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Stability Results for Cohen-Grossberg Neural Networks with Delays*

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Abstract. In this paper we give a sufficient condition to imply local and global asymptotic attractivity of the equilibrium of the Cohen-Grossberg neural network with time-dependent delays of the form

$$\dot{x}_i(t) = c_i(x(t)) \left(-d_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_{ij}(t))) - I_i \right)$$

$t \geq 0$ $(i = 1, \ldots, n)$ independently of the delays.

AMS Subject Classifications: 34K20, 34D23, 92B20

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1. Introduction

The notion of cellular neural networks (CNNs) was introduced by Chua and Yang ([6]), and since then, CNN models have been used in many engineering applications, e.g., in signal processing and especially in static image treatment [7]. As a generalization of CNNs, cellular neural networks with delays (DCNNs) were introduced by Roska and Chua [15].

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In [8] Cohen and Grossberg proposed a neural network model (CGNN) described by the following system of ordinary differential equations

$$\dot{x}_i(t) = c_i(x_i(t)) \left(-d_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - I_i \right), \quad t \geq 0 \tag{1.1}$$

$(i = 1, \ldots, n)$. Here n is the number of neurons in the network; $x_i(t)$ is the potential of the ith neuron; $c_i(x_i(t))$ represents the amplification function; $d_i(x_i(t))$ is an appropriately behaved function such that the solution remains bounded; $f_j(x_j)$ is the activation function of the ith neuron; a_{ij} denotes the strengths of the jth unit on the ith unit at time t; and I_i is an external input to the ith neuron.

In this paper we study the asymptotic stability of the CGNN model with time-dependent delays of the form

$$\dot{x}_i(t) = c_i(x_i(t)) \left(-d_i(x_i(t)) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_{ij}(t))) - I_i \right) \tag{1.2}$$

for $t \geq 0$ $(i = 1, \ldots, n)$. Here $\tau_{ij}(t)$ corresponds to delay of signals from the ith neuron to the jth neuron. We associate the initial conditions

$$x_i(t) = \varphi_i(t), \qquad t \in [-r, 0], \quad i = 1, \ldots, n, \tag{1.3}$$

to (1.2), where $r = \max\{\sup_{t \geq 0} \tau_{ij}(t) \colon i, j = 1, \ldots, n\}$.

The delayed CGNN model (1.2) includes as a special case (using $c_i(x) = 1$, $d_i(x) = \gamma_i x$) the delayed Hopfield CNN model

$$\dot{x}_i(t) = -\gamma_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau_{ij}(t))) - I_i, \tag{1.4}$$

$t \geq 0, \quad (i = 1, \ldots, n)$.

We assume throughout this paper that

(H1) $c_i \colon \mathbb{R} \to (0, \infty)$ is continuous for $i = 1, \ldots, n$;

(H2) $d_i \colon \mathbb{R} \to \mathbb{R}$ is continuous and increasing for $i = 1, \ldots, n$;

(H3) $f_i \colon \mathbb{R} \to \mathbb{R}$ is continuous and increasing, and $|f_i(x)| \leq M$ for $i = 1, \ldots, n$.

A typical, widely used activation function is the Hopfield output function $f \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \frac{1}{2}(|t+1| - |t-1|) = \begin{cases} 1, & t > 1, \\ t, & -1 \leq t \leq 1, \\ -1, & t < -1 \end{cases} \tag{1.5}$$

satisfies (H3). Another frequently used activation function in applications is a sigmoid-type smooth function, like $f(x) = \tanh x$.

The stability of (1.2) and more general classes of CGNNs has been intensively studied, see, e.g., [1], [3]–[5], [10]–[11], [16]–[20], and the references therein. Note that in these references (H1)–(H3) (together with some additional conditions) are used as standard assumptions on the parameters of (1.2).

Arik and Orman in [1] proved that if (H1)–(H3) holds and

$$0 < \underline{\alpha}_i \le c_i(x) \le \overline{\alpha}_i \quad (x \in \mathbb{R}) \tag{1.6}$$

$$\frac{d_i(x) - d_i(y)}{x - y} \ge \gamma_i > 0, \quad |g_i(x) - g_i(y)| \le L_i|x - y|, \quad (x \ne y, \ \ x, y \in \mathbb{R}), \tag{1.7}$$

$$\|A\|_1 + \|B\|_1 < \frac{\gamma_m \underline{\alpha}_m}{\overline{\alpha}_M L_M}, \tag{1.8}$$

where $\gamma_m = \min\{\gamma_1, \ldots, \gamma_n\}$, $\underline{\alpha}_m = \min\{\underline{\alpha}_1, \ldots, \underline{\alpha}_n\}$, $\overline{\alpha}_M = \max\{\overline{\alpha}_1, \ldots, \overline{\alpha}_n\}$, $L_M = \max\{L_1, \ldots, L_n\}$, $\|\cdot\|_1$ is the matrix norm generated by the $\|\mathbf{x}\|_1 = \sum_{k=1}^n |x_i|$ vector norm, then (1.2) has a unique equilibrium, which is globally exponentially stable. Hwang, Cheng and Liao [11] proved a similar result, but instead of (1.8) they assumed

$$\|A\|_2 + \|B\|_2 < \frac{\gamma_m \underline{\alpha}_m}{\overline{\alpha}_M L_M}.$$

