International Jr, of Qualitative Theory of Differential Equations and Applications
Vol. 1 No. 1 (June, 2015)
ISSN: 0973-3590

On the parameter-dependence of the solutions of functional differential equations with unbounded state-dependent delay I. The upper-semicontinuity of the resolvent function

Bernát Slezák*

University of Pannonia, H-8201, P. O. Box 158, Veszprém, Hungary

Abstract. The continuous parameter- and initial value- dependence of the multivalued resolvent function of retarded functional differential equations with unbounded state-dependent delay are investigated, without assuming uniqueness. In the finite dimensional case some results are obtained under weaker conditions than the classical ones, even in the special case of ordinary differential equations.

AMS Subject Classifications: 34K05, 34A12

Keywords: Functional differential equations; Solutions reaching to the boundary; State-dependent delay; Upper-semicontinuous dependence on the parameters

1. Introduction and preliminaries

Let $\mathcal{M}$ and Ω be metric spaces, Y be Banach-space, let σ, r be real numbers, $\sigma < b \leq \infty$, $0 \leq r < \infty$, let $W \subset \mathbb{R} \times \mathcal{C}([-r,0], Y) \times \mathcal{M}$ and $U \subset \mathbb{R} \times Y \times Y \times \Omega$ be nonempty open sets and let $\tau : W \to \mathbb{R}_0^+$ and $f : U \to Y$ be continuous functions, $\phi \in \mathcal{C}([-r,0], Y)$, $(\sigma, \phi, \mu) \in W$ and $(\sigma, \phi, \phi(\sigma - \tau(\sigma, \phi, \mu)), \omega) \in U$. If $x \in \mathcal{C}([\sigma - r, b[, Y)$ and $t \in [\sigma, b[$ then denote

$$x_t : [-r, 0] \to Y, \ s \longmapsto x(t+s)$$

the so called segment function.

State-dependent functional differential equations with initial condition

$$x'(t) = f(t, x(t), x(t - \tau(t, x_t, \mu)), \omega), \ t \in [\sigma, b[, \ x_\sigma = \varphi, \tag{1.1}$$

E-mail addresses: slezakb@almos.vein.hu

*Research supported by Hungarian National Foundation for Scientific Research Grant No. T046929.

had been investigated for instance in [8], [9], [11], [12], [1], [13], [14], [18], where $Y = \mathbb{R}^n$, $\varphi \in \mathcal{C}([-r,0],\mathbb{R}^n)$ is Lipschitzian, τ and f are continuous, Lipschitzian in their second and third variables in some sense, or they are continuously differentiable in their second and third variable, under the following further boundedness supposition:

$$\tau(t,\psi,\mu) \le r, \ if \ (t,\psi,\mu) \in Dom(\tau). \tag{1.2}$$

This condition will be written usually in the shorter form $\tau \le r$.

Under these conditions existence and uniqueness of the solutions, moreover Lipschitz-continuous and smooth parameter-dependence of the solutions was proved, respectively. If $\varphi \in \mathcal{C}([-r,0],\mathbb{R}^n)$ is not Lipschitzian, or τ (or f) is not Lipschitzian in its second variable then the uniqueness cannot be proved (see [8]).

It is known that if supposition (1.2) holds then (1.1) is equivalent to a classical equation of the type

$$x'(t) = \tilde{f}(t,x_t,p), \ x_\sigma = \varphi, \tag{1.3}$$

where the parameter $p \in \mathcal{M}\times\ \Omega$ (see for instance in [9] or Lemma 2.1 in Section 2). However, (1.1) is not a generalization of equation (1.3): it is easily seen that if $n \ge 2$ then there does not exist an equation of the type (1.1), equivalent to (1.3). In this paper we consider more general parametric functional differential equations with state-dependent unbounded delay and with initial condition, formulated below, which is a generalization also of (1.3):

Let $Dom(f) = U \subset \mathbb{R}\times\mathcal{C}([-r,0],Y)\times Y\times\Omega$ be an open set. The following initial value problem will be considered:

$$\begin{aligned} x'(t) &= f(t,x_t,x(t-\tau(t,x_t,\mu)),\omega), \ t \in [\sigma,b[, \\ x(t) &= \phi(t-\sigma) \ \text{if } t \in [\sigma-r,\sigma], \end{aligned} \tag{1.4}$$

where $x'(\sigma)$ denotes the right-hand derivative of x at σ. Sometimes the solutions will be considered on a compact interval $[\sigma-r,b]$; in this case $x'(b)$ denotes the left-hand derivative of x at b. In the special case when τ is the constant zero function we get the classical functional differential equations, if furthermore $r = 0$ then we get ordinary differential equations.

The boundedness condition (1.2) is a standard assumption in state-dependent delay equations (in [9] it is not mentioned explicitly, but it is used all over the paper). However, it has a technical character: it makes possible to use the equivalence between (1.4) and (1.3). But in lot of problems (1.2) is not valid. For instance several paper investigate equations of the type

$$x'(t) = a(t)\,x(t-|x(t)|), \tag{1.5}$$

where this supposition is too strong. Similarly, assumption (1.2) does not hold in the case of the pantograph equation

$$\begin{aligned} x'(t) &= f(t,x(t),x(\alpha t)), \ \sigma \le t, \\ x(t) &= \phi(t-\sigma), \ \alpha\sigma \le t < \sigma \end{aligned}$$

where $0 < \alpha < 1$, $\sigma \geq 0$ and $\phi \in \mathcal{C}([(\alpha-1)\sigma, 0], Y)$ are fixed. In this paper property (1.2) is sometimes omitted, sometimes it is replaced by property

$$\sigma - r \leq t - \tau(t, \psi, \mu) \leq t, \text{ whenever } (t, \psi, \mu) \in Dom(\tau),$$

where σ is fixed.

The following notations will be used: $\Phi(\sigma, \phi, \mu, \omega)$ denotes the set of the noncontinuable solutions of the initial value problem (1.4) and $\Phi^{[a,b]}$ denotes the multivalued function, having $(\sigma, \phi, \mu, \omega)$ in its domain if and only if each element x of $\Phi(\sigma, \phi, \mu, \omega)$ is defined on the interval $[a, b]$, and $\Phi^{[a,b]}(\sigma, \phi, \mu, \omega)$ denotes the set of the restrictions of these solutions to the interval $[a, b]$. The function Φ will be named the **resolvent function** of the corresponding equation.

In Section 2 we investigate the relation between the classical functional differential equations of type (1.3) and equations of type (1.4). By the results of Section 2 the initial value problem (1.4) is locally equivalent to the parametric version of the initial value problem (1.3) under some not too strong assumptions, and using these facts we can generalize the basic theorems. However, the proofs are not simple adaptations: the statements usually give stronger results than the classical ones even in the case of ordinary differential equations.

The weaker suppositions involve some difference between problems (1.1) and (1.4). For instance, in the case of the classical functional differential equations, and also if condition (1.2) holds, under our suppositions (and assuming that Y is finite dimensional) every initial value problem has a solution ([7],Theorem 2.3). In our more general problem (1.4) this statement is not valid (Lemma 2.1). So it is not evident that the noncontinuable solutions reach to the boundary on the right or not. By definition the solution x of the equation

$$x'(t) = f(t, x_t) \tag{1.6}$$

reaches to the boundary of $Dom(f)$ on the right if for each compact subset $C \subset Dom(f)$ there exists an element $t_C \in Dom(x)$ such that $(t, x_t) \in Dom(f) \setminus C$ for all $t \in I$, $t \geq t_C$. In Section 3 we investigate the question that in what sense and under which assumptions the noncontinuable solutions of (1.4) reach to the boundary on the right. It is known that every noncontinuable solution of equation (1.6) reaches to the boundary of $Dom(f)$ on the right, if f is continuous and its domain is open ([17], Theorem 10.). F. Hartung et. al. has given an example ([10] Example 3.4) for neutral equation with state-dependent delay, having noncontinuable solution with compact trajectory. Example 3.2 below shows that in our simpler case this situation can also occur. In Proposition 3.2 and Theorem 3.4 sufficient conditions are given under which a noncontinuable solution either reaches to the boundary on the right or its trajectory is compact.

The functions f and τ are supposed only to be continuous. Obviously, noncontinuable solutions are not necessary unique. In Section 4 it is proved that $\Phi^{[a,b]}(C) = \cup_{(\sigma,\phi,\mu,\omega)\in C}\Phi^{[a,b]}(\sigma, \phi, \mu, \omega)$ is compact, whenever C is compact.

The continuous dependence on the parameters can be only upper semicontinuous dependence, because the noncontinuable solutions usually are not unique.

The following classical theorem is valid for ordinary differential equations:

Theorem *(Theorem 2.10.3 [5]) Let Y be finite dimensional, let Λ be a metric space, let J be a compact interval in $\mathbb{R}$, and let $S = J \times Y \times \Lambda$. Let also $f : S \to Y$ be bounded and continuous, and for each point $\gamma = (t_0, y_0, \lambda) \in S$ let $\sum(\gamma)$ be the subset of $C(J, Y)$ consisting of all solutions of the equation $y'(t) = f(t, y(t), \lambda)$ on J taking the value y_0 at t_0. Then for each point $\bar{\gamma} = (\bar{t}_0, \bar{y}_0, \bar{\lambda}) \in S$ and each open subset G of $C(J, Y)$ containing $\sum(\bar{\gamma})$, there exist a neighborhood V of $\bar{\gamma}$ in S such that $\sum(\gamma) \subset G$ whenever $\gamma \in V$.*

Similar result is Theorem 2.10.4 of [5], where the domain of f is an arbitrary open set, but the uniqueness of the solutions of the ordinary differential equation is supposed. Theorem 2.2 in Chapter 2 of [7] gives a generalization of this second theorem for functional differential equations under weaker conditions, because the uniqueness of the noncontinuable solutions is supposed only for the fixed initial condition (σ_0, φ_0). In Section 5 the common generalizations of these theorems are given (Theorems 5.1, 5.2 and 5.3). One of the main results of this paper is that we prove the upper-semicontinuous dependence of the solutions of (1.4) on the initial conditions and on the parameters. In the special case of ordinary differential equations and also in the case of the classical functional differential equation (1.3) stronger results are obtained than the original ones in some further senses, for instance the domain of f can be different from $S = J \times Y$ and f can be unbounded on its domain.

The questions like this belong to the basic theory, and quite few articles are devoted to this topic. Analogous problems were investigated in the paper [6] for semilinear functional differential inclusions with infinite delay and in [15] with respect to attractors for parametric delay differential equations without uniqueness.

2. Equivalences of initial value problems.

In this section we use the following notations:

Let Y be Banach space, let $\mathcal{M}$ and Ω be metric spaces, let $\sigma \in \mathbb{R}$ and let $r \geq 0$ be fixed, let $f : U \subset \mathbb{R} \times \mathcal{C}([-r, 0], Y) \times Y \times \mathcal{M} \to Y$ and $\tau : W \subset \mathbb{R} \times \mathcal{C}([-r, 0], Y) \times \Omega \to \mathbb{R}_0^+$ be continuous functions, defined on the open sets U and W, respectively.

