nonlinear mathematical problems. While the linearized oscillation, roughly speaking, it is to find some appropriate hypotheses under which certain nonlinear equations have the same oscillatory character as their associated linearized equations.

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Following paper [13], we also consider the nonlinear neutral delay differential equation

$$[x(t)-p(t)g(x(t-\tau))]^{(n)}+q(t)h(x(t-\sigma))=0, \quad t\geq t_0, \tag{1.1}$$

where $n\geq 1$ is an odd integer,

$$p,q\in C([t_0,\infty),\mathbb{R}), \quad g,h\in C(\mathbb{R},\mathbb{R}), \quad \tau>0, \ \ \sigma\geq 0. \tag{1.2}$$

For Eq.(1.1), the following two linearized oscillation results were obtained by Ladas and Qian [7] and Shen, Yu and Qian [9], respectively, (see also [1, 3, 5]):

Theorem A[7]. *Assume that*

$$\liminf_{t\to\infty} p(t)=p_0\in(0,1), \quad p(t)\leq P_0\leq 1, \ \ \textit{for large } t, \tag{1.3}$$

$$\lim_{t\to\infty} q(t)=q_0\in(0,\infty), \tag{1.4}$$

$$0\leq \frac{g(u)}{u}\leq \frac{1}{P_0}, \ \ \textit{for} \ \ u\neq 0, \quad \textit{and} \quad \lim_{u\to 0}\frac{g(u)}{u}=1, \tag{1.5}$$

$$uh(u)>0, \ \ \textit{for} \ \ u\neq 0, \ \ |h(u)|\geq h_0>0, \ \ \textit{for large} \ \ |u|, \tag{1.6}$$

and

$$\lim_{u\to 0}\frac{h(u)}{u}=1. \tag{1.7}$$

Suppose that every solution of the linearized equation

$$[x(t)-p_0x(t-\tau)]^{(n)}+q_0x(t-\sigma)=0, \quad t\geq t_0 \tag{1.8}$$

oscillates. Then every solution of Eq.(1.1) also oscillates.

Theorem B[9]. *Assume that*

$$\limsup_{t\to\infty} p(t)=p_0\in(1,\infty), \quad p(t)\geq P_0\geq 1 \ \ \textit{for large } t, \tag{1.9}$$

$$\lim_{t\to\infty} q(t)=q_0\in(0,\infty), \tag{1.10}$$

$$\frac{g(u)}{u}\geq \frac{1}{P_0}, \ \ \textit{for} \ \ u\neq 0, \quad \textit{and} \quad \lim_{|u|\to\infty}\frac{g(u)}{u}=1, \tag{1.11}$$

$$uh(u)>0, \ \ \textit{for} \ \ u\neq 0, \quad \textit{and} \quad \lim_{|u|\to\infty}\frac{h(u)}{u}=1. \tag{1.12}$$

Suppose that every solution of the linearized equation (1.8) oscillates. Then every solution of Eq.(1.1) also oscillates.

In recent paper [13], Tang established the next theorem which is a converse of Theorem A,

Theorem C[13]. *Assume that there exist $p_0,\ q_0 \in [0,+\infty)$ such that*

$$0 \le p(t) \le p_0 < 1, \quad 0 \le q(t) \le q_0, \quad \text{for large } t, \tag{1.13}$$

and that the following conditions (H_1) and (H_2) hold:

(H_1) *either*

$$p(t) > 0, \quad \text{for large } \ t, \tag{1.14}$$

or

$$\sigma > 0 \quad \text{and} \quad q(s) \not\equiv 0, \quad s \in [t, t+\sigma], \quad \text{for large } t; \tag{1.15}$$

(H_2) *there exist $r > 0$, $\delta > 0$ and $K > 0$ such that either $g(u)$ and $h(u)$ are nondecreasing in $[0,\delta)$ and*

$$0 \le \min\{g(u), h(u)\} \le \max\{g(u), h(u)\} \le u + K|u|^{1+r}, \quad u \in [0,\delta), \tag{1.16}$$

or $g(u)$ and $h(u)$ are nondecreasing in $(-\delta, 0]$ and

$$0 \ge \max\{g(u), h(u)\} \ge \min\{g(u), h(u)\} \ge u - K|u|^{1+r}, \quad u \in (-\delta, 0]. \tag{1.17}$$

Suppose also that Eq.(1.8) has an eventually positive solution. Then Eq.(1.1) has a nonoscillatory solution.