Wang, Zou [17] showed that under (H1)–(H3), (1.6), (1.7), $a_{ij} = 0$ $(i, j = 1, \ldots, n)$, and

$$\underline{\alpha}_i \gamma_i > L_i \sum_{j=1}^n |b_{ij}| \overline{\alpha}_j, \qquad i = 1, \ldots, n \tag{1.9}$$

yields that (1.2) has a unique equilibrium, which is globally exponentially stable. Liao, Yang and Guo [14] and also Li and Yang [13] proved the same result (for a slightly more general equation).

In Section 2 we give a sufficient condition which implies global attractivity of the unique equilibrium of the delayed CGNN (1.2). In our results we do not assume boundedness of the functions $c_1, \ldots, c_n$, so (1.6) is not necessarily satisfied. We also present some other sufficient conditions, where, instead of the global Lipschitz type condition (1.7), we assume nonlinear estimates, which imply local attractivity of the equilibrium. We also present sufficient conditions implying global attractivity of the equilibrium. In Section 3 we give examples which illustrate that our main results are applicable to a larger class of CGNNs than the existing ones cited above. Section 4 contains the proofs of the main results.

First we introduce some notations. Let $\mathbb{R}_+$ be the set of nonnegative real numbers. We use the relation $\mathbf{x} \le \mathbf{y}$ ($\mathbf{x} \ll \mathbf{y}$, respectively) for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, if $x_i \le y_i$ ($x_i < y_i$, respectively) for all $i = 1, \ldots, n$, where $\mathbf{x} = (x_1, \ldots, x_n)^T$ and $\mathbf{y} = (y_1, \ldots, y_n)^T$. We introduce the vectors $\mathbf{0} = (0, 0, \ldots, 0)^T \in \mathbb{R}^n$ and $\mathbf{1} = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$. Any fixed norm on $\mathbb{R}^n$ is denoted by $\|\cdot\|$. The positive part of a real number a is denoted by a^+, i.e., $a^+ = \max(a, 0)$.

We say that an $n \times n$ matrix H is an M-matrix, if all of its diagonal elements are nonnegative, and its off-diagonal elements are nonpositive, and all of its principal minors are nonnegative (see, e.g., [2]). It is known (see, e.g., [2]) that if H is a nonsingular M-matrix, then it is monotone, i.e., $H\mathbf{x} \geq \mathbf{0}$ implies $\mathbf{x} \geq \mathbf{0}$.

Remark 1.1. Let K be a matrix such that the diagonal elements of K are all positive and the off-diagonal elements are all nonpositive. Then it is known (see, e.g., Theorem 2.3 in [2]) that if K is diagonally dominant, then it is a nonsingular M-matrix, as well. Moreover, K is a nonsingular M-matrix, if and only if, there exists a positive diagonal matrix D such that KD is a diagonally dominant matrix. This yields that the diagonal elements of a nonsingular M-matrix are all positive. We note that there are 50 conditions listed in [2] which are all equivalent to that a matrix is a nonsingular M-matrix.

2. Main results

The positivity of the functions c_i yields that $\mathbf{x}^* = (x_1^*, \dots, x_n^*)^T$ is an equilibrium of (1.2), if and only if it satisfies

$$d_i(x_i^*) - \sum_{j=1}^{n} a_{ij} f_j(x_j^*) - \sum_{j=1}^{n} b_{ij} f_j(x_j^*) + I_i = 0, \qquad i = 1, \dots, n.$$

It follows from the assumptions that (1.2) has at least one equilibrium. For the proof see, e.g., [16] or [17].

Lemma 2.1. *Suppose (H1)–(H3). Then there exists at least one equilibrium of* (1.2).

Let $\mathbf{x}^* = (x_1^*, \dots, x_n^*)^T$ be any equilibrium of (1.2), which will be fixed throughout this paper. Substituting the new functions $y_i(t) = x_i(t) - x_i^*$ in (1.2) leads to the system

$$\dot{y}_i(t) = \alpha_i(y_i(t)) \left(-\beta_i(y_i(t)) + \sum_{j=1}^{n} a_{ij} g_j(y_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(y_j(t - \tau_{ij}(t))) \right) \tag{2.1}$$

for $t \geq 0$ $(i = 1, \dots, n)$ $\alpha_i(y) = c_i(y + x_i^*)$, $\beta_i(y) = d_i(y + x_i^*) - d_i(x_i^*)$, and $g_i(y) = f_i(y + x_i^*) - f_i(x_i^*)$, $(i = 1, \dots, n)$.