Lemma 2.1 (I) *Suppose that the initial value problem (1.4) has a solution, whenever $(\sigma, \phi, \mu) \in Dom(\tau)$ and $(\sigma, \phi, \phi(-\tau(\sigma, \phi, \mu)), \omega) \in Dom(f)$. Then the following statements hold:*
(i) $\tau \leq r$.
(ii) *Let Λ stand for the function defined by the following formula:*

$$\Lambda : Dom(\tau) \to Y, \ (t, \psi, \mu) \longmapsto \psi(-\tau(t, \psi, \mu)), \tag{2.1}$$

and let $\tilde{f}$ defined on the following way:

$$\begin{aligned} \tilde{f}(t, \psi, \mu, \omega) \quad &: \quad = f(t, \psi, \Lambda(t, \psi, \mu), \omega), \\ (t, \psi, \mu) \quad &\in \quad Dom(\tau), (t, \psi, \Lambda(t, \psi, \mu)) \in Dom(f). \end{aligned}$$

The domain of $\tilde{f}$ is open and $\tilde{f}$ is continuous. The function x is a solution of (1.4) if and only if it is a solution of the initial value problem

$$x'(t) = \tilde{f}(t, x_t, \mu, \omega), \text{ if } t \in [\sigma, b[, \quad x(t) = \phi(t - \sigma), \text{ if } t \in [\sigma - r, \sigma]. \tag{2.2}$$

(iii) If x is a noncontinuable solution of (1.4) then x reaches to the boundary of $Dom\left(\tilde{f}(\cdot, \cdot, \mu, \omega)\right)$ on the right.
(iv) Let $G \subset \mathbb{R} \times \mathcal{C}([-r, 0], Y) \times \mathcal{M} \times \Omega$ be a compact set such that if $(\sigma, \phi, \mu, \omega) \in G$ then $(\sigma, \phi, \mu) \in Dom(\tau)$ and $(\sigma, \phi, \phi(-\tau(t, \phi, \mu)), \omega) \in Dom(f)$. Denote $\Phi(\sigma, \phi, \mu, \omega)$ the set of the noncontinuable solutions x of (1.4), satisfying the initial condition $x_\sigma = \phi$.
For every neighborhood $V \subset \mathbb{R} \times \mathcal{C}([-r, 0], Y)$ of the compact set $pr_{\mathbb{R}\times\mathcal{C}([-r,0],Y)}(G)$ there exist a neighborhood E of G, and positive numbers α and K such that

$$((\sigma, \phi, \mu, \omega) \in E, \ x \in \Phi(\sigma, \phi, \mu, \omega))$$
$$\Longrightarrow \begin{cases} [\sigma - r, \sigma + \alpha] \subset Dom(x), \ \{(t, x_t) \mid t \in [\sigma, \sigma + \alpha]\} \subset V, \\ \text{and } x \text{ is Lipschitzian on the interval } [\sigma, \sigma + \alpha] \text{ with the constant } K. \end{cases}$$

(II) Suppose furthermore that $Y = \mathbb{R}^n$. The initial value problem (1.4) has a solution, whenever $(\sigma, \phi, \mu) \in Dom(\tau)$ and $(\sigma, \phi, \phi(-\tau(\sigma, \phi, \mu)), \omega) \in Dom(f)$, if and only if $\tau \leq r$.

Proof. *I*. (i) If x is a solution of (1.4) then $x'(\sigma) = f(\sigma, \phi, \phi(-\tau(\sigma, \phi, \mu)), \omega)$, that is $\phi(-\tau(\sigma, \phi, \mu))$ is defined, consequently $\tau(\sigma, \phi, \mu) \leq r$.

(ii) The function

$$H : Dom(\tau) \times \Omega \to \mathbb{R} \times \mathcal{C}([-r, 0], Y) \times \mathcal{M} \times \Omega,$$
$$(t, \psi, \mu, \omega) \longmapsto (t, \psi, \psi(-\tau(\sigma, \psi, \mu)), \omega)$$

is continuous, moreover $Dom\left(\tilde{f}\right) = H^{-1}(Dom(f))$, therefore $Dom\left(\tilde{f}\right)$ is open. The statement follows immediately from the fact that by (i) the function Λ is well defined, and it is continuous, as it is easily seen.

(iii) This statement follows immediately from (ii) and Theorem 10 (ii) of *[17]*.

(iv) By (ii) if $x \in \Phi(\sigma, \phi, \mu, \omega)$ then $x \in \eta(\sigma, \phi, \mu, \omega)$, where $\eta(\sigma, \phi, \mu, \omega)$ denotes the set of the solutions of the initial value problem (2.2). Moreover, $\tilde{f}$ is continuous. By Theorem *12 [17]* our statement holds.

II. By (I) (i) we have to prove only that *if* $\tau \leq r$ then the initial value problem (1.4) has a solution, whenever $(\sigma, \phi, \mu) \in Dom(\tau)$ *and* $(\sigma, \phi, \phi(-\tau(\sigma, \phi, \mu)), \omega) \in Dom(f)$. *By* (I) (ii) it is equivalent to the statement that the initial value problem (2.2) has a solution, whenever $(\sigma, \phi, \mu, \omega) \in Dom\left(\tilde{f}\right)$. By a classical theorem (see for instance [7] Chapter 2.2) the statement holds. □

Considering equations of the type (1.4), the standard trick is to use the equivalence of the initial value problems (1.4) and (2.2) (see for instant in [9], [11] and [12]). The difficulties follow from the fact that it is possible that f and τ have some "good" property but $\tilde{f}$ has not. For instance it may happen that f and τ are Lipschitzian or continuously differentiable but $\tilde{f}$ is not. However, if f and τ are continuous then $\tilde{f}$ is also continuous, so every statement, following from the continuity of the function f on the right of (1.3), can be generalized for the problem (1.4). However, if the assumption $\tau \leq r$ does not hold then this trick cannot be used directly. In Lemma 2.2, and in Corollary 2.1 below we give further relations between the classical functional differential equation of the type (1.3) and the state-dependent problem (1.4), not using the supposition $\tau \leq r$.

As the parameters μ and ω do not play any role in the further part of this section, we omit them.

Lemma 2.2 *Let $r \leq \tilde{r}$ and let $\tilde{\tau}$ stand for the function, defined by the following formula:*

$$\begin{aligned} \tilde{\tau} \quad &: \quad \left\{ \left(t, \tilde{\psi}\right) \mid \tilde{\psi} \in \mathcal{C}\left([-\tilde{r}, 0], Y\right), \; \left(t, \tilde{\psi}_{|[-r,0]}\right) \in Dom\left(\tau\right) \right\} \to \mathbb{R}, \\ \left(t, \tilde{\psi}\right) \quad &\longmapsto \quad \tau\left(t, \tilde{\psi}_{|[-r,0]}\right), \end{aligned}$$

and let $\tilde{f}$ stand for the function, defined on the following way:

$$\begin{aligned} \tilde{f} \quad &: \quad \left\{ \left(t, \tilde{\psi}, y\right) \mid \tilde{\psi} \in \mathcal{C}\left([-\tilde{r}, 0], Y\right), \; \left(t, \tilde{\psi}_{|[-r,0]}, y\right) \in Dom\left(f\right) \right\} \to Y, \\ \left(t, \tilde{\psi}, y\right) \quad &\longmapsto \quad f\left(t, \tilde{\psi}_{|[-r,0]}, y\right). \end{aligned}$$

(i) $\tilde{\tau}$ and $\tilde{f}$ are continuous or Lipschitzian or (n times, continuously) differentiable functions in some of their variables if τ and f have these properties, respectively. $Dom(\tilde{\tau})$ and $Dom(\tilde{f})$ are open.
(ii) Let the function x be a solution of the initial value problem

$$x'(t) = f\left(t, x_t, x\left(t - \tau\left(t, x_t\right)\right)\right), \; x_\sigma = \phi. \tag{2.3}$$

If $\tilde{\phi} \in \mathcal{C}\left([-\tilde{r}, 0], Y\right)$ and $\tilde{\phi}_{|[-r,0]} = \phi$ then the function

$$\tilde{x}(t) = \begin{cases} \tilde{\phi}(t - \sigma), & \text{if } t \in [\sigma - \tilde{r}, \sigma - r] \\ x(t), & \text{if } t \geq \sigma - r \text{ and } t \in Dom(x) \end{cases}$$

is a solution of the initial value problem

$$\tilde{x}'(t) = \tilde{f}\left(t, \tilde{x}_t, \tilde{x}\left(t - \tilde{\tau}\left(t, \tilde{x}_t\right)\right)\right), \; \tilde{x}_\sigma = \tilde{\phi}, \tag{2.4}$$

where $\tilde{x}_t : [-\tilde{r}, 0] \to Y, \; s \longmapsto \tilde{x}(t + s)$.
(iii) If the function $\tilde{x}$ is a solution of the initial value problem (2.4), then $x = \tilde{x}_{|Dom(\tilde{x}) \cap [\sigma - r, \infty[}$ is a solution of the initial value problem (2.3), where $\phi := \tilde{\phi}_{|[-r,0]}$, if and only if for each point $t \geq \sigma$, $t \in Dom(x)$ the inequality $\tau(t, x_t) \leq t - \sigma + r$ holds.

Proof. (i) These statements follow immediately from the fact that the function

$$\Psi : \mathcal{C}\left([-\tilde{r}, 0], Y\right) \to \mathcal{C}\left([-r, 0], Y\right),\ \tilde{\psi} \longmapsto \tilde{\psi}_{|[-r,0]}$$

is linear and continuous, and because $\tilde{\tau} = \tau \circ (id_{\mathbb{R}}, \Psi)$ and $\tilde{f} = f \circ (id_{\mathbb{R}}, \Psi, id_Y)$.

(ii) By definition $\tilde{x}_\sigma = \tilde{\phi}$. If the function x is a solution of the initial value problem (2.3) then for every point $t \in Dom\,(x)$, $t \geq \sigma$, the equation $t - \tilde{\tau}(t, \tilde{x}_t) = t - \tau(t, x_t)$ holds, which implies immediately that

$$\tilde{x}'(t) = x'(t) = f\left(t, x_t, x\left(t - \tau\left(t, x_t\right)\right)\right) = \tilde{f}\left(t, \tilde{x}_t, \tilde{x}\left(t - \tilde{\tau}\left(t, \tilde{x}_t\right)\right)\right),$$

whenever $t \geq \sigma$ and $t \in Dom\,(x)$.

(iii) Suppose that $\tilde{x}$ is a solution of the initial value problem (2.4).
If $t - \tau(t, x_t) < \sigma - r$ then x is not defined at the point $t - \tau(t, x_t)$. Hence our supposition is necessary.

If $t \geq \sigma$ and $t \in Dom\,(\tilde{x})$, moreover, if $\sigma - r \leq t - \tau(t, x_t)$, then $\tilde{x}(t - \tilde{\tau}(t, \tilde{x}_t)) = x(t - \tau(t, x_t))$, by the definition of x. The equalities

$$\begin{aligned} x'(t) &= \tilde{x}'(t) = \tilde{f}\left(t, \tilde{x}_t, \tilde{x}\left(t - \tilde{\tau}\left(t, \tilde{x}_t\right)\right)\right) \\ &= f\left(t, (\tilde{x}_t)_{|[-r,0]}, x\left(t - \tau\left(t, (\tilde{x}_t)_{|[-r,0]}\right)\right)\right) = f\left(t, x_t, x\left(t - \tau\left(t, x_t\right)\right)\right) \end{aligned}$$

hold evidently. □

Corollary 2.1 *Let σ be fixed. Suppose that $\tau(t, \psi) \leq t - \sigma + r$, whenever $(t, \psi) \in Dom\,(\tau)$ and $\sigma \leq t$. Let $b \in\ ]\sigma, \infty[$ and denote $\tilde{r} := b - \sigma + r$, moreover, let $\tilde{\phi} \in \mathcal{C}([-\tilde{r}, 0], Y)$ and $\phi = \tilde{\phi}_{|[-r,0]}$.*
The function x is a solution of the initial value problem

$$x'(t) = f\left(t, x_t, x\left(t - \tau\left(t, x_t\right)\right)\right),\ x_\sigma = \phi \tag{2.5}$$

on the interval $[\sigma - r, b]$ if and only if the function

$$\tilde{x} : [\sigma - \tilde{r}, b] \to Y,\ \tilde{x}(t) = \begin{cases} x(t), & \text{if } t \in [\sigma - r, b] \\ \tilde{\phi}(t - \sigma), & \text{if } t \in [\sigma - \tilde{r}, \sigma - r] \end{cases}$$

is a solution of the initial value problem

$$\tilde{x}'(t) = h\left(t, \tilde{x}_t\right),\ \tilde{x}_\sigma = \tilde{\phi} \tag{2.6}$$

on the interval $[\sigma - \tilde{r}, b]$, where

$$h\left(t, \tilde{\psi}\right) := f\left(t, \tilde{\psi}_{|[-r,0]}, \tilde{\psi}\left(-\tau\left(t, \tilde{\psi}_{|[-r,0]}\right)\right)\right), \tag{2.7}$$

whenever $\left(t, \tilde{\psi}\right) \in \mathbb{R} \times \mathcal{C}([\sigma - \tilde{r}, b], Y)$, $\left(t, \tilde{\psi}_{|[-r,0]}\right) \in Dom\,(\tau)$ and $\left(t, \tilde{\psi}_{|[-r,0]}, \tilde{\psi}\left(-\tau\left(t, \tilde{\psi}_{|[-r,0]}\right)\right)\right) \in Dom\,(f)$.