By combining Theorems A and C, we have the following theorems:

Theorem D[13]. *Assume that (1.6), (1.7) and (H_2) hold, and that*

$$0 \le p(t) \le p_0 = \lim_{t\to\infty} p(t) < 1, \quad 0 \le q(t) \le q_0 = \lim_{t\to\infty} q(t), \quad \text{for large } t, \tag{1.18}$$

and

$$0 \le \frac{g(u)}{u} \le \frac{1}{p_0}, \quad \text{for } \ u \ne 0, \quad \text{and} \quad \lim_{u\to 0} \frac{g(u)}{u} = 1. \tag{1.19}$$

Then every solution of Eq.(1.1) oscillates if and only if every solution of its linearized equation (1.8) oscillates.

Theorem E[13]. *Assume that (1.6), (1.7), (1.18) and the following (H_3) hold:*

(H_3) *there exist $r > 0$, $\delta > 0$ and $K > 0$ such that either $h(u)$ is nondecreasing in $[0,\delta)$ and*

$$0 \le h(u) \le u + K|u|^{1+r}, \quad u \in [0,\delta),$$

or $h(u)$ is nondecreasing in $(-\delta, 0]$ and

$$0 \ge h(u) \ge u - K|u|^{1+r}, \quad u \in (-\delta, 0].$$

Then every solution of the following equation

$$[x(t) - p(t)x(t-\tau)]^{(n)} + q(t)h(x(t-\sigma)) = 0, \quad t \ge t_0 \tag{1.20}$$

oscillates if and only if every solution of its linearized equation (1.8) oscillates.

The above Theorems A-E are the slightly modified versions of the corresponding results in [7, 9, 13]. In this paper, our main purpose is to prove the following theorem which is converse of Theorem B.

Theorem 1.1. *Assume that there exist $p_0, q_0 \in [0, \infty)$ such that*

$$p(t) \geq p_0 > 1, \quad 0 \leq q(t) \leq q_0, \quad \textit{for large } t, \tag{1.21}$$

and that the following conditions (H_4) holds:

(H_4) *there exist $r \in (0,1)$, $M > 0$ and $K > 0$ such that either $h(0) \geq g(0) = 0$, $g(u)$ is increasing and $h(u)$ is nondecreasing in $[0, \infty)$ and*

$$u - K|u|^{1-r} \leq g(u), \quad h(u) \leq u + K|u|^{1-r}, \quad u \in [M, \infty), \tag{1.22}$$

or $h(0) \leq g(0) = 0$, $g(u)$ is increasing and $h(u)$ is nondecreasing in $(-\infty, 0]$ and

$$g(u) \leq u + K|u|^{1-r}, \quad h(u) \geq u - K|u|^{1-r}, \quad u \in (-\infty, -M]. \tag{1.23}$$

Suppose also that Eq.(1.8) has an eventually positive solution. Then Eq.(1.1) has a nonoscillatory solution.

By combining Theorems B and 1.1, we have the following theorems:

Theorem 1.2. *Assume that (1.12) and (H_4) hold, and that*

$$p(t) \geq p_0 = \lim_{t\to\infty} p(t) > 1, \quad 0 \leq q(t) \leq q_0 = \lim_{t\to\infty} q(t), \quad \textit{for large } t, \tag{1.24}$$

and

$$\frac{g(u)}{u} \geq \frac{1}{p_0}, \quad \textit{for } u \neq 0, \quad \textit{and} \quad \lim_{u\to 0} \frac{g(u)}{u} = 1. \tag{1.25}$$

Then every solution of Eq.(1.1) oscillates if and only if every solution of its linearized equation (1.8) oscillates.

Theorem 1.3. *Assume that (1.12), (1.24) and the following condition (H_5) holds:*

(H_5) *there exist $r \in (0,1)$, $M > 0$ and $K > 0$ such that either $h(u)$ is nondecreasing in $[0, \infty)$ and*

$$h(u) \leq u + K|u|^{1-r}, \quad u \in [M, \infty),$$

or $h(u)$ is nondecreasing in $(-\infty, 0]$ and

$$h(u) \geq u - K|u|^{1-r}, \quad u \in (-\infty, -M].$$

Then every solution of Eq.(1.20) oscillates if and only if every solution of its linearized equation (1.8) oscillates.