It is easy to check that assumptions (H1)-(H3) yield the following properties for the new parameters:

(P1) $\alpha_i(y) > 0$ $(y \in \mathbb{R})$ for $i = 1, \dots, n$;

(P2) $\beta_i(0) = 0$, and $\beta_i(y) \operatorname{sign} y > 0$, $(y \neq 0, \quad y \in \mathbb{R})$, β_i is increasing, for $i = 1, \dots, n$;

(P3) $g_i(0) = 0$, g_i is increasing, and $0 \leq g_i(y) \operatorname{sign} y \leq 2M_i$ $(y \in \mathbb{R})$ for $i = 1, \dots, n$.

In addition to (H1)–(H3), we asume that the fixed equilibrium $\mathbf{x}^*$ satisfies

(H4) there exists a function $\eta_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\eta_i(0) = 0$ and $0 < \eta_i(|y|) \le \beta_i(y) \operatorname{sign} y$, $(y \ne 0, \quad y \in \mathbb{R})$ for $i = 1, \dots, n$;

(H5) there exists a function $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega_i(0) = 0$, $\omega_i(y) > 0$ for $y > 0$, ω_i is increasing, and $0 \le g_i(y) \operatorname{sign} y \le \omega_i(|y|) \le \tilde{M}_i$ $(y \in \mathbb{R})$ for $i = 1, \dots, n$.

We note that the dependence of the functions α_i, β_i, η_i and ω_i on $\mathbf{x}^*$ is omitted in their notations, but always should be kept in mind.

Theorem 2.1. *Assume (H1)–(H3), the equilibrium $\mathbf{x}^*$ satisfies (H4)–(H5), moreover, there exist positive numbers $R_1, \dots, R_n$ such that*

$$\eta_i(R_i) > a_{ii}^+ \omega_i(R_i) + \sum_{\substack{j=1,\\ j\ne i}}^{n} |a_{ij}| \omega_j(R_j) + \sum_{j=1}^{n} |b_{ij}| \omega_j(R_j), \qquad i = 1, \dots, n, \tag{2.2}$$

$$\eta_i(y) \ge e_i \omega_i(y), \qquad 0 \le y \le R_i, \quad i = 1, \dots, n, \tag{2.3}$$

and the $n \times n$ matrix $H = (h_{ij})$ defined by

$$h_{ij} = \begin{cases} e_i - a_{ii}^+ - |b_{ii}|, & i = j, \\ -|a_{ij}| - |b_{ij}|, & i \ne j \end{cases} \tag{2.4}$$

is a nonsingular M-matrix. Then the equilibrium $\mathbf{x}^*$ is locally attractive, i.e., for any initial functions $\varphi_1, \dots, \varphi_n$ satisfying $|\varphi_i(s) - x_i^*| < R_i$, $s \in [-r, 0]$ $(i = 1, \dots, n)$ it follows that the corresponding solution $\mathbf{x} = (x_1, \dots x_n)^T$ of (1.2)-(1.3) *satisfies*

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*.$$

Under slightly more restrictive conditions we get global attractivity of the equilibrium, which yields the uniqueness of the equilibrium.

Theorem 2.2. *Assume (H1)–(H3), the equilibrium $\mathbf{x}^*$ satisfies (H4)–(H5), moreover,*

$$\lim_{y\to\infty} \eta_i(y) > a_{ii}^+ \tilde{M}_i + \sum_{\substack{j=1,\\ j\ne i}}^{n} |a_{ij}| \tilde{M}_j + \sum_{j=1}^{n} |b_{ij}| \tilde{M}_j, \qquad i = 1, \dots, n, \tag{2.5}$$

$$\eta_i(y) \ge e_i \omega_i(y), \qquad y \ge 0, \quad i = 1, \dots, n, \tag{2.6}$$

and the $n \times n$ matrix $H = (h_{ij})$ defined by (2.4) *is a nonsingular M-matrix. Then $\mathbf{x}^*$ is the only equilibrium of* (1.2)*, and it is globally attractive, i.e., any solution $\mathbf{x} = (x_1, \dots x_n)^T$ of* (1.2)-(1.3) *satisfies*

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*.$$

Now we consider a special class of (1.2), where η_i and ω_i are linear.

Theorem 2.3. *Assume (H1)–(H3), the equilibrium* $\mathbf{x}^*$ *satisfies (H4)–(H5), moreover,*

$$\eta_i(y) = \gamma_i y \qquad \textit{and} \qquad \omega_i(y) = L_i y, \qquad y \geq 0, \quad i = 1, \ldots, n, \tag{2.7}$$

and the $n \times n$ *matrix* $\tilde{H} = (\tilde{h}_{ij})$ *defined by*

$$\tilde{h}_{ij} = \begin{cases} \gamma_i - a_{ii}^+ L_i - |b_{ii}| L_i, & i = j, \\ -|a_{ij}| L_j - |b_{ij}| L_j, & i \neq j \end{cases} \tag{2.8}$$

is a nonsingular M-matrix. Then $\mathbf{x}^*$ *is the only equilibrium of* (1.2)*, and it is globally attractive.*

Next we consider the special case of (1.2), the Hopfield DCNN (1.4). Theorem 2.3 has the following immediate consequence.

