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Stability Conditions for Differential-Difference Systems of Retarded- and Neutral-Type: The Single Delay Case

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Abstract. Sufficient conditions are given for exponential stability of the solution semigroup associated with certain systems of retarded and neutral differential-difference equations. The conditions also provide for an estimate of the exponential decay rate.

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1. Introduction

We consider systems of linear differential-difference equations of retarded ($C = 0$) - and neutral ($C \neq 0$) -type

$$\frac{d}{dt}[x(t) + Cx(t-r)] = Ax(t) + Bx(t-r), \tag{1.1}$$

where A, B, C are $n \times n$ matrices with complex entries, and $r \in \mathbb{R}$. In recent years there has been a considerable interest in obtaining conditions on the matrices A, B, and C which guarantee exponential stability of the solution semigroup associated with (1.1). With regard to this question, sufficient conditions which depend only on the

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matrices A, B and C are called delay independent, while those which also depend on the delay r are called delay dependent. Here we consider only delay independent conditions. Associated with (1.1) is the characteristic equation

$$\Delta(\lambda) = \det(A - \lambda I + Be^{-\lambda r} - \lambda Ce^{-\lambda r}) = 0. \tag{1.2}$$

It is known that exponential stability of (1.1)) is equivalent to the condition that

$$\sup \{\mathrm{Re}\, \lambda : \ \Delta(\lambda) = 0\} = \alpha < 0. \tag{1.3}$$

Roughly speaking, results in the literature on the question of sufficient conditions for exponential stability follow two distinct approaches. The first approach involves direct analyis of the characteristic equation $\Delta(\lambda) = 0$ to show that certain conditions (ideally, conditions which are easy to check) on the matrices A, B, and C, usually involving the matrix norm or matrix measure, are sufficient for $\alpha < 0$. One of the earliest results in this direction is [1] for the somewhat restrictive case when A, B, and C are real and symmetric. Other results in this direction are found in [3], [7], [9], [11], [12], [13], [14], [15], [17], and [18]. The second approach uses Liapunov-like or Razumikhin type theorems (see [10]), and this usually leads to a linear matrix inequality as a sufficient condition (see [8] and [16] for further discussion and references). We shall proceed instead by renorming the underlying state space in order to obtain a dissipative inequality for the infinitesimal generator of the solution semigroup, which will imply its exponential stability. Our work here is motivated by the fact that for scalar retarded and neutral equations the renormed state space provided the right setting for the construction of semidiscrete approximation schemes which are preserving exponential stability uniformly in the discretization parameter (see [5] and [6] for details). We note that the renorming idea is closely related to the second approach, since a new norm can be viewed as a quadratic Liapunov function in the original norm. The sufficient conditions we obtain directly lead to stability preserving semidiscrete approximation schemes which is not the case with those derived using the first approach. Furthermore, in some cases, our results give sharper stability results, at least in the $\|(\cdot)\|_2$ matrix norm, and in the $\mu_2(\cdot)$ matrix measure, than to those found in [11], [14], and [15] (see (2.17), (3.2), and (3.3) below). Note that the conditions (2.17), (3.2), and (3.3) are valid in any matrix norm, although from a practical perspective the matrix measure $\mu(\cdot)$ is only easy to check in the 1, 2, and ∞ norms. We stress that our most general new sufficient condition (3.18) below is an improvement over some other sufficient conditions, *and* it is easy to check - one need only verify that the eigenvalues of the symmetric matrix in (3.18) are all positive. An additional significant feature of our approach is that we also get an estimate of the rate of exponential decay, which is lacking in other results.

The paper is organized as follows. We first obtain a new sufficient condition for exponential stability of the retarded system (1.1) with $C = 0$ and show how this improves previous results. Then we obtain new sufficient conditions for exponential stability of the neutral system (1.1) with $C \neq 0$ and compare with other sufficient conditions. We include a result for the special case in which the matrix A is self-adjoint because this yields a sufficient condition which is a significant improvement compared to the case of general A.

2. Retarded Systems

In this section we consider (1.1) with $C = 0$, commonly referred to as a retarded delay-differential equation. In particular, consider

$$\frac{d}{dt}x(t) = Ax(t) + Bx(t-r), \tag{2.1}$$
$$x(0) = \eta_0,\ x(\theta) = \varphi_0(\theta) \text{ for } -r \leq \theta < 0,$$

where $r > 0$, $\eta_0 \in \mathbb{C}^n$, $\varphi_0 \in L_2(-r, 0; \mathbb{C}^n)$, and A and B are $n \times n$ matrices with complex entries.