It is a known fact, see [1, 5, 8], that every solution of Eq.(1.8) oscillates if and only if its characteristic equation

$$F(\lambda) \equiv \lambda^n(1 - p_0 e^{-\lambda\tau}) + q_0 e^{-\lambda\sigma} = 0 \tag{1.26}$$

has no real roots. Note that

$$F(\lambda) > 0 \quad \text{for} \quad \lambda \leq 0, \qquad \text{and} \qquad F(\infty) = \lim_{\lambda\to\infty} F(\lambda) = \infty.$$

So, Eq.(1.8) has an eventually positive solution implies two possible cases:

Case (i). there exists a $\lambda^* \in (0,\infty)$ such that

$$F(\lambda^*) < 0; \tag{1.27}$$

Case (ii). there exists a $\lambda_0 \in (0,\infty)$ such that

$$F(\lambda_0) = 0 \;\text{ and }\; F(\lambda) \geq 0 \;\text{ for }\; \lambda \in [0,\lambda_0) \cup (\lambda_0,\infty). \tag{1.28}$$

As is customary, if Case (ii) holds, we say Eq.(1.8) is in a critical state. if Case (i) holds, Eq.(1.8) is called in a non-critical state. A solution is called oscillatory if it has arbitrary large zeros. Otherwise it is called nonoscillatory.

2. Non-Critical Case

Lemma 2.1[5]. *Every solution of Eq.(1.8) oscillates if and only if the characteristic equation (1.26) has no real roots.*

Theorem 2.1. *Assume that (1.21), Case (i) and the following condition (H_6) hold:*

(H_6) *either $h(0) \geq g(0) = 0$, $g(u)$ is increasing and $h(u)$ is nondecreasing in $[0,\infty)$ and*

$$\liminf_{u\to\infty} \frac{g(u)}{u} \geq 1, \qquad \limsup_{u\to\infty} \frac{h(u)}{u} \leq 1, \tag{2.1}$$

or $h(0) \leq g(0) = 0$, $g(u)$ is increasing and $h(u)$ is nondecreasing in $(-\infty,0]$ and

$$\liminf_{u\to-\infty} \frac{g(u)}{u} \geq 1, \qquad \limsup_{u\to-\infty} \frac{h(u)}{u} \leq 1. \tag{2.2}$$

Then Eq.(1.1) has a nonoscillatory solution.

Proof. We only consider the case where (2.1) in (H_6) holds. The case where (2.2) in (H_6) holds can be dealt with by a similar fashion. By Case (i), we can choose an $\varepsilon_0 \in (0,1)$ such that $(1-\varepsilon_0)p_0 e^{-\lambda^*\tau} > 1$ and that

$$\varepsilon_0 \left(\lambda^{*n} p_0 e^{-\lambda^*\tau} + q_0 e^{-\lambda^*\sigma}\right) < -F(\lambda^*). \tag{2.3}$$

Set

$$F^+_{\varepsilon_0}(\lambda) = \lambda^n \left[1 - (1-\varepsilon_0)p_0 e^{-\lambda\tau}\right] + (1+\varepsilon_0)q_0 e^{-\lambda\sigma}.$$

Then by (2.3), we have

$$\begin{aligned} F^+_{\varepsilon_0}(\lambda^*) &= \lambda^{*n}\left[1-(1-\varepsilon_0)p_0 e^{-\lambda^*\tau}\right] + (1+\varepsilon_0)q_0 e^{-\lambda^*\sigma} \\ &= F(\lambda^*) + \varepsilon_0\left(\lambda^{*n} p_0 e^{-\lambda^*\tau} + q_0 e^{-\lambda^*\sigma}\right) \\ &< 0. \end{aligned}$$