Theorem 2.4. *Suppose* $\gamma_1, \ldots, \gamma_n > 0$*,* $f_1(x_1), \ldots, f_n(x_n)$ *satisfy (H3), they are Lipschitz continuous with Lipschitz constants* $L_1, \ldots, L_n$*, respecitvely, and the matrix* $\hat{H} = (\hat{h}_{ij})$ *defined by*

$$\hat{h}_{ij} = \begin{cases} \gamma_i - a_{ii}^+ L_i - |b_{ii}| L_i, & i = j, \\ -|a_{ij}| L_j - |b_{ij}| L_j, & i \neq j \end{cases} \tag{2.9}$$

is a nonsingular M-matrix. Then (1.4) *has a unique equilibrium, which is globally attractive.*

Note that a similar result was proved in [9], where it was shown that if all activation functions are equal to the Hopfield function (1.5), the matrix K with elements

$$k_{ij} = \begin{cases} \gamma_i - a_{ii} - |b_{ii}|, & i = j, \\ -|a_{ij}| - |b_{ij}|, & i \neq j \end{cases}$$

is diaginally dominant, and

$$|I_i| \leq \gamma_i - a_{ii} - \sum_{\substack{j=1, \\ j \neq i}}^{n} |a_{ij}| - \sum_{j=1}^{n} |b_{ij}|, \qquad i = 1, \ldots, n,$$

then (1.4) has a unique equilibrium, which is globally attractive. Theorem 2.4 improves this result in the case when A has only nonnegative diagonal elements.

3. Examples

Example 3.1. To illustrate our results, consider first the two-dimensional delayed CGNN model equations

$$\begin{aligned}\dot{x}_1(t) &= \Big(\sin(x_1(t)) + 1.5\Big)\Big(-2x_1(t) - 0.5\tanh(x_1(t-1)) \\ &\qquad + 0.5\tanh(x_2(t-2)) - I_1\Big) \qquad (3.1)\\ \dot{x}_2(t) &= \Big(\cos(x_2(t)) + 1.25\Big)\Big(-3x_2(t) + \tanh(x_1(t-1)) \\ &\qquad - \tanh(x_2(t-2)) - I_2\Big), \qquad (3.2)\end{aligned}$$

for $t \geq 0$. It is easy to see that in this example $\underline{\alpha}_1 = 0.5$, $\overline{\alpha}_1 = 2.5$, $\underline{\alpha}_2 = 0.25$, $\overline{\alpha}_2 = 2.25$, $\gamma_1 = 2$, $\gamma_2 = 3$, $L_1 = L_2 = 1$, and for any equilibrium $\mathbf{x}^*$, (H4) and (H5) are satisfied with $\eta_1(x) = \gamma_1 x = 2x$, $\eta_2(x) = \gamma_2 x = 3x$, $\omega_1(x) = L_1 x = x$, $\omega_2(x) = L_2 x = x$. Therefore the matrix $\tilde{H}$ defined by (2.8) equals to

$$\tilde{H} = \begin{pmatrix} 1.5 & -0.5 \\ -1 & 2 \end{pmatrix},$$

and it is a nonsingular M-matrix. Hence Theorem 2.3 yields that for any input $(I_1, I_2)^T$, (3.1)-(3.2) has a unique equilibrium, which is globally attractive.

We can check that $\|B\|_1 = 1.5$, $\|B\|_2 = 1.5811$, $\frac{\gamma_m \underline{\alpha}_m}{\overline{\alpha}_M L_M} = \frac{2\cdot 0.25}{2.5\cdot 1} = 0.2$, therefore the results of [1], [11], [13], [14] and [17] can not be applied.

Example 3.2. Consider now the following two dimensional CGNN model.

$$\begin{aligned}\dot{x}_1(t) &= \frac{1}{x_1^2(t)+1}\Big(-d(x_1(t)) - 0.4f(x_1(t-1+0.1\sin(t))) \\ &\qquad + 0.5f(x_2(t-2)) - I_1\Big) \qquad (3.3)\\ \dot{x}_2(t) &= (x_2^2(t)+1)\Big(-2d(x_2(t)) + f(x_1(t-1)) \\ &\qquad - 0.5f(x_2(t-3+\cos t)) - I_2\Big), \qquad (3.4)\end{aligned}$$

where

$$d(x) = \begin{cases} \sqrt{x}, & x \geq 0, \\ -\sqrt{-x}, & x < 0, \end{cases}$$

and f is the Hopfield activation function defined by (1.5). If we select $I_1 = I_2 = 0$ inputs, then $\mathbf{x}^* = \mathbf{0}$ is an equilibrium of (3.3)-(3.4). Around this equilibrium (H4) and (H5) hold with $\eta_1(x) = \sqrt{x}$, $\eta_2(x) = 2\sqrt{x}$ and $\omega_i(x) = x$ for $i = 1, 2$. Select

$R_1 = R_2 = 1$. Then (2.2) is satisfied, and (2.3) also holds with $e_1 = 1$ and $e_2 = 2$, the matrix H defined by (2.4) is

$$H = \begin{pmatrix} 0.6 & -0.5 \\ -1 & 1.5 \end{pmatrix},$$

and it is a nonsingular M-matrix. Therefore Theorem 2.1 yields that $\mathbf{0}$ is locally attractive, all solutions starting from initial functions satisfying $|\varphi_i(s)| < 1$, $s \in [-4, 0]$, $i = 1, 2$ will tend to 0 as $t \to \infty$.