In a standard fashion (2.1) can be reformulated as an abstract Cauchy problem on a Hilbert space, and it will be in the Hilbert space setting that we conduct the stability analysis. To proceed, define the Hilbert space $Z = \mathbb{C}^n \times L^2(-r, 0; \mathbb{C}^n)$ endowed with the norm

$$\|(\eta,\varphi)\|_Z^2 = \|\eta\|_n^2 + \int_{-r}^0 \|\varphi(\theta)\|_n^2 \, d\theta, \tag{2.2}$$

and compatible inner product

$$\langle(\eta,\varphi),(\xi,\psi)\rangle_Z = \xi^T\overline{\eta} + \int_{-r}^0 \psi(\theta)^T\overline{\varphi(\theta)}\, d\theta. \tag{2.3}$$

In (2.2), $\|\cdot\|_n$ represents the usual Euclidean norm on $\mathbb{C}^n$. Next define the linear operator $\mathcal{A} : \operatorname{dom}\mathcal{A} \subset Z \to Z$ on the domain

$$\operatorname{dom}\mathcal{A} = \{(\eta,\varphi) \in Z : \varphi \in H^1(-r,0;\mathbb{C}^n),\ \varphi(0) = \eta\}, \tag{2.4}$$

by

$$\mathcal{A}(\eta,\varphi) = (A\eta + B\varphi(-r), \varphi'). \tag{2.5}$$

It is well known that $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on Z, and that (2.1) can be reformulated as the Cauchy problem

$$\frac{d}{dt}z(t) = \mathcal{A}z(t) \tag{2.6}$$
$$z(0) = (\eta_0, \varphi_0). \tag{2.7}$$

Our approach will be to define a new norm $\|\cdot\|_e$ on Z, with compatible inner product $\langle\cdot,\cdot\rangle_e$, which is equivalent to the norm $\|\cdot\|_Z$ (two norms on Z are equivalent if there are positive constants c_1, c_2 for which $c_1\|z\|_Z^2 \leq \|z\|_e^2 \leq c_2\|z\|_Z^2$ for all $z \in Z$) and for which there is a constant $\alpha < 0$ such that $\operatorname{Re}\langle\mathcal{A}z, z\rangle_e \leq \alpha\|z\|_e^2$ for all $z \in \operatorname{dom}\mathcal{A}$. It then follows from linear semigroup theory and the Lumer-Phillips theorem in particular that

$$\|T(t)z\|_e \leq e^{\alpha t}\|z\|_e \text{ for all } z \in Z, \tag{2.8}$$

and it further follows from the equivalence of norms that

$$\|T(t)z\|_Z \leq Me^{\alpha t}\|z\|_Z \text{ for all } z \in Z, \tag{2.9}$$

where $M = \sqrt{c_2/c_1}$. We note that due to the equivalence of norms it is sufficient to show that there is a constant $\alpha < 0$ such that $\mathrm{Re}\,\langle \mathcal{A}z, z\rangle_e \leq \alpha \|z\|_Z^2$ for all $z \in \mathrm{dom}\,A$ in order to conclude

$$\|T(t)z\|_e \leq e^{\alpha t/c_2}\|z\|_e \text{ for all } z \in Z. \tag{2.10}$$

Let us introduce the following condition, which we will show is a sufficient condition for exponential stability of the solution semigroup $T(t)$ associated with (2.1).

(C1) There exist matrices $\widetilde{A} = \mathrm{diag}(a_1, \ldots, a_n)$ and $\widetilde{B} = \mathrm{diag}(b_1, \ldots, b_n)$ for which the following hold.

1. $\mathrm{Re}\; x^T A\,\overline{x} \leq x^T \widetilde{A}\,\overline{x}$ for all $x \in \mathbb{C}^n$,
2. $\mathrm{Re}\; x^T B\,\overline{y} \leq \frac{1}{2} x^T \widetilde{B}\,\overline{x} + \frac{1}{2} y^T \widetilde{B}\,\overline{y}$ for all $x, y \in \mathbb{C}^n$,
3. $a_i < -|b_i|$ for $i = 1, \ldots, n$.

Theorem 2.1 *If (C1) holds, then the semigroup $T(t)$ associated with* (2.1) *is exponentially stable.*

Proof. Observe that if $a < -|b|$ then the function $f(x) = a + |b|e^{-rx} - x$ has a unique negative real root, since $f(0) < 0$, $f(-\infty) = +\infty$, and $f'(x) < 0$ for $x < 0$. Thus for $i = 1, \ldots, n$ let γ_i be the unique negative real root of

$$a_i + |b_i|e^{-\gamma r} - \gamma = 0. \tag{2.11}$$

Set $\gamma = \max\{\gamma_1, \ldots, \gamma_n\} < 0$ and define the $n \times n$ matrix function