Note that $\lim_{\lambda\to\infty} F^+_{\varepsilon_0}(\lambda) = \infty$, it follows that there exists a $\lambda_1 \in (\lambda^*, \infty)$ such that $F^+_{\varepsilon_0}(\lambda_1) = 0$. By (2.1) in (H_6), we may choose $M_1 > M$ such that $g(u)$ is increasing and $h(u)$ is nondecreasing in $[M_1, \infty)$ and

$$(1-\varepsilon_0)u \le g(u), \quad 0 \le h(u) \le (1+\varepsilon_0)u, \quad u \in [M_1, \infty). \tag{2.4}$$

Set $x_0(t) = e^{\lambda_1 t}$. Then

$$\frac{1}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}x_0(s-\sigma)ds = \lambda_1^{-n}e^{\lambda_1(t-\sigma)} - e^{\lambda_1(t_0-\sigma)}\sum_{i=1}^{n}\frac{\lambda_1^{-i}(t-t_0)^{n-i}}{(n-i)!}.$$

It follows from the fact that $F^+_{\varepsilon_0}(\lambda_1) = 0$ that

$$\lambda_1^{-n}e^{\lambda_1(t_0-\sigma)} + x_0(t) \le (1-\varepsilon_0)p_0 x_0(t-\tau) - \frac{(1+\varepsilon_0)q_0}{(n-1)!}\int_{t_0}^{t}(t-s)^{n-1}x_0(s-\sigma)ds, \quad t \ge t_0. \tag{2.5}$$

Choose $T > t_0$ such that (1.21) holds for $t \ge T$, and

$$x_0(t-\tau-\sigma) > M_1, \quad t \ge T, \tag{2.6}$$

and

$$x_0(t) > g^{-1}\left(\frac{a}{p_0}\right) \ge g^{-1}\left(\frac{a}{p(t+\tau)}\right), \quad t \ge T-\sigma, \tag{2.7}$$

where $a = \lambda_1^{-n}e^{\lambda_1(t_0-\sigma)}$. Then from (1.21), (2.4), (2.5) and (2.6), we obtain

$$\begin{aligned} & p(t+\tau)g(x_0(t)) \\ \ge\ & (1-\varepsilon_0)p_0 x_0(t) \\ \ge\ & \lambda_1^{-n}e^{\lambda_1(t_0-\sigma)} + x_0(t+\tau) + \frac{(1+\varepsilon_0)q_0}{(n-1)!}\int_T^{t+\tau}(t+\tau-s)^{n-1}x_0(s-\sigma)ds \\ \ge\ & \lambda_1^{-n}e^{\lambda_1(t_0-\sigma)} + x_0(t+\tau) + \frac{1}{(n-1)!}\int_T^{t+\tau}(t+\tau-s)^{n-1}q(s)h(x_0(s-\sigma))ds \\ =\ & a + x_0(t+\tau) + \frac{1}{(n-1)!}\int_T^{t+\tau}(t+\tau-s)^{n-1}q(s)h(x_0(s-\sigma))ds, \quad t \ge T. \end{aligned}$$

This is

$$x_0(t) \geq g^{-1}\left(\frac{1}{p(t+\tau)}\left[a+x_0(t+\tau) + \frac{1}{(n-1)!}\int_T^{t+\tau}(t+\tau-s)^{n-1}q(s)h(x_0(s-\sigma))ds\right]\right), \quad t \geq T. \quad (2.8)$$

Define a sequence $\{x_k(t)\}_{k=1}^{\infty}$ as follows:

$$x_k(t)=\begin{cases} g^{-1}\left(\frac{1}{p(t+\tau)}\left[a+x_{k-1}(t+\tau) + \frac{1}{(n-1)!}\int_T^{t+\tau}(t+\tau-s)^{n-1}q(s)h(x_{k-1}(s-\sigma))ds\right]\right), & t \geq T, \\ g^{-1}\left(\frac{a}{p(t+\tau)}\right)+\frac{x_k(T)-g^{-1}(a/p(T+\tau))}{x_0(T)-g^{-1}(a/p(T+\tau))}\left[x_0(t)-g^{-1}\left(\frac{a}{p(t+\tau)}\right)\right], & T-\sigma \leq t < T, \end{cases}$$

$$k = 1, 2, \ldots. \quad (2.9)$$

From (2.7), (2.8) and (2.9), by induction, we have

$$g^{-1}\left(\frac{a}{p(t+\tau)}\right) \leq x_k(t) \leq x_{k-1}(t), \quad t \geq T-\sigma, \quad k = 1, 2, \ldots.$$