Example 3.3. Finally, consider

$$\begin{aligned} \dot{x}_1(t) &= e^{x_1(t)}\Big(-3x_1(t) - 6f(x_1(t)) + 0.5f(x_2(t)) - 0.4f(x_1(t-2)) \\ &\qquad + 0.5f(x_2(t-2)) - I_1\Big) \qquad (3.5) \\ \dot{x}_2(t) &= \frac{1}{x_2^2(t)+1}\Big(-2x_2(t) - 0.5f(x_1(t)) + f(x_2(t)) + 0.7f(x_1(t-3)) \\ &\qquad - 0.5f(x_2(t-1)) - I_2\Big), \qquad (3.6) \end{aligned}$$

$t \geq 0$, where f is the Hopfield activation function defined by (1.5). For this equation we can apply Theorem 2.3, since the matrix

$$\tilde{H} = \begin{pmatrix} 2.6 & -1 \\ -1.2 & 0.5 \end{pmatrix},$$

defined by (2.4) is a nonsingular M-matrix, and we get that (3.5)-(3.6) has a unique globally attractive equilibrium for all inputs.

Note the importance of taking the positive part of a_{ii} in the definition of $\tilde{H}$ in (1.5) instead of using $|a_{ii}|$, since otherwise the condition would be false for our equation. We also comment that α_1 and α_2 do not satisfy (1.6) in this example, therefore the results of [1], [11], [13], [14] and [17] can not be applied.

4. Proofs

Let y be a fixed solution y of (2.1) and $v_1, \ldots, v_n$ be positive numbers. Then consider the associated auxiliary system

$$\begin{aligned} \dot{z}_i(t) &= \alpha_i(y_i(t))\Bigg(-\eta_i(z_i(t)) + a_{ii}^+\omega_i(z_i(t)) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}|\omega_j(z_j(t)) \\ &\qquad + \sum_{j=1}^{n} |b_{ij}|\omega_j(z_j(t-\tau_{ij}(t))) + v_i\Bigg), \qquad t \geq 0 \quad (i = 1, \ldots, n), \qquad (4.1) \end{aligned}$$

and the initial condition

$$z_i(t) = \psi_i(t) \qquad t \in [-r, 0], \quad i = 1, \dots, n. \tag{4.2}$$

The proof of our main results will be based on the following lemmas, which collect some properties of the solutions of (4.1)-(4.2).

Lemma 4.1. *Suppose (H1)–(H5), and there exist positive numbers* $R_1, \dots, R_n$ *such that* (2.2) *holds. Let* $\mathbf{v} = (v_1, \dots, v_n)^T$ *be such that* $\mathbf{0} \ll \mathbf{v}$ *and*

$$\eta_i(R_i) > a_{ii}^+ \omega_i(R_i) + \sum_{\substack{j=1,\\ j\neq i}}^n |a_{ij}| \omega_j(R_j) + \sum_{j=1}^n |b_{ij}| \omega_j(R_j) + v_i, \qquad i = 1, \dots, n. \tag{4.3}$$

Let $\psi_i : [-r, 0] \to \mathbb{R}_+$ *satisfying* $0 < \psi_i(s) < R_i$, $s \in [-r, 0]$ *(*$i = 1, \dots, n$*), and let* $z_1, \dots, z_n$ *be the corresponding solution of* (4.1)-(4.2)*. Then*

$$0 < z_i(t) < R_i, \qquad t \geq 0, \quad i = 1, \dots, n.$$

Proof. Since $z_i(0) > 0$ and z_i is continuous on $[0, \infty)$ for all $i = 1, \dots, n$, $z_i(t) > 0$ for small enough $t \geq 0$. Suppose there exist i and $T > 0$ such that

$$z_i(T) = 0 \qquad \text{and} \qquad z_j(t) > 0 \quad \text{for } t \in [-r, T),\ j = 1, \dots, n.$$

Then $\dot{z}_i(T) \leq 0$. On the other hand, the positivity of v_i implies

$$\begin{aligned} \dot{z}_i(T) &= \alpha_i(y_i(T)) \Bigg(-\eta_i(z_i(T)) + a_{ii}^+ \omega_i(z_i(T)) + \sum_{\substack{j=1,\\ j\neq i}}^n |a_{ij}| \omega_j(z_j(T)) \\ &\quad + \sum_{j=1}^n |b_{ij}| \omega_j(z_j(T - \tau_{ij}(T))) + v_i \Bigg) \\ &= \alpha_i(y_i(T)) \Bigg(\sum_{\substack{j=1,\\ j\neq i}}^n |a_{ij}| \omega_j(z_j(T)) + \sum_{j=1}^n |b_{ij}| \omega_j(z_j(T - \tau_{ij}(T))) + v_i \Bigg) \\ &> 0, \end{aligned}$$

which is a contradiction. Therefore $z_i(t) > 0$ for all $t > 0$ and $i = 1, \dots, n$.