$$G(\theta) = \mathrm{diag}\left(|b_i|e^{-\gamma r}e^{-2\gamma\theta}\right).$$

Let us now define a norm on Z by

$$\|(\eta, \varphi)\|_e^2 = \|\eta\|_n^2 + \int_{-r}^{0} \varphi(\theta)^T G(\theta)\overline{\varphi(\theta)}\, d\theta, \tag{2.12}$$

$$= \sum_{i=1}^{n} \left\{ |\eta_i|^2 + \int_{-r}^{0} |b_i|e^{-\gamma r}e^{-2\gamma\theta}|\varphi_i(\theta)|^2\, d\theta \right\} \tag{2.13}$$

with a compatible inner product given by

$$\langle(\eta, \varphi), (\xi, \psi)\rangle_e = \xi^T\overline{\eta} + \int_{-r}^{0} \psi(\theta)^T G(\theta)\overline{\varphi(\theta)}\, d\theta, \tag{2.14}$$

for all $(\eta, \varphi), (\xi, \psi)$ in Z. It is straightforward to check that $\|\cdot\|_e$ is equivalent to the original norm $\|\cdot\|_Z$ on Z. We claim that $\mathrm{Re}\langle \mathcal{A}z, z\rangle_e \leq \gamma\|z\|_e^2$ for all $z \in \mathrm{dom}\,\mathcal{A}$, from which the result will follow. To verify this claim, for $z = (\eta, \varphi) \in \mathrm{dom}\,\mathcal{A}$ we have

$$\begin{aligned}
\mathrm{Re}\,\langle \mathcal{A}(\eta,\varphi),(\eta,\varphi)\rangle_e &= \mathrm{Re}\left\{\eta^T A\overline{\eta} + \eta^T B\overline{\varphi(-r)} + \int_{-r}^{0} \varphi(\theta)^T G(\theta)\overline{\varphi(\theta)'}\, d\theta\right\} \\
&= \mathrm{Re}\left\{\eta^T A\overline{\eta} + (e^{-\gamma r/2}\eta)^T B(e^{\gamma r/2}\overline{\varphi(-r)})\right. \\
&\quad \left. + \int_{-r}^{0} \varphi(\theta)^T G(\theta)\overline{\varphi(\theta)'}\, d\theta\right\} \\
&\leq \eta^T \widetilde{A}\overline{\eta} + \frac{1}{2}e^{\gamma r}\varphi(-r)^T \widetilde{B}\overline{\varphi(-r)} + \frac{1}{2}e^{-\gamma r}\eta^T\widetilde{B}\overline{\eta} \\
&\quad +\mathrm{Re}\left\{\int_{-r}^{0} \varphi(\theta)^T G(\theta)\overline{\varphi(\theta)'}\, d\theta\right\} \\
&= \sum_{i=1}^{n}\left\{a_i|\eta_i|^2 + \frac{1}{2}b_i e^{\gamma r}|\varphi_i(-r)|^2 + \frac{1}{2}b_i e^{-\gamma r}|\eta_i|^2\right\}. \\
&\quad +\mathrm{Re}\left\{\int_{-r}^{0} |b_i| e^{-\gamma r}e^{-2\gamma\theta}\varphi_i(\theta)\overline{\varphi_i(\theta)'}\, d\theta\right\}. \qquad (2.15)
\end{aligned}$$

Now use the integration by parts formula

$$\begin{aligned}
\mathrm{Re}\int_{-r}^{0} g(\theta)\varphi(\theta)\overline{\varphi(\theta)'}\, d\theta &= \frac{1}{2}g(0)|\varphi(0)|^2 - \frac{1}{2}g(-r)|\varphi(-r)|^2 \\
&\quad -\frac{1}{2}\int_{-r}^{0} g'(\theta)|\varphi(\theta)|^2\, d\theta
\end{aligned}$$

and continue from (2.15) to get

$$\begin{aligned}
\mathrm{Re}\,\langle \mathcal{A}(\eta,\varphi),(\eta,\varphi)\rangle_e &\leq \sum_{i=1}^{n}\left\{(a_i + \frac{1}{2}(b_i + |b_i|)e^{-\gamma r})|\eta_i|^2\right. \\
&\quad +\frac{1}{2}(b_i - |b_i|)e^{\gamma r}|\varphi_i(-r)|^2 \\
&\quad \left. + \gamma\int_{-r}^{0} |b_i| e^{-\gamma r}e^{-2\gamma\theta}|\varphi_i(\theta)|^2\, d\theta\right\}. \qquad (2.16)
\end{aligned}$$