3. Solutions, reaching to the boundary

Let Y be a Banach space, let $r \geq 0$, let $f : U \subset \mathbb{R}\times\mathcal{C}([-r,0],Y) \times Y \to Y$ and $\tau : W \subset \mathbb{R}\times\mathcal{C}([-r,0],Y) \to \mathbb{R}_0^+$ be continuous functions, defined on the open sets U and W, respectively.

Definition 3.1 *Let $\sigma < b \leq \infty$, let $W \subset \mathbb{R} \times \mathcal{C}([-r,0],Y)$ be a non empty set, let I be an interval and let $x : I \to Y$ be a function, $(\sigma, x_\sigma) \in W$. We say that x reaches to the boundary of $W = Dom(\tau)$ on the right if for each compact subset $C \subset W$ there exists an element $t_C \in I$ such that $(t, x_t) \in W \diagdown C$ for all $t \in I$, $t \geq t_C$.*

Definition 3.2 *Let x be a solution of the initial value problem*

$$x'(t) = f(t, x_t, x(t - \tau(t, x_t))), \ x_\sigma = \phi \tag{3.1}$$

on the interval I. We say that x reaches to the boundary of the domain of f on the right, if for each compact subset $C \subset Dom(f)$ there exists an element $t_C \in I$ such that $(t, x_t, x(t - \tau(t, x_t))) \in Dom(f) \diagdown C$ for all $t \in I$, $t \geq t_C$. We say that x reaches to the boundary on the right if either x reaches to the boundary of the domain of τ on the right, or x reaches to the boundary of the domain of f on the right.

Theorem 3.1 *Suppose that for every initial condition $x_\sigma = \phi$ (3.1) has a solution. Let x be a solution of the initial value problem (3.1) on the interval $[\sigma - r, b[$. Denote*

$$\Gamma_x := \{(t, x_t) \mid t \in [\sigma, b[\}$$

and

$$G_x := \{(t, x_t, x(t - \tau(t, x_t))) \mid t \in [\sigma, b[\}.$$

Denote $\mathcal{C}^\tau$ the set of all bounded and closed subsets of $Dom(\tau)$ and $\mathcal{C}^f$ the set of all bounded and closed subsets of $Dom(f)$. If Γ_x (respectively G_x) is relatively compact then the following statements are equivalent:
(i) x reaches to the boundary of $Dom(\tau)$ (respectively of $Dom(f)$) on the right;
(i') for every $C \in \mathcal{C}^\tau$ (respectively $C \in \mathcal{C}^f$) there exists a number $t_C \in [\sigma, b[$ such that

$$t_C < t < b \Longrightarrow (t, x_t) \notin C; \text{ (respectively } (t, x_t, x(t - \tau(t, x_t))) \notin C);$$

(ii) Γ_x (respectively G_x) is not contained in any compact subset of $Dom(\tau)$ (respectively of $Dom(f)$)
(ii') Γ_x (respectively G_x) is not contained in any $C \in \mathcal{C}^\tau$ (respectively $C \in \mathcal{C}^f$);
(iii) $\lim_{t\to b^-} x_t$ exists and $(b, \lim_{t\to b^-} x_t) \in \partial(Dom(\tau))$ (respectively $(b, \lim_{t\to b^-} x_t, \lim_{t\to b^-} x(t - \tau(t, x_t))) \in \partial(Dom(f))$), where ∂H denotes the boundary of the set H;
(iv) the distance between the set Γ_x (respectively G_x) and the complement of $Dom(\tau)$ (respectively the complement of $Dom(f)$) is equal to zero.

Proof. The function x is continuous, and Γ_x is its trajectory. The function

$$g := (\mathcal{R}(x), x \circ (id_{\mathbb{R}} - \tau \circ (id_{\mathbb{R}}, \mathcal{R}(x))))$$

is also continuous, where $\mathcal{R}(x)(t) := x_t$. G_x is the graph of g. The statement follows immediately from Theorem 9 of [17]. □

The following example shows that there exists a differential equation of the type (1.4) with a noncontinuable solution x, not reaching to the boundary on the right and having a compact interval as its domain.

Example 3.2 *Let $r \geq 0$, let Y be Banach space, let $g : \mathbb{R} \times \mathcal{C}([-r, 0], Y) \to \mathbb{R}_0^+$ be an arbitrary nonnegative and continuous function,*

$$\tau : [0, \infty[\times \mathcal{C}([-r, 0], Y) \to \mathbb{R}_0^+, \ (t, \psi) \longmapsto 2t + \|\psi\| g(t, \psi)$$

and

$$f := pr_3 : \mathbb{R} \times \mathcal{C}([-r, 0], Y) \times \mathbb{R} \to Y, \ (t, \psi, y) \longmapsto y.$$

Then a noncontinuable solution of the initial value problem

$$\begin{aligned} x'(t) &= f(t, x_t, x(t - \tau(t, x_t))) = x(-t - \|x_t\| g(t, x_t)), \ t \geq 0 \\ x(t) &= 0 \in Y, \ t \in [-r, 0] \end{aligned} \tag{3.2}$$

is equal to $0_{|[-r,r]}$, and it does not reach to the boundary on the right. For instance, if $g = 0$ then the initial value problem (3.2) has the following very simple form:

$$x'(t) = x(-t), \ t \geq 0, \ x(t) = 0 \in Y, \ t \in [-r, 0].$$

Lemma 3.1 *Let $b < \infty$, let $x : [\sigma - r, b[\to Y$ be a continuous function, let $x(t) = \phi(t - \sigma)$ if $t \in [\sigma - r, \sigma]$, and let $x'(t) = f(t, x_t, x(t - \tau(t, x_t)))$ if $\sigma < t < b$. Suppose that the trajectory of $x_{|[\sigma,b[}$ is relatively compact, moreover*

$$\left(\sigma, \lim_{t\to\sigma^+} x_t\right) \in Dom(\tau), \ \left(b, \lim_{t\to b^-} x_t\right) \in Dom(\tau)$$

and

$$\left(\sigma, \lim_{t\to\sigma^+} x_t, x\left(\sigma - \tau\left(\sigma, \lim_{t\to\sigma^+} x_t\right)\right)\right) \in Dom(f),$$

$$\left(b, \lim_{t\to b^-} x_t, x\left(b - \tau\left(b, \lim_{t\to b^-} x_t\right)\right)\right) \in Dom(f).$$

Let $x(b) := \lim_{t\to b^-} x(t)$ by definition. The function x is differentiable at σ from the right, it is differentiable at b from the left, moreover,

$$x'(\sigma) = f(\sigma, \phi, x(\sigma - \tau(\sigma, \phi))),$$

where $x'(\sigma)$ denotes the right-hand derivative of x at σ, and

$$x'(b) = f(b, x_b, x(b - \tau(b, x_b))),$$

where $x'(b)$ denotes the left hand derivative of x at b.

Proof. The functions f and τ are continuous, hence

$$\lim_{t\to\sigma^+} f\left(t, x_t, x\left(t-\tau\left(t, x_t\right)\right)\right) = f\left(\sigma, \phi, x\left(\sigma-\tau\left(\sigma, \phi\right)\right)\right).$$

Denote $A := \lim_{t\to\sigma^+} x'(t) = f\left(\sigma, \phi, x\left(\sigma-\tau\left(\sigma, \phi\right)\right)\right)$. We prove that $x'(\sigma)$ exists and is equal to A. The function

$$\Delta : [\sigma, b] \to Y,\ s \longmapsto x(s) - sA$$

is continuous, moreover it is differentiable on $]\sigma, b[$. For every positive number ε there is a positive number δ such that if $|s-\sigma| \le \delta$, $s > \sigma$, then $\|\Delta'(s)\| = \|x'(s) - A\| \le \varepsilon$. By the Mean Value Theorem if $s, t \in]\sigma, \delta]$ then $\|\Delta(s) - \Delta(t) - (s-t)A\| \le \varepsilon|s-t|$, therefore $\|x(t) - x(\sigma) - (t-\sigma)A\| \le \varepsilon|t-\sigma|$, because x is continuous on $[\sigma, b]$. It means that

$$x'(\sigma) = f\left(\sigma, \phi, x\left(\sigma-\tau\left(\sigma, \phi\right)\right)\right).$$

The differentiability of x at b and the equality

$$x'(b) = f\left(b, x_b, x\left(b-\tau\left(b, x_b\right)\right)\right)$$

can be proved in a similar way. □

Proposition 3.1 *Let $r > 0$, let x be a noncontinuable solution of the equation*

$$x'(t) = f\left(t, x_t, x\left(t-\tau\left(t, x_t\right)\right)\right).$$

Either x reaches to the boundary on the right, or x is defined on a compact interval $[\sigma - r, b]$.

Proof. If x is defined on an interval $[\sigma - r, \infty[$ then x reaches to the boundary on the right, obviously. So we can suppose without loss of generality that its domain is bounded.

Suppose that x does not reach to the boundary on the right. It is enough to show that Γ_x is relatively compact, because in this case by Lemma 3.1

$$x'(b) = f\left(b, \lim_{t\to b^-}\left(t, x_t\right), \lim_{t\to b^-}\left(t, x_t, x\left(t-\tau\left(t, x_t\right)\right)\right)\right),$$

therefore x is a solution on $[\sigma - r, b]$. By Corollary 1 of [17] Γ_x is relatively compact. □

Remark 3.1 *The solutions of the equation*

$$x'(t) = f\left(t, x(t), x\left(t-\tau\left(t, x(t)\right)\right)\right), \tag{3.3}$$

where $r = 0$, are obviously the same as the solutions of the equation

$$x'(t) = \tilde{f}(t, x_t, x(t - \tilde{\tau}(t, x_t))), \tag{3.4}$$

where $r > 0$ is arbitrary fixed and

$$\tilde{f}(t, \psi, y) := f(t, \psi(0), y), \ \psi \in \mathcal{C}([-r, 0], Y), \ (t, \psi(0), y) \in Dom(f),$$
$$\tilde{\tau}(t, \psi) := \tau(t, \psi(0)), \quad \psi \in \mathcal{C}([-r, 0], Y), \ (t, \psi(0)) \in Dom(\tau).$$

If x is considered as the solution of (3.4) then its trajectory $\Gamma_x := \{(t, x_t) \mid t \in Dom(x)\}$ can be closed and bounded. In this case by Corollary 1 [17] x reaches to the boundary of every set on the right. However, u can be a continuously differentiable function on an interval $[a - r, b[$ such that $\Gamma_u := \{(t, u_t) \mid t \in [a, b[\}$ is closed and bounded, but the graph of u is not closed. For instance, if $r = 1$ then the trajectory $\Gamma_u = \{(t, u_t) \mid t \in [-1, 0[\}$ of the function

$$u : [-2, 0[\to [-1, 1], \ t \longmapsto \sin\left(\frac{1}{t}\right)$$

is closed and bounded, but graph (u) is not closed. It means that the equations (3.3) and (3.4) may be not equivalent from the point of view that the noncontinuable solutions reach to the boundary on the right or not. That is why the following question arises:

Problem 3.3 *Does a differential equation of type (3.3) exist such that it has a noncontinuable solution x, not reaching to the boundary on the right and not having a compact interval as its domain?*

Lemma 3.2 *Let $Y = \mathbb{R}^n$, let x be a solution of the equation*

$$x'(t) = f(t, x(t), x(t - \tau(t, x(t))))$$

on the interval $[\sigma, b[$, $b < \infty$, let $t_n \in [\sigma, b[$ be a sequence tending to b such that $\lim_{n\to\infty} x(t_n) = v$, $(b, v) \in Dom(\tau)$ and

$$(t_n, x(t_n), x(t_n - \tau(t_n, x(t_n)))) \to (b, v, y) \in Dom(f).$$

If there is a positive number c such that for every natural number n $0 < c \leq \tau(t_n, x(t_n))$ holds then x is Lipschitzian on $[\sigma, b[$ and its graph is relatively compact.