To prove that $z_i(t) < R_i$ for all $i = 1, \dots, n$, suppose there exists $t^* > 0$ and i such that

$$z_i(t^*) = R_i, \qquad \text{and} \qquad z_j(t) < R_j, \quad t \in [-r, t^*), \quad j = 1, \dots, n.$$

Then $\dot{z}_i(t^*) \geq 0$. On the other hand, the monotonicity of ω_j yields

$$\begin{aligned}
0 \leq\ & \dot{z}_i(t^*) \\
=\ & \alpha_i(y_i(t^*))\Big(-\eta_i(z_i(t^*)) + a_{ii}^{+}\omega_i(z_i(t^*)) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}|\omega_j(z_j(t^*)) \\
& + \sum_{j=1}^{n} |b_{ij}|\omega_j(z_j(t^* - \tau_{ij}(t^*))) + v_i\Big) \\
\leq\ & \alpha_i(y_i(t^*))\Big(-\eta_i(R_i) + a_{ii}^{+}\omega_i(R_i) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}|\omega_j(R_j) + \sum_{j=1}^{n} |b_{ij}|\omega_j(R_j) + v_i\Big) \\
<\ & 0
\end{aligned}$$

This contradiction concludes the proof. □

Lemma 4.2. *Assume (H1)–(H5), and suppose there exist positive numbers* $R_1, \ldots, R_n$ *and* $e_1, \ldots, e_n$ *such that* (2.2) *and* (2.3) *hold, and the matrix* $H = (h_{ij})$ *defined by* (2.4) *is a nonsingular M-matrix. Let* $\mathbf{v} = (v_1, \ldots, v_n)^T$ *be such that* $\mathbf{0} \ll \mathbf{v}$ *and* (4.3) *hold. Let* $\psi_i\colon [-r, 0] \to \mathbb{R}_+$ *satisfying* $0 < \psi_i(s) < R_i$, $s \in [-r, 0]$ *(*$i = 1, \ldots, n$*), and let* $\mathbf{z}(t) = (z_1(t), \ldots, z_n(t))^T$ *be the corresponding solution of* (4.1)-(4.2). *Then*

$$\omega(\limsup_{t\to\infty} \mathbf{z}(t)) \leq H^{-1}\mathbf{v}, \tag{4.4}$$

where $\omega(\mathbf{z}) = (\omega_1(z_1), \ldots, \omega_n(z_n))^T$.

Proof. It follows from Lemma 4.1 that

$$\overline{m_i} = \limsup_{t\to\infty} z_i(t) \qquad \text{and} \qquad \underline{m_i} = \liminf_{t\to\infty} z_i(t)$$

satisfy $0 \leq \underline{m_i} \leq \overline{m_i} \leq R_i$. Consider first the case when $\underline{m_i} = \overline{m_i}$, i.e., $\lim_{t\to\infty} z_i(t) = \overline{m_i}$. Then let $t_k^{(i)}$ be an arbitrary sequence such that $t_k^{(i)} \to \infty$ as $k \to \infty$. We may also assume that

$$\lim_{k\to\infty} z_j(t_k^{(i)}) = m_{ij}^* \qquad \text{and} \qquad \lim_{k\to\infty} z_j(t_k - \tau_{ij}(t_k^{(i)})) = m_{ij}^{**} \tag{4.5}$$

for all $j = 1, \ldots, n$ for some $m_{ij}^*, m_{ij}^{**} \in [\underline{m}_j, \overline{m}_j]$, since otherwise we can select a subsequence of $t_k^{(i)}$ with this property. Then we get

$$\begin{aligned}
0 =\ & \lim_{k\to\infty} \dot{z}_i(t_k^{(i)}) \\
=\ & \alpha_i(\overline{m}_i)\Big(-\eta_i(\overline{m}_i) + a_{ii}^{+}\omega_i(\overline{m}_i) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}|\omega_j(m_{ij}^*) + \sum_{j=1}^{n} |b_{ij}|\omega_j(m_{ij}^{**}) + v_i\Big),
\end{aligned}$$

and therefore

$$
\begin{aligned}
0 &= -\eta_i(\overline{m}_i) + a_{ii}^{+}\omega_i(\overline{m}_i) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}|\omega_j(m_{ij}^{*}) + \sum_{j=1}^{n} |b_{ij}|\omega_j(m_{ij}^{**}) + v_i \\
&\leq -\eta_i(\overline{m}_i) + a_{ii}^{+}\omega_i(\overline{m}_i) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}|\omega_j(\overline{m}_j) + \sum_{j=1}^{n} |b_{ij}|\omega_j(\overline{m}_j) + v_i.
\end{aligned}
$$