Then for $t \geq T-\sigma$, $x(t) = \lim_{k\to\infty} x_k(t)$ exists and

$$g^{-1}\left(\frac{a}{p(t+\tau)}\right) \leq x(t) \leq x_0(t), \quad t \geq T-\sigma, \quad (2.10)$$

and

$$x(t) \geq g^{-1}\left(\frac{1}{p(t+\tau)}\left[a+x(t+\tau)+\frac{1}{(n-1)!}\int_T^{t+\tau}(t+\tau-s)^{n-1}q(s)h(x(s-\sigma))ds\right]\right), \quad t \geq T. \quad (2.11)$$

That is

$$x(t) = -a + p(t)g(x(t-\tau)) - \frac{1}{(n-1)!}\int_T^t (t-s)^{n-1}q(s)h(x(s-\sigma))ds, \quad t \geq T+\tau. \quad (2.12)$$

It is easy to show that $x(t)$ is an eventually positive solution of Eq.(1.1). The proof is complete. □

3. Critical Case

Theorem 3.1. *Assume that (1.21), Case (ii) and (H_4) hold. Then Eq.(1.1) has a nonoscillatory solution.*

Proof. We only consider the case where (1.22) in (H_4) holds. The case where (1.23) in (H_4) holds is similar and so omit it. It follows from Case (ii) that

$$F(\lambda_0)=0 \quad\text{and}\quad F'(\lambda_0)=0,$$

which yields that

$$\lambda_0^n=\lambda_0^n p_0 e^{-\lambda_0\tau}-q_0e^{-\lambda_0\sigma} \tag{3.1}$$

and

$$n\lambda_0^{n-1}=n\lambda_0^{n-1}p_0e^{-\lambda_0\tau}-\lambda_0^n p_0\tau e^{-\lambda_0\tau}+q_0\sigma e^{-\lambda_0\sigma}. \tag{3.2}$$

From (3.1) and (3.2), it is easy to see that $\lambda_0>0$ and that

$$(n+\lambda_0\sigma)\left(p_0e^{-\lambda_0\tau}-1\right)=\lambda_0p_0\tau e^{-\lambda_0\tau}. \tag{3.3}$$