Proof. It is enough to show that x is Lipschitzian. Indeed, in this case for every sequence $t_n \to b^-$ the sequence $x(t_n)$ is a Cauchy sequence, i.e. it is convergent. It means that the graph of x is relatively compact.

The positive number δ can be chosen so that

$$B := \{(t, z) \mid |b - t| \leq 2\delta, \ \|v - z\| \leq 2\delta\} \subset Dom(\tau)$$

and τ is uniformly continuous on B. Denote m the modulus of the uniform continuity of τ on B. We can suppose without loss of generality that the function f is bounded on $V := B \times \{w \in Y \mid \|y - w\| \leq 2\delta\}$, i.e. there is a number K such that

$$\begin{aligned} |b-t| &\leq 2\delta,\ \|v-z\| \leq 2\delta,\ \|y-w\| \leq 2\delta \\ &\Longrightarrow (t,z,w) \in Dom\,(f) \text{ and } \|f\,(t,z,w)\| \leq K. \end{aligned} \tag{3.5}$$

The function $x_{|[\sigma,b-c]}$ is Lipschitzian because f is bounded on its compact graph; let us denote by L its Lipschitz constant.

Let the natural number n be chosen so that $b - \delta < t_n,\ ,\ \|x\,(t_n) - v\| < \delta$, $\|x\,(t_n - \tau\,(t_n, x\,(t_n))) - y\| < \delta$ and $L\,(b - t_n + m\,(b - t_n + K\,(b - t_n))) < \delta$. Denote

$$\rho := \sup\{\beta \mid t \in [t_n,\beta] \Longrightarrow \|x\,(t) - v\| \leq 2\delta \textit{ and } \|x\,(t - \tau\,(t, x\,(t))) - y\| \leq 2\delta\}.$$

If $\rho = b$ then x' is bounded, which implies that x is Lipschitzian.

Suppose that $\rho < b$. Then

$$\rho = \max\{\beta \mid t \in [t_n,\beta] \Longrightarrow \|x\,(t) - v\| \leq 2\delta \text{ and } \|x\,(t - \tau\,(t, x\,(t))) - y\| \leq 2\delta\}$$

and

$$\text{either } \|x\,(\rho) - v\| = 2\delta \text{ or } \|x\,(\rho - \tau\,(\rho, x\,(\rho))) - y\| = 2\delta. \tag{3.6}$$

On the other hand, it follows from(3.5) that

$$\|x\,(\rho) - x\,(t_n)\| \leq K\,(\rho - t_n) \leq K\,(b - t_n)$$

and

$$\begin{aligned} &\|x\,(\rho - \tau\,(\rho, x\,(\rho))) - x\,(t_n - \tau\,(t_n, x\,(t_n)))\| \\ &\quad\leq L\,(\rho - t_n + |\tau\,(\rho, x\,(\rho)) - \tau\,(t_n, x\,(t_n))|) \\ &\quad\leq L\,(\rho - t_n + m\,(\rho - t_n + \|x\,(\rho) - x\,(t_n)\|)) \\ &\quad\leq L\,(b - t_n + m\,(b - t_n + K\,(b - t_n))). \end{aligned}$$

It implies that

$$\begin{aligned} &\|x\,(\rho - \tau\,(\rho, x\,(\rho))) - y\| \\ &\quad\leq \|x\,(\rho - \tau\,(\rho, x\,(\rho))) - x\,(t_n - \tau\,(t_n, x\,(t_n)))\| - \|x\,(t_n - \tau\,(t_n, x\,(t_n))) - y\| \\ &\quad\leq L\,(b - t_n + m\,(b - t_n + K\,(b - t_n))) + \delta < 2\delta, \end{aligned}$$

and

$$\|x\,(\rho) - v\| \leq \|x\,(\rho) - x\,(t_n)\| + \|x\,(t_n) - v\| \leq K\,(b - t_n) + \delta < 2\delta.$$

This contradicts (3.6). Therefore $\rho = b$, which completes the proof. □

Proposition 3.2 *Let $Y=\mathbb{R}^n$ and let x be a noncontinuable solution of the equation*

$$x'(t)=f(t,x(t),x(t-\tau(t,x(t))))$$

on the interval $[\sigma-r,b)$ where $[\sigma-r,b)$ denotes either the interval $[\sigma-r,b[$ or the compact interval $[\sigma-r,b]$. If $\tau(b,v)>0$, whenever $(b,v)\in Dom(\tau)$ then one of the following three statements holds:
(1) *x reaches to the boundary on the right;*
(2) *the distance between the graph of x and the complement of $Dom(\tau)$ is equal to zero;*
(3) *$b<\infty$ and x is defined on a compact interval $[\sigma-r,b]$.*

Proof. If x is defined on an interval $[\sigma-r,\infty[$ then x reaches to the boundary on the right, obviously. So we can suppose without loss of generality that $b<\infty$.

Suppose that x does not reach to the boundary on the right. It is enough to show that Γ_x is relatively compact, because in this case by Lemma 3.1

$$x'(b)=f\left(b,\lim_{t\to b^-}(t,x(t)),\lim_{t\to b^-}(t,x(t),x(t-\tau(t,x(t))))\right),$$

therefore x is a solution on $[\sigma-r,b]$.

Suppose that (1) and (2) do not hold. Let $C\subset Dom(f)$ be a compact set and $t_n\in[\sigma,b[$ be a sequence tending to b such that for each natural number n $(t_n,x(t_n))\in Dom(\tau)$ and $(t_n,x(t_n),x(t_n-\tau(t_n,x(t_n))))\in C$. We can suppose without loss of generality that

$$(t_n,x(t_n),x(t_n-\tau(t_n,x(t_n))))\to(b,v,y)\in C\subset Dom(f).$$

By supposition $(b,v)\notin\partial(Dom(\tau))$ and $\tau(b,v)>0$, therefore by Lemma 3.2 Γ_x is relatively compact. □

Theorem 3.4 *Let Y be Banach space, let σ be fixed and let x be a noncontinuable solution of the initial value problem*

$$\begin{aligned} x'(t) &= f(t,x(t),x(t-\tau(t,x(t)))),\ if\ t\in[\sigma,b[, \\ x(t) &= \phi(t-\sigma),\ if\ t\in[\sigma-r,\sigma]. \end{aligned} \tag{3.7}$$

If $\tau(t,v)\le t+r-\sigma$, whenever $(t,v)\in Dom(\tau)$ and $t\ge\sigma$, moreover graph (x) is relatively compact, then x reaches to the boundary on the right.

Proof. By supposition $b<\infty$. Let us define

$$\tilde{\tau}:\{(t,\psi)\in\mathbb{R}\times\mathcal{C}([-r,0],Y)\mid(t,\psi(0))\in Dom(\tau)\}\to\mathbb{R},\ (t,\psi)\longmapsto\tau(t,\psi(0)),$$

$$\begin{aligned} \tilde{f} &: \{(t,\psi,y)\in\mathbb{R}\times\mathcal{C}([-r,0],Y)\times Y\mid(t,\psi(0))\in Dom(\tau)\}, \\ (t,\psi,y) &\longmapsto f(t,\psi(0),y), \end{aligned}$$

It is easily seen that

$$\begin{aligned} x'(t) &= f(t, x(t), x(t-\tau(t,x(t)))),\ \text{if } t \in [\sigma, b[\,, \\ x(t) &= \phi(t-\sigma),\ \text{if } t \in [\sigma - r, \sigma] \end{aligned}$$

if and only if

$$\begin{aligned} \tilde{x}'(t) &= \tilde{f}(t, \tilde{x}_t, \tilde{x}(t-\tilde{\tau}(t,\tilde{x}_t))),\ \text{if } t \in [\sigma, b[\,, \\ \tilde{x}(t) &= \phi(t-\sigma),\ \text{if } t \in [\sigma - r, \sigma]\,, \end{aligned}$$

where

$$\tilde{x}(t) = x(t)\ \ if\ \ t \in [\sigma, b[\,, \text{ and } \tilde{x}_t : [-\tilde{r}, 0] \to Y,\ s \longmapsto \tilde{x}(t+s)\,.$$

By Lemma 3 [17] $\Gamma_{\tilde{x}}$ is relatively compact if and only if $graph(\tilde{x})$ is relatively compact. Denote $\psi_0 := \lim_{t\to b^-} \tilde{x}_t$. It follows from the definition of $\tilde{\tau}$ that $(b, \psi_0(0)) \in \partial(Dom(\tau))$ if and only if $(b, \psi_0) \in \partial(Dom(\tilde{\tau}))$. By Theorem 3.1 $(b, \psi_0) \in \partial(Dom(\tilde{\tau}))$, which proves our statement. □

Example 3.5 *Let $U \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ be open set, $f : U \to \mathbb{R}^n$ be continuous function. Let us consider the so called pantograph equation*

$$\begin{aligned} x'(t) &= f(t, x(t), x(\alpha t)),\ \sigma \le t, \\ x(t) &= \phi(t-\sigma),\ \alpha\sigma \le t < \sigma \end{aligned}$$

where $0 < \alpha < 1$, $\sigma \ge 0$ and $\phi \in \mathcal{C}([(\alpha-1)\sigma, 0], Y)$ are fixed. This initial value problem also can be written in the form

$$x'(t) = f(t, x(t), x(t-\tau(t,x(t)))),\ x_\sigma = \phi, \tag{3.8}$$

where

$$\tau = (1-\alpha)\, pr_1 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R},\ (t, \psi) \longmapsto (1-\alpha)\, t.$$

If $b > \sigma$ and $t \in \left[\frac{b-\sigma}{2}, b\right[$ then $\tau(t, x(t)) \ge (1-\alpha)\frac{b-\sigma}{2} > 0$. By Proposition 3.2 if x is a noncontinuable solution then either x reaches to the boundary on the right or its graph is compact. By Theorem 3.4 x reaches to the boundary on the right.

4. The compactness of $\Phi^{[a,b]}(\sigma, \phi, \mu, \omega)$.

Definition 4.1 *Let I be a non-empty set and for every point $i \in I$ let D_i and R_i be metric spaces with distances d_{D_i} and d_{R_i}, respectively. The family of uniformly continuous functions $\{x_i : D_i \to R_i \mid i \in I\}$ is said to be uniformly equicontinuous if there is a function $m : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying the following properties: $\lim_0 m = 0$ and $m_i \le m$, where m_i denotes the modulus of the uniform continuity of x_i. It means that $d_{R_i}(x_i(t), x_i(s)) \le m(d_{D_i}(t,s))$, whenever $i \in I$ and $t, s \in D_i$.*

Invoke the following classical theorem we need:

Theorem 4.1 *(Theorem 3.6.1. [3]) Let U be a convex open subset of a Banach space E, let $f_n : U \to F$ (F is a Banach space) be a sequence of differentiable functions. If*
(i) there is a point $a \in U$ such that $f_n(a)$ has a limit,
(ii) the sequence $f_n' : U \to \mathcal{L}(E,F)$ uniformly converges to the function $g : U \to \mathcal{L}(E,F)$,
then for each point $t \in U$ the sequence $f_n(t)$ has a limit (denoted by $f(t)$); the sequence $\{f_n\}$ tends to f uniformly on each bounded subset of U. Finally, the limit function f is differentiable and $f' = g$.

The following lemma is essential in the proof of the main theorem of this section.