Now $(b_i + |b_i|)e^{-\gamma r}/2 \leq |b_i|e^{-\gamma r} \leq |b_i|e^{-\gamma_i r}$ and $(b_i - |b_i|) \leq 0$, so it follows from (2.16) that

$$\begin{aligned}
\mathrm{Re}\,\langle \mathcal{A}z, z\rangle_e &\leq \sum_{i=1}^{n}\left\{(a_i + |b_i|e^{-\gamma_i r})|\eta_i|^2 + \gamma\int_{-r}^{0} |b_i| e^{-\gamma r}e^{-2\gamma\theta}|\varphi_i(\theta)|^2\, d\theta\right\} \\
&= \sum_{i=1}^{n}\left\{\gamma_i|\eta_i|^2 + \gamma\int_{-r}^{0} |b_i| e^{-\gamma r}e^{-2\gamma\theta}|\varphi_i(\theta)|^2\, d\theta\right\} \\
&\leq \gamma\|(\eta,\varphi)\|_e^2.
\end{aligned}$$

It follows from the discussion surrounding (2.8)-(2.10) that $\|T(t)\|_e \leq e^{\gamma t}$ and that $\|T(t)\|_Z \leq Me^{\gamma t}$ for some $M \geq 1$. The result follows. □

Next we recall Mori's result in [15], where it was shown that a sufficient condition for exponential stability is

$$\mu(A) + \|B\| < 0. \tag{2.17}$$

In (2.17) $\mu(A)$ refers to the measure of the matrix A, which is defined by

$$\mu(A) \equiv \lim_{\theta \searrow 0} \frac{\|I + \theta A\| - 1}{\theta}.$$

See [4] for properties of the matrix measure. It turns out that it is only easy to compute $\mu(A)$ in the 1, 2, and ∞ norms, and

$$\begin{aligned} \mu_1(A) &= \max_j \left[\mathrm{Re}\,(a_{jj}) + \sum_{i \neq j} |a_{ij}| \right], \quad \mu_2(A) = \max_i [\lambda_i(A + A^*)/2], \\ \mu_\infty(A) &= \max_i \left[\mathrm{Re}\,(a_{ii}) + \sum_{j \neq i} |a_{ij}| \right]. \end{aligned}$$

Here $\lambda_i(A + A^*)$ represents the ith eigenvalue of the Hermitian matrix $A + A^*$. Our condition (C1) is an improvement over Mori's result in the 2-norm, as the next result verifies.

Lemma 2.1 *Inequality* (2.17) *implies (C1) in the* $\|\cdot\|_2$ *norm.*

Proof. Assume that $\mu(A) < -\|B\|$ in the 2-norm. Set $\widetilde{B} = \|B\|I$ and $\widetilde{A} = \mu(A)I$. Then $a_i = \mu(A) < -\|B\| = -b_i$ for each $i = 1, \ldots, n$, so part 3 from (C1) holds. The Cauchy-Schwarz inequality yields that part 2 from (C1) is true. To see that part 1 from (C1) is true, recall that in the 2-norm

$$\mu(A) = \max_i \{\lambda_i(A + \overline{A}^T)\}/2,$$

so we have

$$\mathrm{Re}\, x^T A \overline{x} \leq \mu(A)\|x\|^2 = x^T \widetilde{A} \overline{x}$$

for all $x \in \mathbb{C}^n$. The result follows. □

Thus our condition is sharper than Mori's in the 2-norm, and in some particular cases it becomes possible to take advantage of the structure of the matrices to obtain significant improvements over Mori's result, even in other norms.

Example. Consider (2.1) with

$$A = \begin{bmatrix} -3.1 & 0 \\ 0 & -1.6 \end{bmatrix} \quad B = \begin{bmatrix} -2.1 & -1 \\ -1 & -.6 \end{bmatrix} \delta$$

This is similar to an example in [15], where the idea is to determine the range of $\delta \geq 0$ for which the system is exponentially stable. It is easy to calculate that

$$\|B\|_1 = 3.1\delta, \quad \|B\|_2 = 2.6\delta, \quad \|B\|_\infty = 3.1\delta,$$

and that

$$\mu_1(A) = \mu_2(A) = \mu_\infty(A) = -1.6.$$

If we apply Mori's condition with the 1 or ∞ norm, stability is guaranteed for $\delta < \frac{1.6}{3.1} \approx .516$, and with the 2-norm stability is guaranteed for $\delta < \frac{1.6}{2.6} \approx .6154$. However if we apply the Cauchy-Schwarz inequality appropriately we can obtain

$$\operatorname{Re} x^T B\overline{y} \leq \frac{1}{2}\left\{(2.1+\varepsilon)|x_1|^2 + (.6+\frac{1}{\varepsilon})|x_2|^2 + (2.1+\varepsilon)|y_1|^2 + (.6+\frac{1}{\varepsilon})|y_2|^2\right\}\delta$$

for any $\varepsilon > 0$. Thus we can choose

$$\widetilde{B} = \begin{bmatrix} 2.1+\varepsilon & 0 \\ 0 & .6+\frac{1}{\varepsilon} \end{bmatrix} \delta$$

and for $\varepsilon = 1$ condition (C1) guarantees stability for $\delta < 1$, a significant improvement over Mori's condition for this example.