Set $x_0(t)=\sqrt{t}e^{\lambda_0 t}$. Then from (3.1), (3.2) and (3.3), we have

$$\begin{aligned}
&[x_0(t)-p_0x_0(t-\tau)]^{(n)}+q_0x_0(t-\sigma)\\
&\quad= \left(\sqrt{t}e^{\lambda_0 t}\right)^{(n)}-p_0e^{-\lambda_0\tau}\left(\sqrt{t-\tau}e^{\lambda_0 t}\right)^{(n)}+q_0e^{-\lambda_0\sigma}\sqrt{t-\sigma}e^{\lambda_0 t}\\
&\quad= e^{\lambda_0 t}\Bigg\{q_0e^{-\lambda_0\sigma}\sqrt{t-\sigma}+\Bigg[\lambda_0^n\sqrt{t}+\frac{n\lambda_0^{n-1}}{2\sqrt{t}}-\frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{t^3}}\\
&\qquad-\sum_{k=3}^{n}\frac{(-1)^k n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{t^{2k-1}}}\Bigg]\\
&\qquad-p_0e^{-\lambda_0\tau}\Bigg[\lambda_0^n\sqrt{t-\tau}+\frac{n\lambda_0^{n-1}}{2\sqrt{t-\tau}}-\frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{(t-\tau)^3}}\\
&\qquad-\sum_{k=3}^{n}\frac{(-1)^k n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{(t-\tau)^{2k-1}}}\Bigg]\Bigg\}\\
&\quad= e^{\lambda_0 t}\Bigg\{\Bigg[-\frac{q_0\sigma e^{-\lambda_0\sigma}}{\sqrt{t}+\sqrt{t-\sigma}}+\frac{n\lambda_0^{n-1}}{2\sqrt{t}}-\frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{t^3}}\\
&\qquad-\sum_{k=3}^{n}\frac{(-1)^k n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{t^{2k-1}}}\Bigg]\\
&\qquad+p_0e^{-\lambda_0\tau}\Bigg[\frac{\lambda_0^n\tau}{\sqrt{t}+\sqrt{t-\tau}}-\frac{n\lambda_0^{n-1}}{2\sqrt{t-\tau}}+\frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{(t-\tau)^3}}\\
&\qquad+\sum_{k=3}^{n}\frac{(-1)^k n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{(t-\tau)^{2k-1}}}\Bigg]\Bigg\}\\
&\quad= e^{\lambda_0 t}\Bigg\{\Bigg[-\frac{q_0\sigma^2 e^{-\lambda_0\sigma}}{2\sqrt{t}(\sqrt{t}+\sqrt{t-\sigma})^2}-\frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{t^3}}\\
&\qquad-\sum_{k=3}^{n}\frac{(-1)^k n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{t^{2k-1}}}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&+p_0e^{-\lambda_0\tau}\left[\frac{\lambda_0^n\tau^2}{2\sqrt{t}(\sqrt{t}+\sqrt{t-\tau})^2}-\frac{n\lambda_0^{n-1}\tau}{2\sqrt{t}\sqrt{t-\tau}(\sqrt{t}+\sqrt{t-\tau})}+\frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{(t-\tau)^3}}\right.\\
&\quad\left.\left.+\sum_{k=3}^{n}\frac{(-1)^k n(n-1)\cdots(n-k+1)\cdot 1\cdot 3\cdots(2k-3)\lambda_0^{n-k}}{2^k k!\sqrt{(t-\tau)^{2k-1}}}\right]\right\}\\
\le\;& e^{\lambda_0 t}\left\{\left[-\frac{q_0\sigma^2e^{-\lambda_0\sigma}}{2\sqrt{t}(\sqrt{t}+\sqrt{t-\sigma})^2}-\frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{t^3}}\right]+\frac{C}{\sqrt{t^5}}\right.\\
&\left.+p_0e^{-\lambda_0\tau}\left[\frac{\lambda_0^n\tau^2}{2\sqrt{t}(\sqrt{t}+\sqrt{t-\tau})^2}-\frac{n\lambda_0^{n-1}\tau}{2\sqrt{t}\sqrt{t-\tau}(\sqrt{t}+\sqrt{t-\tau})}+\frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{(t-\tau)^3}}\right]\right\}\\
\le\;& e^{\lambda_0 t}\left\{-\frac{1}{8\sqrt{t^3}}\left[q_0\sigma^2e^{-\lambda_0\sigma}+2n\lambda_0^{n-1}p_0\tau e^{-\lambda_0\tau}+n(n-1)\lambda_0^{n-2}\right]+\frac{C}{\sqrt{t^5}}\right.\\
&\left.+p_0e^{-\lambda_0\tau}\left[\frac{\lambda_0^n\tau^2}{2\sqrt{t}(\sqrt{t}+\sqrt{t-\tau})^2}+\frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{(t-\tau)^3}}\right]\right\}\\
=\;& e^{\lambda_0 t}\left\{-\frac{\lambda_0^{n-2}}{8\sqrt{t^3}}\left[\left(p_0e^{-\lambda_0\tau}-1\right)(\lambda_0\sigma)^2+2n\lambda_0p_0\tau e^{-\lambda_0\tau}+n(n-1)\right]+\frac{C}{\sqrt{t^5}}\right.\\
&\left.+p_0e^{-\lambda_0\tau}\left[\frac{\lambda_0^n\tau^2}{2\sqrt{t}(\sqrt{t}+\sqrt{t-\tau})^2}+\frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{(t-\tau)^3}}\right]\right\}\\
=\;& e^{\lambda_0 t}\left\{-\frac{\lambda_0^{n-2}}{8\sqrt{t^3}}\left[\left(p_0e^{-\lambda_0\tau}-1\right)\left(\frac{\lambda_0p_0\tau e^{-\lambda_0\tau}}{p_0e^{-\lambda_0\tau}-1}-n\right)^2+2n\lambda_0p_0\tau e^{-\lambda_0\tau}+n(n-1)\right]\right.\\
&\left.+\frac{C}{\sqrt{t^5}}+p_0e^{-\lambda_0\tau}\left[\frac{\lambda_0^n\tau^2}{2\sqrt{t}(\sqrt{t}+\sqrt{t-\tau})^2}+\frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{(t-\tau)^3}}\right]\right\}\\
=\;& e^{\lambda_0 t}\left\{-\frac{\lambda_0^{n-2}}{8\sqrt{t^3}}\left[\frac{\left(\lambda_0p_0\tau e^{-\lambda_0\tau}\right)^2}{p_0e^{-\lambda_0\tau}-1}+n^2p_0e^{-\lambda_0\tau}-n\right]+\frac{C}{\sqrt{t^5}}\right.\\
&\left.+p_0e^{-\lambda_0\tau}\left[\frac{\lambda_0^n\tau^2}{2\sqrt{t}(\sqrt{t}+\sqrt{t-\tau})^2}+\frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{(t-\tau)^3}}\right]\right\}\\
=\;& e^{\lambda_0 t}\left\{-\frac{\lambda_0^{n-2}}{8\sqrt{t^3}}\left[\frac{\left(\lambda_0p_0\tau e^{-\lambda_0\tau}\right)^2}{p_0e^{-\lambda_0\tau}-1}+n(n-1)p_0e^{-\lambda_0\tau}+n\left(p_0e^{-\lambda_0\tau}-1\right)\right]+\frac{C}{\sqrt{t^5}}\right.\\
&\left.+p_0e^{-\lambda_0\tau}\left[\frac{\lambda_0^n\tau^2}{2\sqrt{t}(\sqrt{t}+\sqrt{t-\tau})^2}+\frac{n(n-1)\lambda_0^{n-2}}{8\sqrt{(t-\tau)^3}}\right]\right\}\\
=\;& e^{\lambda_0 t}\left\{-\frac{\lambda_0^{n-2}}{8\sqrt{t^3}}\left[(\lambda_0\tau)^2p_0e^{-\lambda_0\tau}\left(\frac{p_0e^{-\lambda_0\tau}}{p_0e^{-\lambda_0\tau}-1}-\frac{4t}{(\sqrt{t}+\sqrt{t-\tau})^2}\right)\right.\right.\\
&\left.\left.+n(n-1)p_0e^{-\lambda_0\tau}\left(1-\frac{\sqrt{t^3}}{\sqrt{(t-\tau)^3}}\right)+n\left(p_0e^{-\lambda_0\tau}-1\right)\right]+\frac{C}{\sqrt{t^5}}\right\}\\
\le\;& e^{\lambda_0 t}\left[-\frac{\lambda_0^{n-2}n\left(p_0e^{-\lambda_0\tau}-1\right)}{8\sqrt{t^3}}+\frac{C}{\sqrt{t^5}}\right],
\end{aligned}
$$