Lemma 4.1 *Consider the initial value problem (1.4), let Y be Banach space. Suppose that*

$$(\sigma_n, \phi_n, \mu_n) \in Dom(\tau), \ (\sigma_n, \phi_n, \phi_n(-\tau(\sigma_n, \phi_n, \mu_n)), \omega_n) \in Dom(f)$$

and $(\sigma_n, \phi_n, \mu_n, \omega_n) \to (\sigma, \phi, \mu, \omega)$, where

$$(\sigma, \phi, \mu) \in Dom(\tau), \ (\sigma, \phi, \phi(-\tau(\sigma, \phi, \mu)), \omega) \in Dom(f),$$

moreover, $\sigma < b < \infty$, and for every natural number n there is given a solution $x_n : [\sigma_n - r, b] \to Y$ of the initial value problem

$$\begin{aligned} x_n'(t) &= f(t, (x_n)_t, x_n(t - \tau(t, (x_n)_t, \mu_n)), \omega_n), \ t \in [\sigma_n, b], \\ x_n(t) &= \phi_n(t - \sigma_n), \ \text{if } t \in [\sigma_n - r, \sigma_n]. \end{aligned}$$

Suppose furthermore that every noncontinuable solution of the initial value problem

$$\begin{aligned} x'(t) &= f(t, x_t, x(t - \tau(t, x_t, \mu)), \omega), \ t \geq \sigma \\ x(t) &= \phi(t - \sigma), \ \text{if } t \in [\sigma - r, \sigma], \end{aligned} \tag{4.1}$$

is defined on the interval $[\sigma - r, b]$.
If the sequence $\left\{ (x_n)_{|[\sigma_n, b]} \mid n \in \mathbb{N} \right\}$ is equicontinuous and there is a function $u :]\sigma, b] \to Y$ such that $x_n(t) \to u(t)$, whenever $t \in]\sigma, b]$, then $\{x_n\}$ has a subsequence, converging uniformly on $[a, b]$ to the function

$$x : [\sigma - r, b] \to Y, \ \begin{cases} u(t), & \text{if } t \in]\sigma, b] \\ \phi(t - \sigma), & \text{if } t \in [\sigma - r, \sigma] \end{cases},$$

whenever $\sigma - r < a < b$.
Moreover, x is a solution of the initial value problem (4.1) on the interval $[\sigma - r, b]$.

Proof. As the sequence $\left\{ (x_n)_{|[\sigma_n, b]} \mid n \in \mathbb{N} \right\}$ is uniformly equicontinuous there is a function $m : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying the following properties: $\lim_0 m = 0$ and $m_n \leq m$, where m_n denotes the modulus of the uniform continuity of $(x_n)_{|[\sigma_n, b]}$. We can suppose that $m_\phi \leq m$ also holds, where m_ϕ denotes the modulus of the uniform continuity of ϕ.

If $n \in \mathbb{N}$ and $t, s \in [\sigma_n - r, b]$ then

$$\|x_n(t) - x_n(s)\| \leq 2\left(m(|t-s|) + \|\phi_n - \phi\|\right). \tag{4.2}$$

Indeed, if $t, s \in [\sigma_n - r, \sigma_n]$ then

$$\begin{aligned} \|\phi_n(t) - \phi_n(s)\| &\leq \|\phi_n(t) - \phi(t)\| + \|\phi(t) - \phi(s)\| + \|\phi(s) - \phi_n(s)\| \\ &\leq m(|t-s|) + 2\|\phi_n - \phi\| \end{aligned} \tag{4.3}$$

and, using (4.3) and the equality $\phi_n(0) = x_n(\sigma_n)$,

$$\|x_n(t) - x_n(s)\| \leq \begin{cases} \|\phi_n(t-\sigma_n) - \phi_n(s-\sigma_n)\| \leq m(|t-s|) + 2\|\phi_n - \phi\|, & \text{if } t, s \in [\sigma_n - r, \sigma_n] \\ \|\phi_n(t-\sigma_n) - \phi_n(0)\| + \|x_n(\sigma_n) - x_n(s)\| & \\ \qquad \leq 2m(|t-s|) + 2\|\phi_n - \phi\|, & \text{if } t \leq \sigma_n \leq s \\ m(|t-s|), & \text{if } \sigma_n \leq t, s \end{cases}.$$

We can suppose without loss of generality that $a \leq \sigma$. We prove our statements in four steps.

(1) The function x is continuous on $[a, b]$. Indeed, if $t \geq \sigma$ then

$$\begin{aligned} \|x(t) - \phi(0)\| &\leq \|x(t) - x_n(t)\| + \|x_n(t) - \phi_n(0)\| + \|\phi_n(0) - \phi(0)\| \leq \\ &\leq \|x(t) - x_n(t)\| + m(|t-\sigma|) + \|\phi_n - \phi\|, \end{aligned}$$

hence $\lim_\sigma x = \phi(0) = x(\sigma)$. If $t, s \in [a, b]$ then

$$\|x(t) - x(s)\| \leq \begin{cases} \|\phi(t-\sigma) - \phi(s-\sigma)\| \leq m(|t-s|), & \text{if } t, s \in [\sigma - r, \sigma]; \\ \|\phi(t-\sigma) - \phi(0)\| + \|\phi(0) - x(s)\| \leq & \\ \qquad \leq m(|t-s|) + \|\phi(0) - x(s)\|, & \text{if } t \leq \sigma \leq s; \\ \|x(t) - x_n(t)\| + \|x_n(t) - x_n(s)\| + \|x_n(s) - x(s)\| \leq & \\ \qquad \leq \|x(t) - x_n(t)\| + \|x_n(s) - x(s)\| + m(|t-s|), & \text{if } \sigma \leq t, s. \end{cases}$$

It implies that for every positive ε there is a number δ such that $\|x(t) - x(s)\| < \varepsilon$, whenever $|t - s| < \delta$.

(2) We show that $(x_n)_{|[a,b]}$ tends uniformly to $x_{|[a,b]}$. Suppose that it is not true, i.e.

$$\exists \varepsilon > 0\ \forall n \in \mathbb{N}\ \exists t_n \in [a, b] : \|x_n(t_n) - x(t_n)\| > \varepsilon.$$

We can suppose that $t_n \to t \in [a, b]$. Then

$$\begin{aligned} \varepsilon &< \|x_n(t_n) - x(t_n)\| \leq \|x_n(t_n) - x_n(t)\| + \|x_n(t) - x(t)\| + \|x(t) - x(t_n)\| \\ &\leq m(|t_n - t|) + \|x_n(t) - x(t)\| + m_x(|t_n - t|), \end{aligned}$$

where m_x denotes the modulus of the uniform continuity of $x_{|[a,b]}$. We have got a contradiction, because the function on the right-hand side of the last inequality tends to zero.

(3) We show that there is a number $\delta > \sigma$ such that x satisfies equation (4.1) on the interval $]\sigma, \delta[$.
Let $c := \sigma$, if $\sigma_n \le \sigma$ for each n, and let $c \in]\sigma, b[$ be fixed in other case. We can suppose that for every natural number n the function x_n is defined on $[c, b]$.

Let $\mathcal{R}^{[c,b]}$ stand for the function defined by the following formula:

$$\begin{aligned} \mathcal{R}^{[c,b]} &: \mathcal{C}([c-r, b], Y) \to \mathcal{C}([c,b], \mathcal{C}([-r,0], Y)), \; u \longmapsto \mathcal{R}^{[c,b]}(u) \\ \mathcal{R}^{[c,b]}(u) &: [c,b] \to Y, \; t \longmapsto u_t. \end{aligned}$$

It is easily seen that $\mathcal{R}^{[c,b]}$ is linear. It is also continuous and $\left\|\mathcal{R}^{[c,b]}\right\| \le 1$, because

$$\left\|\mathcal{R}^{[c,b]}(u)\right\| = \sup_{t\in[c,b]} \left\|\mathcal{R}^{[c,b]}(u)(t)\right\| = \sup_{t\in[c,b]} \sup_{s\in[-r,0]} \|u(t+s)\| \le \|u\|.$$

Therefore

$$\mathcal{R}^{[c,b]}\left((x_n)_{|[c-r,b]}\right) \to \mathcal{R}^{[c,b]}\left(x_{|[c-r,b]}\right). \tag{4.4}$$

If $a \le s_n \to s \le b$ then $x_n(s_n) \to x(s)$, because $(x_n)_{|[a,b]} \to x_{|[a,b]}$ uniformly and

$$\|x_n(s_n) - x(s)\| \le \|x_n(s_n) - x(s_n)\| + \|x(s_n) - x(s)\|. \tag{4.5}$$

$(\sigma, \phi, \mu) \in Dom(\tau)$ and $(\sigma, \phi, \phi(-\tau(\sigma, \phi, \mu)), \omega) \in Dom(f)$ imply that there is a number $\delta > \sigma$ such that if $t \in]\sigma, \delta[$ then

$$(t, x_t, \mu) \in Dom(\tau) \text{ and } (t, x_t, x(t - \tau(t, x_t, \mu)), \omega) \in Dom(f).$$

The set $H := [c,b] \times \left(\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}\right) \subset \mathbb{R}^2$ is compact and (using property (4.4)) the function $H \to Y$,

$$\begin{aligned} H\left(t, \frac{1}{n}\right) &= f\left(t, (x_n)_t, x_n(t - \tau(t, (x_n)_t, \mu_n)), \omega_n\right), \text{ if } n > 0 \\ H(t, 0) &= f(t, x_t, x(t - \tau(t, x_t, \mu)), \omega) \end{aligned}$$

is continuous. Therefore it follows from the equalities

$$\begin{aligned} x_n'(t) &= f\left(t, (x_n)_t, x_n(t - \tau(t, (x_n)_t, \mu_n)), \omega_n\right), \; t \in [\sigma_n, b[, \\ x_n(t) &= \phi_n(t - \sigma_n) \text{ if } t \in [\sigma_n - r, \sigma_n] \end{aligned}$$

that the sequence $x_n'(t)$ is uniformly convergent on the interval $[c, b]$ and tends to $f(t, x_t, x(t - \tau(t, x_t, \mu)), \omega)$. By Theorem 4.1 x is differentiable on $]\sigma, \delta[$ and if $t \in]\sigma, \delta[$ then

$$\begin{aligned} x'(t) &= \lim_{n\to\infty} f\left(t, (x_n)_t, x_n(t - \tau(t, (x_n)_t, \mu_n)), \omega_n\right) \\ &= f(t, x_t, x(t - \tau(t, x_t, \mu)), \omega). \end{aligned}$$

(4) Finally, we have to prove that if $t \in [\sigma, b]$ then

$$x'(t) = f(t, x_t, x(t - \tau(t, x_t, \mu)), \omega). \tag{4.6}$$

Denote β the supremum of the numbers $\delta \leq b$ such that the equation (4.6) is satisfied on the interval $]\sigma, \delta[$. By definition $x_\sigma = \phi$. By Lemma 3.1 the function x is continuously differentiable from the right at σ and

$$x'(\sigma) = f(\sigma, \phi, x(\sigma - \tau(\sigma, \phi, \mu)), \omega).$$

If $t \in [\sigma, \beta[$ then (4.6) holds, therefore there exists a function $\tilde{x} : [\sigma - r, b] \to Y$ such that $\tilde{x}_{|[\sigma - r, \beta[} = x_{|[\sigma - r, \beta[}$, $\tilde{x}_\sigma = \phi$, and for each element $t \in [\sigma, b]$ the equality

$$\tilde{x}'(t) = f(t, \tilde{x}_t, \tilde{x}(t - \tau(t, \tilde{x}_t, \mu)), \omega)$$

also holds. It implies that $(\beta, x_\beta, \mu) \in Dom(\tau)$ and $(\beta, x_\beta, x(\beta - \tau(\beta, x_\beta, \mu)), \omega) \in Dom(f)$, furthermore, $x(\beta) = \tilde{x}(\beta)$. If $\beta < b$ then there is a positive number ε such that if $\beta \leq t \leq \beta + \varepsilon$ then $(t, x_t, \mu) \in Dom(\tau)$ and $(t, x_t, x(t - \tau(t, x_t, \mu)), \omega) \in Dom(f)$. Repeating our argumentation in step (3), replacing σ by β, we get a contradiction with the definition of β. Therefore $\beta = b$. By Lemma 3.1

$$x'(b) = f(b, \phi, x(b - \tau(b, \phi, \mu)), \omega).$$

Consequently (4.6) holds whenever $t \in [\sigma, b]$. □

Lemma 4.2 *Let $[\sigma, b] \subset \mathbb{R}$ be a compact interval and Y be a Banach space, $r \geq 0$. Let $H \subset \mathcal{C}([\sigma - r, b], Y)$. Denote*

$$\Gamma := \{(t, x_t) \mid t \in [\sigma, b], \ x \in H\}.$$

Let $\tau : \Gamma \to \mathbb{R}$ be a continuous function such that $t - \tau(t, x_t) \in Dom(x)$, whenever $x \in H$ and $t \in [\sigma, b]$.
If the set $H \subset \mathcal{C}([\sigma - r, b], Y)$ is compact then the sets Γ and

$$G := \{(t, x_t, x(t - \tau(t, x_t))) \mid t \in [\sigma, b], \ x \in H\}$$

are compact.