One advantage of Mori's result is that exponential stability follows if $\mu(A) < -\|B\|$ in *any* matrix norm, although as previously noted it is only easy to compute $\mu(A)$ in the 1, 2, and ∞ norms. As the above lemma indicates, in the 2 norm our result is an improvement over Mori's result, and as the example indicates, in some cases our result provides an improvement in any norm. An important point is that our result gives an estimate of the exponential decay rate (the number $\gamma < 0$), whereas the results of Mori and others do not.

3. Neutral Systems

Consider the system of linear delay-differential equations

$$\begin{aligned} \frac{d}{dt}[x(t) + Cx(t-r)] &= Ax(t) + Bx(t-r), \\ x(0) + Cx(-r) = \eta_0, \quad x(\theta) &= \varphi_0(\theta), \quad -r \leq \theta < 0, \end{aligned} \tag{3.1}$$

where A, B, C are $n \times n$ matrices with complex entries, and $r > 0$. We are interested in conditions on A, B, C which are sufficient for exponential stability of (3.1) for all

r. Let us recall two recent noteworthy conditions. In [14] Li proved that exponential stability follows if

$$\mu(A) + \frac{\|C\|\|A\| + \|B\|}{1 - \|C\|} < 0 \text{ and } \|C\| < 1. \tag{3.2}$$

In [11] Hu and Hu improved Li's result and showed that exponential stability follows if

$$\mu(A) + \|B\| + \frac{\|CA\| + \|CB\|}{1 - \|C\|} < 0 \text{ and } \|C\| < 1. \tag{3.3}$$

In this section we shall derive new sufficient conditions for exponential stability, including the special case in which A is Hermitian, and our conditions will be compared with (3.2) and (3.3). As we did for retarded systems, we conduct our stability analysis in a Hilbert space setting with the tools of linear semigroup theory. To proceed, consider the previously defined Hilbert space $Z = \mathbb{C}^n \times L^2(-r, 0; \mathbb{C}^n)$ with the norm and inner product given by (2.2) and (2.3). If we make the identification $z(t) = (x(t) + Cx(t - r), x(t + \theta))$, then as introduced in [2] equation (3.1) can be reformulated as the Cauchy problem

$$\frac{d}{dt}z(t) = \mathcal{A}z(t) \tag{3.4}$$

$$z(0) = (\eta_0, \varphi_0), \tag{3.5}$$

on Z. Here the operator $\mathcal{A} : \operatorname{dom}\mathcal{A} \subset Z \to Z$ is defined on the domain

$$\operatorname{dom}\mathcal{A} = \{(\eta, \varphi) \in Z :\ \varphi \in H^1(-r, 0; \mathbb{C}^n),\ \eta = \varphi(0) + C\varphi(-r)\}, \tag{3.6}$$

by

$$\mathcal{A}(\eta, \varphi) = (A\varphi(0) + B\varphi(-r), \varphi'). \tag{3.7}$$

It is well known from [2] that $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$ on Z.

3.1. The case A is Hermitian

We first restrict our attention to the case in which the matrix A is Hermitian (equivalently, A is self-adjoint). We note that for a self-adjoint matrix, $A < 0$ means that A is negative definite.

Theorem 3.1 *In the case in which the matrix A is self-adjoint, if $\mu(A) < 0$ and*

$$1 - \|C\|^2 - \frac{1}{|\mu(A)|^2}\|B\|^2 > 0, \tag{3.8}$$

then the semigroup $T(t)$ associated with (3.1) is exponentially stable.

Proof. In the statement of the theorem, $\|\cdot\|$ refers to the usual matrix norm induced by the Euclidean vector norm $\|\cdot\|_n$. Assume that $\mu(A) < 0$ and (3.8) is true. Since $\mu(A) < 0$ the matrix A is negative definite, and so $\Lambda \equiv (-A)^{-1}$ is a positive definite, self-adjoint matrix. Furthermore, $\|\Lambda\| = 1/|\mu(A)|$. Let us define a norm on Z by

$$\|(\eta,\varphi)\|_e^2 = \eta^T\Lambda\overline{\eta} + \int_{-r}^0 e^{-2\gamma\theta}\varphi(\theta)^T\overline{\varphi(\theta)}\,d\theta$$

for all $(\eta,\varphi) \in Z$, with a compatible inner product given by

$$\langle(\eta,\varphi),(\xi,\psi)\rangle_e = \xi^T\Lambda\overline{\eta} + \int_{-r}^0 e^{-2\gamma\theta}\psi(\theta)^T\overline{\varphi(\theta)}\,d\theta, \tag{3.9}$$