Now consider the case when $\underline{m}_i < \overline{m_i}$. Then there exists a sequence $t_k^{(i)}$ such that

$$
\lim_{k\to\infty} t_k^{(i)} = \infty, \qquad \dot{z}_i(t_k^{(i)}) \geq 0,\ k = 1, 2\ldots, \qquad \text{and} \qquad \lim_{k\to\infty} z_i(t_k^{(i)}) = \overline{m}_i.
$$

We may again assume that (4.5) holds. It is easy to argue that in this case

$$
0 \leq -\eta_i(\overline{m}_i) + a_{ii}^{+}\omega_i(\overline{m}_i) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}|\omega_j(m_{ij}^{*}) + \sum_{j=1}^{n} |b_{ij}|\omega_j(m_{j}^{**}) + v_i \tag{4.6}
$$

$$
\leq -e_i\omega_i(\overline{m}_i) + a_{ii}^{+}\omega_i(\overline{m}_i) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}|\omega_j(\overline{m}_j) + \sum_{j=1}^{n} |b_{ij}|\omega_j(\overline{m}_j) + v_i \tag{4.7}
$$

is satisfied, as well. Therefore (4.6) and (4.7) hold for all $i = 1, \ldots, n$. We can rewrite (4.7) as

$$
H\Omega(\overline{m})\mathbf{1} \leq \mathbf{v},
$$

$\Omega(\overline{m}) = \operatorname{diag}(\omega_1(\overline{m}_1), \ldots, \omega_n(\overline{m}_n))$, $\mathbf{1} = (1, \ldots, 1)^T$. Since H is a nonsingular M-matrix, it is monotone (see, e.g., [2]), therefore it implies $\Omega(\overline{m})\mathbf{1} \leq H^{-1}\mathbf{v}$, or equivalently, $\omega(\overline{m}) \leq H^{-1}\mathbf{v}$, and the proof of the lemma is complete. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)^T$ be any fixed equilibrium of (1.2), $\varphi_1, \ldots \varphi_n$ be given initial functions satisfying $|\varphi_i(s) - x_i^*| < R_i$ for $s \in [-r, 0]$, $\mathbf{x} = (x_1, \ldots, x_n)^T$ be the corresponding solution of (1.2)-(1.3), and $\mathbf{y}(t) = \mathbf{x}(t) - \mathbf{x}^*$. Then fix initial functions $\psi_i \colon [-r, 0] \to \mathbb{R}_+$ such that

$$
|\varphi_i(s) - x_i^*| < \psi_i(s) < R_i, \qquad s \in [-r, 0], \quad i = 1, \ldots, n.
$$

Let $\mathbf{0} \ll \mathbf{v} = (v_1, \ldots, v_n)^T$ be such that (4.3) is satisfied. Let $\mathbf{z} = (z_1, \ldots, z_n)^T$ denote the solution of the corresponding IVP (4.1)-(4.2). Since $z_i(0) > |y_i(0)|$, relation $|y_i(t)| < z_i(t)$ holds for sufficiently small $t > 0$ and $i = 1, \ldots, n$. Suppose there exists i and $T > 0$ such that

$$
|y_i(T)| = z_i(T), \quad \text{and} \quad |y_j(t)| < z_j(t), \quad t \in [-\tau, T), \quad j = 1, \ldots, n. \tag{4.8}
$$

It follows from Lemma 4.1 that $|y_i(T)| = z_i(T) \neq 0$, therefore $\frac{d}{dt}|y_i(t)|$ exists at T, and $\frac{d}{dt}(|y_i(t)|)_{|t=T} = \dot{y}_i(T)\,\mathrm{sign}\,y_i(T)$. Hence

$$\begin{aligned}\frac{d}{dt}(|y_i(t)|)_{|t=T} &= \alpha_i(y_i(T))\Bigg(-\beta_i(y_i(T)) + \sum_{j=1}^{n} a_{ij}g_j(y_j(T)) \\ &\quad + \sum_{j=1}^{n} b_{ij}g_j(y_j(T-\tau_{ij}(T)))\Bigg)\,\mathrm{sign}\,y_i(T).\end{aligned}$$

Since $\beta_i(y_i(T))\,\mathrm{sign}\,y_i(T) \geq 0$ it follows from (P1) and (P2) that

$$\beta_i(y_i(T))\,\mathrm{sign}\,y_i(T) \geq \eta_i(|y_i(T)|).$$

If $a_{ii} \geq 0$, then (P3) yields

$$0 \leq a_{ii}g_i(y_i(T))\,\mathrm{sign}\,y_i(T) \leq a_{ii}\omega_i(|y_i(T)|).$$