Proof. We show that Γ and G are sequentially compact.

If $\left(t_n, (x_n)_{t_n}, x_n\left(t_n - \tau\left(t_n, (x_n)_{t_n}\right)\right)\right) \in G$ is a sequence then we can suppose that there is a function $x \in H$ such that $x_n \to x$. We can also suppose that $t_n \to t$. For every positive number ε there is a positive number δ such that if $u, v \in [\sigma - r, b]$ and $|u - v| \leq \delta$ then $\|x(u) - x(v)\| \leq \varepsilon$. Therefore

$$\begin{aligned}
\left\|(x_n)_{t_n} - x_t\right\| &= \sup_{s \in [-r,0]} \|x_n(t_n + s) - x(t + s)\| \\
&\leq \sup_{s \in [-r,0]} \{\|x_n(t_n + s) - x(t_n + s)\| + \|x(t_n + s) - x(t + s)\|\} \\
&\leq \|x_n - x\| + \varepsilon \leq 2\varepsilon,
\end{aligned}$$

if $|t_n - t| \le \delta$ and $\|x_n - x\| \le \varepsilon$. It implies that

$$\left(t_n, (x_n)_{t_n}\right) \to (t, x_t) \in \Gamma$$

and if $s_n \to s$ then $x_n(s_n) \to x(s)$. Consequently

$$\left(t_n, (x_n)_{t_n}, x_n\left(t_n - \tau\left(t_n, (x_n)_{t_n}\right)\right)\right) \to (t, x_t, x(t - \tau(t, x_t))) \in G.$$

□

Recall that considering the initial value problem

$$x'(t) = f(t, x_t, x(t - \tau(t, x_t, \mu)), \omega), \ x_\sigma = \varphi \tag{4.7}$$

$\Phi^{[a,b]}$ denotes the function defined at the point $(\sigma, \phi, \mu, \omega)$ if and only if every non-continuable solutions of the initial value problem (4.7) is defined on the interval $[a, b]$, and $\Phi^{[a,b]}(\sigma, \phi, \mu, \omega)$ denotes the set of the restriction of these solutions to the interval $[a, b]$.

Theorem 4.2 *Let $Y := \mathbb{R}^n$ and let $[a, b]$ be a compact interval.*
(i) Let $C \subset Dom\left(\Phi^{[a,b]}\right) \subset \mathbb{R} \times \mathcal{C}([-r, 0], \mathbb{R}^n) \times \mathcal{M} \times \Omega$ be a compact set.
The set

$$\Phi^{[a,b]}(C) = \left\{x_{|[a,b]} \mid (\sigma, \phi, \mu, \omega) \in C, \ x \in \Phi(\sigma, \phi, \mu, \omega)\right\}$$

is compact in $\mathcal{C}([a, b], \mathbb{R}^n)$.
(ii) If $(\sigma, \phi, \mu, \omega)$ is fixed then the sets

$$\Gamma := \left\{(t, x_t, \mu, \omega) \mid t \in [\sigma, b], \ x \in \Phi^{[\sigma-r,b]}((\sigma, \phi, \mu, \omega))\right\},$$

$$G := \left\{((t, x_t, x(t - \tau(t, x_t, \mu)), \omega)) \mid t \in [\sigma, b], \ x \in \Phi^{[\sigma-r,b]}((\sigma, \phi, \mu, \omega))\right\}$$

are compact.

Proof. We can suppose that $\Phi^{[a,b]}(C)$ is not empty.

(i) We prove that $\Phi^{[a,b]}(C)$ is sequentially compact. Let $\{x_n\} \subset \Phi^{[a,b]}(C)$ be a sequence. It can be supposed without loss of generality that $(\sigma_n, \phi_n, \mu_n, \omega_n) \to (\sigma, \phi, \mu, \omega)$.

The structure of the proof is the following:

(I) there exists a number $\alpha > 0$ such that x_n is Lipschitzian on $[\sigma_n, \sigma_n + 2\alpha]$ with a constant L, where α and L do not depend on n, if n is great enough;

(II) by Lemma 4.1 and by the Ascoli-Arselà theorem $\{x_n\}$ has a subsequence, tending uniformly to a solution of (4.7) on the interval $[a, \sigma + \alpha]$;

(III) repeating the argumentation of (I) and (II) we prove the statement indirectly.

(I) There exist positive numbers δ and L such that for each $n \in \mathbb{N}$,

$$|t - \sigma| + \|(x_n)_t - \phi\| + d(\mu_n, \mu) + d(\omega_n, \omega) \le 3\delta$$

implies

$$\left\| f\left(t, (x_n)_t, x_n\left(t - \tau\left(t, (x_n)_t\right), \mu_n\right), \omega_n\right)\right\| \leq L.$$

Denote m the modulus of the uniform continuity of ϕ and let n_0 be such that

$$|\sigma_n - \sigma| + 3\left\|\phi - \phi_n\right\| + d\left(\mu_n, \mu\right) + d\left(\omega_n, \omega\right) \leq \delta,$$

whenever $n \geq n_0$.

We show that if $2\alpha + m(2\alpha) + 2L\alpha \leq \delta$, $\sigma_n + 2\alpha \leq b$ and $n \geq n_0$ then x_n is Lipschitzian on the interval $[\sigma_n, \sigma_n + 2\alpha]$ with the constant L, where L and α do not depend on x_n.

Let the number $n \geq n_0$ and $x_n \in \Phi^{[a,b]}(C)$ be fixed. Since the function

$$\mathcal{R}(x_n) : [\sigma_n, b] \to \mathcal{C}\left([-r, 0], \mathbb{R}^n\right), \ t \longmapsto (x_n)_t$$

is continuous ([7], 2.2, Lemma 2.1, or [17] Lemma 2), and

$$\left\|x_n'(t)\right\| = \left\| f\left(t, (x_n)_t, x_n\left(t - \tau\left(t, (x_n)_t, \mu_n\right)\right), \omega_n\right)\right\|, \ t \in [\sigma_n, b],$$

there is a positive number γ such that x_n is Lipschitzian on the interval $[\sigma_n, \sigma_n + \gamma]$ with the constant L. Denote β the supremum of the numbers γ, satisfying this property, i.e.

$$\beta := \sup\left\{\gamma \in [0, b - \sigma_n] \mid t, s \in [\sigma_n, \sigma_n + \gamma] \Longrightarrow \left\|x_n(t) - x_n(s)\right\| \leq L|t - s|\right\}.$$

We show indirectly that $\beta > 2\alpha$.

Denote m_n the modulus of the uniform continuity of ϕ_n. It is easily seen that

$$m_n \leq 2\left\|\phi - \phi_n\right\| + m. \tag{4.8}$$

Indeed, if $t, s \in [-r, 0]$ then

$$\left\|\phi_n(t) - \phi_n(s)\right\| \leq \left\|\phi_n(t) - \phi(t)\right\| + \left\|\phi(t) - \phi(s)\right\| + \left\|\phi(s) - \phi_n(s)\right\|.$$

Using the equality $x_n(\sigma_n) = \phi_n(0)$, we get that for every element $s \in [-r, 0]$

$$\begin{aligned}
&\left\|(x_n)_{\sigma_n+\beta}(s) - \phi_n(s)\right\| \\
&\quad \leq \begin{cases} \left\|\phi_n(\beta + s) - \phi_n(s)\right\| \leq m_n(\beta), & \text{if } \beta + s \leq 0, \\ \left\|x_n(\sigma_n + \beta + s) - x_n(\sigma_n)\right\| + \left\|\phi_n(0) - \phi_n(s)\right\| & \\ \qquad \leq L\beta + m_n(|s|) & \text{if } 0 < \beta + s. \end{cases}
\end{aligned}$$

If $0 < \beta + s$ then $-s = |s| < \beta$ and $m_n(|s|) \leq m_n(\beta)$, because m_n is increasing. Therefore, using inequality (4.8), we get the estimation

$$\left\|(x_n)_{\sigma_n+\beta} - \phi\right\| \leq \left\|(x_n)_{\sigma_n+\beta} - \phi_n\right\| + \left\|\phi_n - \phi\right\| \leq 3\left\|\phi - \phi_n\right\| + m(\beta) + L\beta.$$

If $\beta \leq 2\alpha$ then $m(\beta) + L\beta \leq m(2\alpha) + 2L\alpha$, because m is increasing. It implies that

$$|\sigma_n + \beta - \sigma| + \left\|(x_n)_{\sigma_n+\beta} - \phi\right\| + d\left(\mu_n, \mu\right) + d\left(\omega_n, \omega\right) \leq 3\delta.$$

Consequently, there is a positive number ε such that x_n is Lipschitzian on the interval $[\beta, \beta+\varepsilon]$ with the constant L, which contradicts to the definition of β. Hence x_n is Lipschitzian on the interval $[\sigma_n, \sigma_n+2\alpha]$ with the constant L. Here α depends only on m and on L, if $n \geq n_0$.

(II) We can suppose without loss of generality that $\{\sigma_n\}$ is monotone, moreover that if $\sigma_n \leq \sigma$ for each σ_n then the fixed number a is equal to $\sigma - r$, if $\sigma_n > \sigma$ for each σ_n then $\sigma - r < a < \sigma$. Suppose that if $n \geq n_0$ then $|\sigma_n - \sigma| \leq \alpha$. In this case $\sigma_n + 2\alpha \geq \sigma + \alpha$. Let $c \in]\sigma, \sigma+\alpha]$ be fixed. By (I) we can suppose that the sequence $\left\{(x_n)_{|[c,\sigma+\alpha]} \mid n \in \mathbb{N}\right\}$ is equicontinuous. It is also uniformly bounded, because if $t \in [c, \sigma+\alpha]$, then

$$\|x_n(t)\| \leq \|x_n(t) - x_n(\sigma_n)\| + \|\phi_n(0)\| \leq L2\alpha + \|\phi_n\|.$$

By the Ascoli-Arselà theorem $\{x_n \mid n \in \mathbb{N}\}$ has a subsequence, converging uniformly on $[c, \sigma+\alpha]$. By Lemma 4.1 the sequence $\{x_n \mid n \in \mathbb{N}\}$ has a subsequence, converging uniformly to a solution x of (4.7) on the interval $[a, \sigma+\alpha]$, satisfying the initial condition $x_\sigma = \phi$.

(III) Denote β the supremum of the numbers γ such that the sequence $\left\{(x_n)_{|[a,\gamma]}\right\}$ has a subsequence, tending uniformly on $[a,\gamma] \subset [a,b]$ to a noncontinuable solution x of (4.7). By (II) $\beta > \sigma$ and we can suppose that $(x_n)_{|[a,\gamma]}$ tends uniformly to $x_{|[a,\gamma]}$, whenever $[a,\gamma] \subset [a,\beta[$.