for all $(\eta,\varphi),(\xi,\psi) \in Z$. It is straightforward to check that $\|\cdot\|_e$ is equivalent to the original norm $\|\cdot\|_Z$ on Z for any choice of γ. For any $z = (\eta,\varphi) \in \operatorname{dom}\mathcal{A}$ we have

$$\begin{aligned}
\operatorname{Re}\langle\mathcal{A}z,z\rangle_e &= \operatorname{Re}\Big\{\eta^T\Lambda\overline{[A\eta - AC\varphi(-r)]} + \eta^T\Lambda B\overline{\varphi(-r)} \\
&\quad + \int_{-r}^0 e^{-2\gamma\theta}\varphi^T(\theta)\overline{\varphi(\theta)'}\,d\theta\Big\} \\
&= -\|\eta\|_n^2 + \operatorname{Re}\Big\{\eta^T\overline{[C\varphi(-r)]}\Big\} + \frac{1}{2}\|\eta - C\varphi(-r)\|_n^2 \\
&\quad -\frac{1}{2}\|\varphi(-r)\|_n^2 e^{2\gamma r} + \operatorname{Re}\Big\{\eta^T\Lambda B\overline{\varphi(-r)}\Big\} \\
&\quad +\gamma\int_{-r}^0 e^{-2\gamma r}\|\varphi(\theta)\|_n^2\,d\theta \\
&= -\frac{1}{2}\|\eta\|_n^2 + \frac{1}{2}\varphi(-r)^T[C^TC - e^{2\gamma r}I]\overline{\varphi(-r)} \\
&\quad +\operatorname{Re}\Big\{\eta^T\Lambda B\overline{\varphi(-r)}\Big\} + \gamma\int_{-r}^0 e^{-2\gamma r}\|\varphi(\theta)\|_n^2\,d\theta.
\end{aligned} \tag{3.10}$$

Now

$$\operatorname{Re}\eta^T\Lambda B\overline{\varphi(-r)} \le \frac{k}{2}\|\Lambda\eta\|_n^2 + \frac{1}{2k}\|B\varphi(-r)\|_n^2$$

for any $k > 0$, so from (3.10) we have

$$\begin{aligned}
\operatorname{Re}\langle\mathcal{A}z,z\rangle_e &\le -\frac{1}{2}\|\eta\|_n^2(1 - \frac{k}{|\mu(A)|^2}) + \gamma\int_{-r}^0 e^{-2\gamma r}\|\varphi(\theta)\|_n^2\,d\theta \\
&\quad +\frac{1}{2}\varphi(-r)^T[C^TC - e^{2\gamma r}I + \frac{1}{k}B^TB]\overline{\varphi(-r)}.
\end{aligned} \tag{3.11}$$

If for $0 < k < |\mu(A)|^2$ we can choose $\gamma < 0$ such that

$$e^{2\gamma r}I - C^TC - \frac{1}{k}B^TB \ge 0, \tag{3.12}$$

then we get

$$\mathrm{Re}\,\langle Az, z\rangle_e \le \alpha \|z\|_Z^2, \tag{3.13}$$

for all $z \in \mathrm{dom}\,A$, where $\alpha = \max\{-\frac{1}{2}(1 - \frac{k}{|\mu(A)|^2}), \gamma\}$. A sufficient condition for (3.12) is (3.8), and the result follows from the discussion surrounding (2.8)-(2.10). □

In order to get a more refined estimate of the decay rate, observe that

$$\begin{aligned} -\frac{1}{2}(1 - \frac{k}{|\mu(A)|^2})\|\eta\|_n^2 &\le -\frac{1}{2}(1 - \frac{k}{|\mu(A)|^2})|\mu(A)|\eta^T \Lambda \overline{\eta} \\ &= -\frac{1}{2}(|\mu(A)| - \frac{k}{|\mu(A)|})\eta^T \Lambda \overline{\eta}. \end{aligned}$$

Thus from (3.11) and (3.12) we see that (3.13) can be modified to

$$\mathrm{Re}\,\langle Az, z\rangle_e \le \alpha \|z\|_e^2, \tag{3.14}$$

for all $z \in \mathrm{dom}\,A$, where $\alpha = \max\{-\frac{1}{2}(|\mu(A)| - \frac{k}{|\mu(A)|}), \gamma\} < 0$. The optimal decay rate is obtained by choosing k and γ (satisfying $0 < k < |\mu(A)|^2$, $\gamma < 0$ and (3.12)) to minimize $\alpha < 0$. Observe that if for k satisfying $0 < k < |\mu(A)|^2$ we can choose $\gamma < 0$ such that