where $C > 0$ is a sufficient large constant. On the other hand, we have

$$\begin{aligned}[x_0(t-\sigma)]^{1-r} &= \left(\sqrt{t-\sigma}e^{\lambda_0(t-\sigma)}\right)^{1-r} \\ &\le t^{(1-r)/2}e^{(1-r)\lambda_0(t-\sigma)} \\ &= t^{(1-r)/2}e^{-\lambda_0[rt+(1-r)\sigma]}e^{\lambda_0 t},\end{aligned}$$

and for large t

$$\begin{aligned}&\left\{[x_0(t-\tau)]^{1-r}\right\}^{(n)} \\ =\ & e^{-(1-r)\lambda_0\tau}\left[(t-\tau)^{(1-r)/2}e^{(1-r)\lambda_0 t}\right]^{(n)} \\ =\ & e^{(1-r)\lambda_0(t-\tau)}\Bigg[(1-r)^n\lambda_0^n(t-\tau)^{(1-r)/2}+\frac{n(1-r)^n\lambda_0^{n-1}}{2(t-\tau)^{(1+r)/2}} \\ & +\sum_{k=2}^{n}\frac{(-1)^{k-1}n!\cdot(1+r)\cdot(3+r)\cdots(2k-3+r)(1-r)^{n-k+1}\lambda_0^{n-k}}{2^k k!(n-k)!(t-\tau)^{(2k-1-r)/2}}\Bigg] \\ \le\ & 2(1-r)^n\lambda_0^n t^{(1-r)/2}e^{(1-r)\lambda_0(t-\tau)} \\ =\ & 2(1-r)^n\lambda_0^n t^{(1-r)/2}e^{-\lambda_0[rt+(1-r)\tau]}e^{\lambda_0 t}.\end{aligned}$$