We prove that $\beta = b$. Suppose that $\beta < b$. The set $\{(t, x_t, \mu, \omega) \mid t \in [\sigma, b]\}$ is compact, therefore there are numbers $\delta' > 0$, $n_0 \in \mathbb{N}$ and L' such that if $t \in [\sigma, b]$ then for each number $n \in \mathbb{N}$

$$\begin{aligned} |\beta - t| + \|(x_n)_t - x_t\| + d(\mu_n, \mu) + d(\omega_n, \omega) &\leq 3\delta' \Longrightarrow \\ \|f(t, (x_n)_t, x_n(t - \tau(t, (x_n)_t), \mu_n), \omega_n)\| &\leq L'. \end{aligned}$$

Denote m_x the modulus of the uniform continuity of $x_{|[\sigma-r,b]|}$. Then for every point $t \in [\sigma, b]$ the modulus of continuity of x_t is not greater then m_x. Let $2\alpha' \leq \delta'$ be a positive number such that $m_x(2\alpha') + 2L'\alpha' \leq \delta'$ and let $b' := \beta - \alpha'$. Then we can suppose that $\{x_n \mid n \in \mathbb{N}\}$ converges uniformly to the solution x on $[a, b']$. By (I) x_n is Lipschitzian on the interval $[b', b'+2\alpha'] \cap [b', b]$ with the constant L', if n is great enough, hence it is Lipschitzian on the interval $[a, b'+2\alpha'] \cap [a, b]$. By the Ascoli-Arselà theorem and by Lemma 4.1 we can suppose that $\{x_n \mid n \in \mathbb{N}\}$ converges uniformly to a solution $\tilde{x}$ on $[a, b'+2\alpha'] \cap [a, b]$. As $b' + 2\alpha' > \beta$, we have a contradiction. It means that $b = \beta$ and we get that there is a subsequence of $\{x_n\}$ which tends to a solution x on $[a, b]$.

(ii) By (i) the set

$$H := \Phi^{[a,b]}(C) = \left\{x_{|[a,b]} \mid (\sigma, \phi, \mu, \omega) \in C,\ x \in \Phi(\sigma, \phi, \mu, \omega)\right\}$$

is compact in $\mathcal{C}([a,b], \mathbb{R}^n)$. Lemma 4.2 proves the statement. □

5. The upper-semicontinuity of $\Phi^{[a,b]}$ and the openness of its domain.

Definition 5.1 *(For a similar definition, see [2], Chapter 1 Definition 1) Let X and Y be topological spaces and let $F : X \to Y$ be a multivalued function. F is said to be upper-semicontinuous at $x_0 \in X$ if for any open V containing $F(x_0)$ there exists a neighborhood W of x_0 such that $F(V) := \cup_{x \in V} F(x) \subset W$.*

Theorem 5.1 *Let $Y = \mathbb{R}^n$ and denote Φ the resolvent function of the initial value problem*

$$\begin{aligned} x'(t) &= f(t, x_t, x(t - \tau(t, x_t, \mu)), \omega), \ t \geq \sigma, \\ x(t) &= \phi(t - \sigma), \quad \text{if } t \in [\sigma - r, \sigma]. \end{aligned} \tag{5.1}$$

For every compact interval $[a, b]$ the function $\Phi^{[a,b]}$ (if it is defined) is upper-semicontinuous.

Proof. Suppose that every noncontinuable solution of the initial value problem (5.1) is defined on the compact interval $[\sigma - r, b]$, i.e. $(\sigma, \phi, \mu, \omega) \in Dom\left(\Phi^{[\sigma-r,b]}\right)$, and let $[a, b] \subset]\sigma - r, b]$. Denote $\Phi^{[a,b]}(\sigma, \phi, \mu, \omega)$ the set of the restrictions of the noncontinuable solutions of (5.1) to the interval $[a, b]$. We prove indirectly that $\Phi^{[a,b]}$ is upper-semicontinuous at $(\sigma, \phi, \mu, \omega)$. Suppose that it is not true, i.e. there exists a positive number ε and there is a sequence $\{(\sigma_n, \phi_n, \mu_n, \omega_n) \mid n \in \mathbb{N}\}$, converging to $(\sigma, \phi, \mu, \omega)$, such that every solution in $\Phi(\sigma_n, \phi_n, \mu_n, \omega_n)$ is defined on $[a, b]$, moreover there is a sequence

$$\left\{x_n \mid x_n \in \Phi^{[a,b]}(\sigma_n, \phi_n, \mu_n, \omega_n), \ n \in \mathbb{N}\right\}$$

such that for every element $x \in \Phi^{[a,b]}(\sigma, \phi, \mu, \omega)$ the inequality $\|x_n - x\| \geq \varepsilon$ holds. By Theorem 4.2 $\Phi^{[a,b]}(\{(\sigma_n, \phi_n, \mu_n, \omega_n), \ n \in \mathbb{N}\} \cup (\sigma, \phi, \mu, \omega))$ is compact, therefore we can suppose without loss of generality that there is an element x of $\Phi^{[a,b]}(\sigma, \phi, \mu, \omega)$ such that $x_n \to x$ uniformly, which is a contradiction. □

By Theorem 2.10.4 of [5] if Y is finite dimensional, $r = 0$, $\tau = 0$, f is continuous and every noncontinuable solution is unique then the characteristic function is continuous and its domain is open. Theorem 5.2 generalizes the second part of this statement for functional differential equations with state-dependent delay in infinite dimensional case, proving the openness of the domain of Φ under the supposition that Φ is upper-semicontinuous and $\Phi^{[\sigma-r,b]}(\sigma, \phi, \mu, \omega)$ is compact, whenever $(\sigma, \phi, \mu, \omega) \in Dom\left(\Phi^{[\sigma-r,b]}\right)$.

Theorem 5.2 *Let Y be a Banach space. Suppose that each noncontinuable solution x of the parametric initial value problem (1.4) has a solution, whenever $(\sigma, \phi, \mu) \in Dom(\tau)$ and $(\sigma, \phi, \phi(-\tau(\sigma, \phi, \mu)), \omega) \in Dom(f)$, moreover $(\sigma_0, \phi_0, \mu_0) \in Dom(\tau)$ and $(\sigma_0, \phi_0, \phi_0(-\tau(\sigma_0, \phi_0, \mu_0)), \omega_0) \in Dom(f)$.*

If the set $\Phi^{[\sigma_0-r,b]}(\sigma_0,\phi_0,\mu_0,\omega_0)\subset\mathcal{C}([\sigma_0-r,b],Y)$ *is compact and for each compact subinterval* $[a,\beta]\subset\,]\sigma_0-r,b]$ *the multivalued function* $\Phi^{[a,\beta]}$ *is upper-semicontinuous at the point* $(\sigma_0,\phi_0,\mu_0,\omega_0)$ *then* $(\sigma_0,\phi_0,\mu_0,\omega_0)$ *is an inner point of the domain of the function* $\Phi^{[a,b]}$*, whenever* $a\in\,]\sigma_0-r,b[$.

Proof. (i) By Lemma 4.2 it follows from the compactness of $\Phi^{[\sigma_0-r,b]}(\sigma_0,\phi_0,\mu_0,\omega_0)$ that the set $\Gamma:=\{(t,x_t,\mu_0,\omega_0)\mid t\in[\sigma_0,b],x\in\Phi(\sigma_0,\phi_0,\mu_0,\omega_0)\}$ is compact. By Lemma 2.1 $I.$ (iv) the set Γ has a neighborhood E and there is a positive number α such that

$$(\sigma,\phi,\mu,\omega)\in E,\ x\in\Phi(\sigma,\phi,\mu,\omega)\Longrightarrow[\sigma,\sigma+\alpha]\subset Dom\,(x)\,. \tag{5.2}$$

Let δ be a positive number such that if $|\sigma-\sigma_0|+\|\phi-\phi_0\|+d_{\mathcal{M}}(\mu,\mu_0)+d_{\Omega}(\omega,\omega_0)<\delta$ then $(\sigma,\phi,\mu,\omega)\in E$, where $d_{\mathcal{M}}$ and d_{Ω} denote the distances on $\mathcal{M}$ and Ω, respectively.

We prove our statement by induction.

If $|\sigma-\sigma_0|+\|\phi-\phi_0\|+d_{\mathcal{M}}(\mu,\mu_0)+d_{\Omega}(\omega,\omega_0)<\delta$ and $x\in\Phi(\sigma,\phi,\mu,\omega)$ then by (5.2) $[\sigma,\sigma+\alpha]\subset Dom\,(x)$, i.e. $\Phi^{[\sigma-r,\sigma+\alpha]}$ is defined at (σ,ϕ,μ,ω). Let $k_0\in N$ be such that $\sigma+\alpha k_0<b\le\sigma+(k_0+1)\,\alpha$. If $k_0=0$ then $\sigma+\alpha\ge b$ and $\Phi^{[\sigma,b]}$ is defined at (σ,ϕ,μ,ω). Suppose that $k_0\ge1$. Since $\Phi^{[\sigma,\sigma+\alpha]}$ is upper-semicontinuous at $(\sigma_0,\phi_0,\mu_0,\omega_0)$ the positive number δ_1 can be chosen so that if

$$|\sigma-\sigma_0|+\|\phi-\phi_0\|+d_{\mathcal{M}}(\mu,\mu_0)+d_{\Omega}(\omega,\omega_0)\le\delta_1$$

and

$$x\in\Phi(\sigma,\phi,\mu,\omega)\,,\ t\in[\sigma,\sigma+\alpha]$$

then $(t,x_t,\mu,\omega)\in E$. Using again property (5.2) it implies that (σ,ϕ,μ,ω) is included in the domain of $\Phi^{[\sigma-r,\sigma+2\alpha]}$, and so on. In k_0 steps we get that there is a positive number δ_{k_0} such that (σ,ϕ,μ,ω) is included in the domain of $\Phi^{[\sigma-r,b]}$, whenever

$$|\sigma-\sigma_0|+\|\phi-\phi_0\|+d_{\mathcal{M}}(\mu,\omega)+d_{\Omega}(\omega,\omega_0)\le\delta_{k_0}. \tag{5.3}$$

If $\sigma\le\sigma_0$ then it implies that $[\sigma_0-r,b]$ is included in the domain of x, whenever $x\in\Phi(\sigma,\phi,\mu,\omega)$, i.e. $(\sigma,\phi,\mu,\omega)\in Dom\left(\Phi^{[\sigma_0-r,b]}\right)$. If $\sigma>\sigma_0$ and $\sigma-r<a$ then (σ,ϕ,μ,ω) is included in $Dom\left(\Phi^{[\sigma-r,b]}\right)\subset Dom\left(\Phi^{[a,b]}\right)$.

Therefore $(\sigma,\phi,\mu,\omega)\in Dom\left(\Phi^{[a,b]}\right)$, whenever (5.3) holds . ☐

Theorem 5.3 *Let* $Y=\mathbb{R}^n$*, and suppose*

$$(\sigma_0,\phi_0,\mu_0)\in Dom(\tau)\quad\text{and}\quad(\sigma_0,\phi_0,\phi_0(-\tau(\sigma_0,\phi_0,\mu_0)),\omega_0)\in Dom(f).$$

(i) *If* $\tau(t,\psi,\mu)\le r$*, whenever* $(t,\psi,\mu)\in Dom\,(\tau)$ *, then the point* $(\sigma_0,\phi_0,\mu_0,\omega_0)$ *is an inner point of the domain of the function* $\Phi^{[a,b]}$*, whenever* $a\in\,]\sigma_0-r,b[$.

(ii) *Let the parameter* σ_0 *be fixed. If for every point* $(t,\psi,\mu)\in Dom\,(\tau)$ *the inequality* $\tau(t,\psi,\mu)\le t-\sigma_0+r$ *holds whenever* $\sigma_0\le t$*, then* (ϕ_0,μ_0,ω_0) *is an inner point of the domain of the function* $\Phi^{[\sigma_0-r,b]}(\sigma_0,\cdot,\cdot,\cdot)$.

Proof. (i) By Lemma 2.1 (II) and Theorem 5.1 the suppositions of Theorem 5.2 hold. It proves our statement.