$$e^{2\gamma r} - \|C\|^2 - \frac{1}{k}\|B\|^2 \ge 0, \tag{3.15}$$

then (3.14) holds, and also observe that (3.8) implies (3.15) which in turn implies (3.12). For any feasible choice of k, γ must satisfy (from (3.15))

$$\frac{1}{2r}\ln(\|C\|^2 + \frac{1}{k}\|B\|^2) \le \gamma < 0,$$

and so our optimal decay rate satisfies

$$\begin{aligned} \alpha &= \min_k \max\left\{-\frac{1}{2}(|\mu(A)| - \frac{k}{|\mu(A)|}), \frac{1}{2r}\ln(\|C\|^2 + \frac{1}{k}\|B\|^2)\right\} \\ &= \min_k \max\{f(k), g(k)\}. \end{aligned}$$

Notice that for $k > 0$, $f(k) = -\frac{1}{2}(|\mu(A)| - \frac{k}{|\mu(A)|})$ is an increasing function of k and $g(k) = \frac{1}{2r}\ln(\|C\|^2 + \frac{1}{k}\|B\|^2)$ is a decreasing function of k. Also $f(|\mu(A)|^2) = 0$, $g(|\mu(A)|^2) < 0$, and if (3.8) holds then

$$0 < \frac{\|B\|^2 |\mu(A)|^2}{\|B\|^2 + |\mu(A)|^2(1 - \|C\|^2 - \|B\|^2/|\mu(A)|^2)} < |\mu(A)|^2$$

and

$$g\left(\frac{\|B\|^2 |\mu(A)|^2}{\|B\|^2 + |\mu(A)|^2(1 - \|C\|^2 - \|B\|^2/|\mu(A)|^2)}\right) = 0.$$

Thus (3.8) implies that there is a unique k satisfying

$$0 < \frac{\|B\|^2|\mu(A)|^2}{\|B\|^2 + |\mu(A)|^2(1 - \|C\|^2 - \|B\|^2/|\mu(A)|^2)} < k < |\mu(A)|^2$$

which is a solution of

$$-\frac{1}{2}(|\mu(A)| - \frac{k}{|\mu(A)|}) = \frac{1}{2r}\ln(\|C\|^2 + \frac{1}{k}\|B\|^2), \tag{3.16}$$

and for this k the optimal decay rate is given by

$$\alpha = -\frac{1}{2}(|\mu(A)| - \frac{k}{|\mu(A)|}). \tag{3.17}$$

Therefore we may use (3.16) and (3.17) to get an estimate of the decay rate. Next we shall consider the general neutral equation, in which A is not restricted to be Hermitian, but before doing so we show that our condition (3.8) is sharper than Li's condition (3.2).

Theorem 3.2 *In the case in which the matrix A is self-adjoint,* (3.2) *implies* (3.8).

Proof. Assume that (3.2) is true. Then $\mu(A) < 0$, and also $\|C\| < 1$ implies that $(1 - \|C\|)^2 < 1 - \|C\|^2$. We have

$$|\mu(A)|^2(1 - \|C\|^2) > |\mu(A)|^2(1 - \|C\|)^2 > (\|B\| + \|C\|\,\|A\|)^2 > \|B\|^2$$

and the result follows. □

3.2. The general neutral equation

We provide a sufficient condition for exponential stability of the linear neutral system (3.1).

Theorem 3.3 *Consider the neutral system* (3.1). *Define the matrices $G = -(A + A^T)/2$ and $H = G + A = (A - A^T)/2$. If $\mu(A) < 0$ and*

$$G - C^TGC - \frac{1}{k}C^TH^THC - \frac{1}{|\mu(A)| - k}B^TB > 0 \tag{3.18}$$

for some constant $0 < k < |\mu(A)|$, then the semigroup $T(t)$ associated with (3.1) *is exponentially stable. In the special case $B \equiv 0$ it is sufficient that*

$$G - C^TGC - \frac{1}{|\mu(A)|}C^TH^THC > 0. \tag{3.19}$$

Proof. Observe that G is a positive-definite, self-adjoint matrix. With this matrix define a norm on Z by

$$\|(\eta,\varphi)\|_e^2 = \eta^T\overline{\eta} + \int_{-r}^{0} e^{-2\gamma\theta}\varphi(\theta)^T G\overline{\varphi(\theta)}\,d\theta \tag{3.20}$$

for all $(\eta,\varphi)\in Z$, with a compatible inner product given by

$$\langle(\eta,\varphi),(\xi,\psi)\rangle_e = \xi^T\overline{\eta} + \int_{-r}^{0} e^{-2\gamma\theta}\psi(\theta)^T G\overline{\varphi(\theta)}\,d\theta, \tag{3.21}$$

for all $(\eta,\varphi),(\xi,\psi)\in Z$. It is straightforward to check that $\|\cdot\|_e$ is equivalent to the original norm $\|\cdot\|_Z$ on Z for any choice of γ. We will make use of the integration by parts formula

$$\begin{aligned}\mathrm{Re}\int_{-r}^{0} e^{-2\gamma\theta}\varphi^T(\theta)G\overline{\varphi(\theta)'}\,d\theta &= \frac{1}{2}\varphi(0)^T G\overline{\varphi(0)} - \frac{1}{2}\varphi(-r)^T Ge^{2\gamma r}\overline{\varphi(-r)}\\ &\quad +\gamma\int_{-r}^{0} e^{-2\gamma\theta}\varphi^T(\theta)G\overline{\varphi(\theta)}\,d\theta \end{aligned}\tag{3.22}$$