If $a_{ii} < 0$, then (P3) yields

$$a_{ii}g_i(y_i(T))\,\mathrm{sign}\,y_i(T) \leq 0. \tag{4.9}$$

Consequently,

$$\begin{aligned}\frac{d}{dt}(|y_i(t)|)_{|t=T} &< \alpha_i(y_i(T))\Bigg(-\eta_i(|y_i(T)|) + a_{ii}^{+}\omega_i(|y_i(T)|) \\ &\quad + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}|\omega_j(|y_j(T)|) + \sum_{j=1}^{n} |b_{ij}|\omega_j(|y_j(T-\tau_{ij}(T))|) + v_i\Bigg) \\ &\leq \alpha_i(y_i(T))\Bigg(-\eta_i(z_i(T)) + a_{ii}^{+}\omega_i(z_i(T)) + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}|\omega_j(z_j(T)) \\ &\quad + \sum_{j=1}^{n} |b_{ij}|\omega_j(z_j(T-\tau_{ij}(T))) + v_i\Bigg) \\ &= \dot{z}_i(T).\end{aligned}$$

This contradicts to assumption (4.8), therefore $|y_i(t)| < z_i(t)$ holds for all $t > 0$ and $i = 1, \ldots, n$. Moreover, Lemma 4.2 yields

$$\omega(\limsup_{t\to\infty} \mathbf{z}(t)) \leq H^{-1}\mathbf{v}.$$

Since $\mathbf{v}$ can be arbitraly close to $\mathbf{0}$, it imples

$$\omega(\limsup_{t\to\infty} \mathbf{z}(t)) \leq \mathbf{0},$$

which yields

$$\lim_{t\to\infty} \mathbf{z}(t) = \limsup_{t\to\infty} \mathbf{z}(t) = \mathbf{0}.$$

This concludes the proof. □

Proof of Theorem 2.2. Let $\varphi_1, \ldots, \varphi_n$ be given initial functions, R_i be such that $\sup\{\varphi_i(s)\colon\ s \in [-r, 0]\} < R_i$ and

$$\eta_i(R_i) > a_{ii}^+ \tilde{M}_i + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| \tilde{M}_j + \sum_{j=1}^{n} |b_{ij}| \tilde{M}_j, \tag{4.10}$$

for $i = 1, \ldots, n$. Then (2.2) holds, and so Theorem 2.1 yields that the solution $\mathbf{x}$ of (1.2) corresponding to these initial functions satisfies

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^*.$$

□

The proof of Theorem 2.3 is based on the following version of Lemma 4.2.

Lemma 4.3. *Assume (H1)–(H5),* (2.7) *holds, and the matrix* $\tilde{H} = (h_{ij})$ *defined by* (2.8) *is a nonsingular M-matrix. Let* $\mathbf{v} = (v_1, \ldots, v_n)^T$ *be such that* $\mathbf{0} \ll \mathbf{v}$*. Let* $\psi_i : [-r, 0] \to \mathbb{R}_+$ *satisfying* $0 < \psi_i(s)$*,* $s \in [-r, 0]$ *(*$i = 1, \ldots, n$*). Then the corresponding solution* $\mathbf{z}$ *of* (4.1)-(4.2) *satisfies*

$$\omega(\limsup_{t\to\infty} \mathbf{z}(t)) \le H^{-1}\mathbf{v}.$$

Proof. Let R_i be such that $\sup\{\psi_i(s)\colon\ s \in [-r, 0]\} < R_i$ and

$$\eta_i(R_i) > a_{ii}^+ \tilde{M}_i + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| \tilde{M}_j + \sum_{j=1}^{n} |b_{ij}| \tilde{M}_j + v_i$$

holds for $i = 1, \ldots, n$. Then (2.2) is also satisfied.

It follows from (4.6) and (2.7)

$$0 \quad \le \quad -\gamma_i \overline{m}_i + a_{ii}^+ L_i \overline{m}_i + \sum_{\substack{j=1,\\ j\neq i}}^{n} |a_{ij}| L_j \overline{m}_j + \sum_{j=1}^{n} |b_{ij}| L_j \overline{m}_j + v_i. \tag{4.11}$$

We can rewrite (4.11) as

$$\tilde{H}\overline{M}\mathbf{1} \le \mathbf{v},$$

where $\overline{M} = \mathrm{diag}(\overline{m}_1, \ldots, \overline{m}_n)^T$. Since $\tilde{H}$ is a nonsingular M-matrix, it implies $\overline{M}\mathbf{1} \le H^{-1}\mathbf{v}$, which yields the statement of the lemma, since $\mathbf{v}$ can be arbitrary close to $\mathbf{0}$. □

Proof of Theorem 2.3. The result follows from Lemma 4.3 and the proof of Theorem 2.1 using the fact that (2.2) holds with any large enough $R_1, \ldots, R_n$, as it was argued in the proof of Lemma 4.3. The global attractivity clearly implies the uniqueness of the equilibrium. □

Proof of Theorem 2.4. We apply Theorem 2.3 with $c_i(x) = 1$ and $d_i(x) = \gamma_i x$ ($i = 1, \ldots, n$). □

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