Thus, there exists a large $T > t_0$ such that (1.21) holds for $t \ge T$, and

$$x_0(t-\tau-\sigma) > M, \qquad t \ge T, \tag{3.4}$$

$$\left[x_0(t)-p_0\left(x_0(t-\tau)-K[x_0(t-\tau)]^{1-r}\right)\right]^{(i)} < 0, \qquad i=0,1,\ldots,n, \quad t \ge T, \tag{3.5}$$

and

$$\begin{aligned}&\left[x_0(t)-p_0\left(x_0(t-\tau)-K[x_0(t-\tau)]^{1-r}\right)\right]^{(n)}+q_0\left[x_0(t-\sigma)+K|x_0(t-\sigma)|^{1-r}\right] \\ &\quad= [x_0(t)-p_0x_0(t-\tau)]^{(n)}+q_0x_0(t-\sigma) \\ &\qquad +p_0K\left\{[x_0(t-\tau)]^{1-r}\right\}^{(n)}+q_0K|x_0(t-\sigma)|^{1-r} \\ &\quad\le e^{\lambda_0 t}\left[-\frac{\lambda_0^{n-2}n\left(p_0e^{-\lambda_0\tau}-1\right)}{8\sqrt{t^3}}+\frac{C}{\sqrt{t^5}}\right] \\ &\qquad +2p_0K(1-r)^n\lambda_0^n t^{(1-r)/2}e^{-\lambda_0[rt+(1-r)\tau]}e^{\lambda_0 t}+t^{(1-r)/2}e^{-\lambda_0[rt+(1-r)\sigma]}e^{\lambda_0 t} \\ &\quad\le 0, \qquad t \ge T,\end{aligned}$$

which yields

$$\begin{aligned}&\left\{x_0(t)-p_0\left[x_0(t-\tau)-K|x_0(t-\tau)|^{1-r}\right]\right\}^{(n)} \\ &\qquad +q_0\left[x_0(t-\sigma)+K|x_0(t-\sigma)|^{1-r}\right] \le 0, \qquad t \ge T.\end{aligned} \tag{3.6}$$

It follows from (1.21), (1.22), (3.4), (3.5) and (3.6) that

$$\begin{aligned}
& p(t+\tau)g(x_0(t)) \\
\geq\ & p_0\left[x_0(t) - K|x_0(t)|^{1-r}\right] \\
\geq\ & b + x_0(t+\tau) + \frac{q_0}{(n-1)!}\int_T^{t+\tau}(t+\tau-s)^{n-1}\left[x_0(s-\sigma) + K|x_0(s-\sigma)|^{1-r}\right]ds \\
\geq\ & b + x_0(t+\tau) + \frac{1}{(n-1)!}\int_T^{t+\tau}(t+\tau-s)^{n-1}q(s)h(x_0(s-\sigma))ds, \qquad t \geq T,
\end{aligned}$$

where

$$b = p_0\left(x_0(T-\tau) - K[x_0(T-\tau)]^{1-r}\right) - x_0(T) > 0.$$

This is

$$\begin{aligned}
x_0(t) \geq\ & g^{-1}\left(\frac{1}{p(t+\tau)}\left[b + x_0(t+\tau)\right.\right. \\
& \left.\left. + \frac{1}{(n-1)!}\int_T^{t+\tau}(t+\tau-s)^{n-1}q(s)h(x_0(s-\sigma))ds\right]\right), \qquad t \geq T. \qquad (3.7)
\end{aligned}$$

Similar to the proof of Theorem 2.1, it is easy to show that Eq.(1.1) has an eventually positive solution. The proof is complete. □

4. Some Remarks

It is easy to see that Condition (H_4) implies (H_6). Hence, combining Theorems 2.1 and 3.1, we have immediately Theorem 1.1.

In view of the proofs, the main theorems in [7, 9, 13] are slightly modified as Theorem A-Theorem E, respectively.

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