(ii) Let $\tilde{r} := b - \sigma_0 + r$. By Corollary 2.1 the function x is a solution of the initial value problem

$$x'(t) = f(t, x_t, x(t - \tau(t, x_t))), \ x_{\sigma_0} = \phi \tag{5.4}$$

on the interval $[\sigma_0 - r, b]$ if and only if the function

$$\tilde{x} : [\sigma_0 - \tilde{r}, b] \to \mathbb{R}^n, \ \tilde{x}(t) = \begin{cases} x(t), & \text{if } t \in [\sigma_0 - r, b] \\ \phi(-r), & \text{if } t \in [\sigma_0 - \tilde{r}, \sigma_0 - r] \end{cases}$$

is a solution of the equation

$$\tilde{x}'(t) = h(t, \tilde{x}_t) \tag{5.5}$$

on the interval $[\sigma_0 - \tilde{r}, b]$, satisfying the initial condition $\tilde{x}_{\sigma_0}(s) = \tilde{x}(\sigma_0 - s)$, $s \in [-\tilde{r}, 0]$, where h is continuous. By Theorem 5.2 $(\sigma_0, \tilde{x}_{\sigma_0}, \mu_0, \omega_0)$ is an inner point of $Dom\left(\tilde{\Phi}^{[a,b]}\right)$, where $\tilde{\Phi}$ denotes the resolvent function of the equation (5.5). It implies that $(\phi_0, \mu_0, \omega_0)$ is an inner point of $Dom\left(\Phi^{[a,b]}(\sigma_0, \cdot, \cdot, \cdot)\right)$, because by Corollary 2.1 the solutions of (5.4) and the solutions of (5.5) are the same. Since every solution $x \in \Phi(\sigma_0, \phi, \mu, \omega)$ is defined on $[\sigma_0 - r, \sigma_0]$, hence $(\phi_0, \mu_0, \omega_0)$ is an inner point of $Dom\left(\Phi^{[\sigma_0 - r, b]}(\sigma_0, \cdot, \cdot, \cdot)\right)$ □

If g is an upper-semicontinuous multivalued function between two topological spaces then it is obvious that the image of a compact set can be not compact and the image of a connected set can be not connected. By statements above the upper-semicontinuous function $\Phi^{[a,b]}$ has some good properties, similar to the properties of a continuous function: "it seems to be continuous" from some points of view. However, $\Phi^{[a,b]}$ can be not uniformly upper-semicontinuous on some compact subset, as the following lemma shows.

Lemma 5.1 *Let $U \subset \mathbb{R} \times \mathbb{R}^n$ be open set, $(\sigma_0, \phi_0) \in U$ and $f : U \to Y$ be continuous, and denote $\Phi(\sigma_0, \phi_0)$ the set of the noncontinuable solutions of the ordinary initial value problem*

$$x'(t) = f(t, x(t)), \ x(\sigma_0) = \phi_0. \tag{5.6}$$

Suppose that every element of $\Phi(\sigma_0, \phi_0)$ is defined on the interval $[\sigma_0, b]$ and $\Phi(\sigma_0, \phi_0)$ has more than one element.
There exists a metric space P, an open set $\tilde{U} \subset \mathbb{R} \times \mathbb{R}^n \times P$ and a continuous function $\tilde{f} : \tilde{U} \to Y$ such that every noncontinuable solution $\tilde{x} \in \tilde{\Phi}(\sigma_0, \phi_0, p_0)$ of the initial value problem

$$x'(t) = \tilde{f}(t, x(t), p_0), \ x(\sigma_0) = \phi_0 \tag{5.7}$$

is defined on $[\sigma_0, b]$, $\tilde{\Phi}^{[\sigma_0,b]}(\sigma_0, \phi_0, p_0) = \Phi^{[\sigma_0,b]}(\sigma_0, \phi_0)$ and for every compact neighborhood C of the point (σ_0, ϕ_0, p_0) the function $\tilde{\Phi}^{[\sigma_0,b]}$ is not uniformly upper-semicontinuous on C.

Proof. By Theorem 4.2 and Lemma 4.2 the sets $\Phi^{[\sigma_0,b]}(\sigma_0,\phi_0) \subset (\mathcal{C}[\sigma_0,b],Y)$ and $\Gamma^{[\sigma_0,b]} := \{(t,x(t)) \mid t \in [\sigma_0,b], x \in \Phi\} \subset U$ are compact. Hence $\Gamma^{[\sigma_0,b]}$ has a compact neighborhood $W \subset U$ such that f is bounded on W. By the Stone-Weierstrass theorem for every natural number n there exists a function f_n which is continuously differentiable on W and $\left\|(f-f_n)_{|W}\right\| \leq \frac{1}{n}$. Denote $P := \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \cup \{0\}$ the metric space with the distance inherited from $\mathbb{R}$ and let V be an open neighborhood of $\Gamma^{[\sigma_0,b]}$ such that $V \subset W$,

$$\tilde{U} := V \times P,\ \tilde{f} : \tilde{U} \to \mathbb{R}^n, \begin{cases} \left(t,y,\frac{1}{n}\right) \longmapsto f_n(t,y) \\ (t,y,0) \longmapsto f(t,y) \end{cases}$$

Then $\tilde{f}$ is continuous. Obviously, $\Phi^{[\sigma_0,b]}(\sigma_0,\phi_0) = \tilde{\Phi}^{[\sigma_0,b]}(\sigma_0,\phi_0,0)$, moreover, the set $\tilde{\Phi}^{[\sigma_0,b]}\left(\sigma,\phi,\frac{1}{n}\right)$ has one element, whenever $n \in \mathbb{N}$. By Theorem 5.1 $\tilde{\Phi}^{[\sigma_0,b]}$ is upper-semicontinuous. Let x and u be two different elements of $\Phi^{[\sigma_0,b]}(\sigma_0,\phi_0)$. Suppose that $\Phi^{[\sigma_0,b]}$ is uniformly upper-semicontinuous on a compact neighborhood of $(\sigma_0,\phi_0,0)$. Then for every positive number ε there exists a neighborhood E_ε of $(\sigma_0,\phi_0,0)$ such that x and u are included in the open ball around the element $\tilde{\Phi}^{[\sigma_0,b]}\left(\sigma,\phi,\frac{1}{n}\right)$, with the radius ε, whenever $\left(\sigma,\phi,\frac{1}{n}\right) \in E_\varepsilon$. We conclude to contradiction if $2\varepsilon < \|x-u\|$. □

Proposition 5.1 *Let $r > 0$, let $U \subset \mathbb{R} \times C([-r,0],\mathbb{R}^n)$ be open set. Denote $\Phi(\sigma,\phi)$ the set of the noncontinuable solutions of the initial value problem*

$$x'(t) = f(t,x_t,x(t-\tau(t,x_t))),\ x_\sigma = \phi.$$

Suppose that $b < \infty$, $\emptyset \neq C \subset Dom\left(\Phi^{[a,b[}\right)$ and C is compact.
If (b,ψ) is a boundary point of the set

$$G := \{(t,x_t) \mid t \in [a,b[, (\sigma,\phi) \in C, x \in \Phi(\sigma,\phi)\}$$

then there is an element $(\sigma,\phi) \in C$ and there is a solution $x \in \Phi(\sigma,\phi)$ such that $\lim_{t\to b^-} x_t$ exists and is equal to ψ, moreover $\lim_{t\to b^-} x(t)$ exists and is equal to $\psi(0)$.

Proof. Let $\{(t_n,\psi_n) \mid n \in \mathbb{N}\} \subset G$ be a sequence tending to (b,ψ), $x_n \in \Phi(\sigma_n,\phi_n)$ be such that $(x_n)_{t_n} = \psi_n$. We can suppose that $(\sigma_n,\phi_n) \to (\sigma_0,\phi_0)$ for some element $(\sigma_0,\phi_0) \in C$. If $[a,\beta] \subset [a,b[$ then by Theorem 4.2 the set $\cup_{(\sigma,\phi)\in C}\Phi^{[a,\beta]}(\sigma,\phi)$ is compact. Therefore, using Cantor's diagonal choice method, we can suppose without loss of generality that $\{x_n\}$ converges uniformly on every compact subinterval $[a,\beta]$ of $[a,b[$ to a solution x, and $(x_n)_{t_n}$ is convergent. It implies that $x_{t_n} \to \psi$. By Lemma 3 [17] $\lim_{t\to b^-} x_t = \psi$, moreover $\lim_{t\to b^-} x(t)$ exists and is equal to $\psi(0)$. □

6. Conclusion

We have obtained some new results for state-dependent functional differential equations without boundedness conditions on the delay function, supposing the continuity of the functions on the right-hand side, but not supposing that the solutions are unique. We have established sufficient conditions under which every noncontinuable solutions reach to the boundary on the right of the domain where the equation is defined. To the best of our knowledge this problem has not been studied yet for state-dependent delay equations. We have proved that the so called resolvent function $\Phi^{[a,b]}$ (see the definition in the introduction) is upper-semicontinuous, its domain is open and $\Phi^{[a,b]}(C)$ is compact, whenever $C \subset Dom\left(\Phi^{[a,b]}\right)$ is compact.

In the future we would like to give some applications of the results obtained in this paper. For instance it seems to be possible to generalize the well known Kneser-theorem for state-dependent delay equations. A classical comparison theorem ([5] 2.5.3) also can be generalized in a sharper form then the original one. These questions will be studied in a forthcoming paper.

References

[1] O. Arino and E. Sanchez, A saddle point theorem for functional state-dependent delay equations., Discrete Continuous Dynamical Systems, **12** (2005), 687-722.

[2] J. P. Aubin, A. Cellina, Differential Inclusions, Springer-Verlag, 1984.

[3] H. Cartan, Calcul différentiel, formes différentielles, Hermann Paris 1967.

[4] R. D. Driver, Existence theory for delay differential system, Contributions to Differential Equations, **1** (1961), 317-336.

[5] T. M. Flett, Differential Analysis, Cambridge University Press, 1980.

[6] C. Gori, V. Obukhovskii, M. Ragni, and P. Rubbioni, Existence and continuous dependence results for semilinear functional differential inclusions with infinite delay, Nonlinear Analysis, **51** (2002), 765-782.

[7] J. K. Hale and S. M. V. Lunel, Introduction to Functional Differential Equations, Springer-Verlag, New York, 1993.

[8] F. Hartung, On classes of functional differential equations with state-dependent delays, Ph.D. Dissertation, University Texas at Dallas, 1995.

[9] F. Hartung, On differentiability of solutions with respect to parameters in a class of functional differential equations, Funct. Differential Equations, **4** (1997), 65-79.

[10] F. Hartung, On existence, uniqueness and numerical approximation for neutral equations with state-dependent delays, Applied Numerical Mathematics, **24:2-3** (1997), 393-409.

[11] F. Hartung, T. Krisztin, H.-O. Walter, and J. Wu, Functional Differential Equations with State-Dependent Delays: Theory and Applications, Handbook of Differential Equations: Ordinary Differential Equations, volume 3, Edited by A. Cañada. P. Drábek and A. Fonda, Elsevier, North-Holland, 2006, 435-545.

[12] F. Hartung and J. Turi, On differentiability of solutions with respect to parameters in state-dependent delay equations, J. Diff. Eqns, **135** (1997), 192-237.

[13] T. Krisztin, Invariance and noninvariance of center manifolds of time-t-maps with respect to semiflow, SIAM J. Math. Analysis **36** (2004), 717-739.

[14] T. Krisztin, $\mathcal{C}^1$-smoothness of center manifolds for differential equations with state-dependent delays. Nonlinear Dynamics and Evolution Equations, Fields Institute Communications, **48** (2006), 213-226.

[15] P. Marín-Rubio, Attractors for parametric delay differential equations without uniqueness and their upper semicontinuous behaviour, Nonlinear Analysis, to appear.

[16] B. Slezák, On the maximal solutions of abstract differential equations, Studia Sci. Math. Hungar. **37:3-4** (2001), 471-478.

[17] B. Slezák, On the noncontinuable solutions of retarded functional differential equations, Functional Differential Equations, **13:3-4** (2006), 603-635.

[18] H.-O. Walther, The solution manifold and $\mathcal{C}^1$-smoothness of solution operators for differential equations with state dependent delay, J. Differential Equations, **195** (2003), 46-65.