For any $z=(\eta,\varphi)\in \mathrm{dom}\,\mathcal{A}$ we have

$$\begin{aligned}\mathrm{Re}\,\langle\mathcal{A}z,z\rangle_e &= \mathrm{Re}\left\{\eta^T\overline{[A\eta - AC\varphi(-r)]} + \eta^T B\overline{\varphi(-r)} + \int_{-r}^{0} e^{-2\gamma\theta}\varphi^T(\theta)G\overline{\varphi(\theta)'}\,d\theta\right\}\\
&= \mathrm{Re}\left\{\eta^T A\overline{\eta} - \eta^T\overline{[AC\varphi(-r)]} - \frac{1}{2}\varphi(-r)^T Ge^{2\gamma r}\overline{\varphi(-r)}\right.\\
&\quad +\frac{1}{2}[\eta - C\varphi(-r)]^T G\overline{[\eta - C\varphi(-r)]} + \eta^T B\overline{\varphi(-r)}\\
&\quad \left.+\gamma\int_{-r}^{0} e^{-2\gamma\theta}\varphi^T(\theta)G\overline{\varphi(\theta)}\,d\theta\right\}\\
&= \mathrm{Re}\left\{\frac{1}{2}\eta^T[(A+A^T)+G]\overline{\eta} - \eta^T[A+G]C\overline{\varphi(-r)}\right.\\
&\quad -\frac{1}{2}\varphi(-r)^T[Ge^{2\gamma r} - C^T GC]\overline{\varphi(-r)} + \eta^T B\overline{\varphi(-r)}\\
&\quad \left.+\gamma\int_{-r}^{0} e^{-2\gamma\theta}\varphi^T(\theta)G\overline{\varphi(\theta)}\,d\theta\right\}.\end{aligned}\tag{3.23}$$

Now

$$\frac{1}{2}\mathrm{Re}\,\eta^T[(A+A^T)+G]\overline{\eta} = \frac{1}{2}\mathrm{Re}\,\eta^T[\frac{1}{2}(A+A^T)]\overline{\eta} \le \frac{1}{2}\mu(A)\|\eta\|_n^2,$$

$$\begin{aligned}-\mathrm{Re}\,\eta^T[A+G]C\overline{\varphi(-r)} &= -\mathrm{Re}\,\eta^T HC\overline{\varphi(-r)}\\ &\le \frac{k_1}{2}\|\eta\|_n^2 + \frac{1}{2k_1}\varphi(-r)^T C^T H^T HC\overline{\varphi(-r)}\end{aligned}$$

and

$$\operatorname{Re}\eta^T B\overline{\varphi(-r)} \le \frac{k_2}{2}\|\eta\|_n^2 + \frac{1}{2k_2}\varphi(-r)^T B^T B\overline{\varphi(-r)}$$

for any $k_1, k_2 > 0$. We continue from (3.23) to get

$$\begin{aligned}\operatorname{Re}\langle \mathcal{A}z, z\rangle_e \quad &\le \quad \frac{1}{2}(\mu(A)+k_1+k_2)\|\eta\|_n^2 + \gamma\int_{-r}^{0} e^{-2\gamma\theta}\varphi(\theta)G\overline{\varphi(\theta)}\,d\theta \\ &\quad -\frac{1}{2}\varphi(-r)^T[e^{2\gamma r}G - C^T GC - \frac{1}{k_1}C^T H^T HC - \frac{1}{k_2}B^T B]\overline{\varphi(-r)}.\end{aligned} \tag{3.24}$$

If for $k_1, k_2 > 0$ satisfying $0 < k_1 + k_2 < |\mu(A)|$ we can choose $\gamma < 0$ such that

$$e^{2\gamma r}G - C^T GC - \frac{1}{k_1}C^T H^T HC - \frac{1}{k_2}B^T B > 0 \tag{3.25}$$

then we get

$$\operatorname{Re}\langle \mathcal{A}z, z\rangle_e \le \alpha\|z\|_e^2, \tag{3.26}$$

for all $z \in \operatorname{dom}\mathcal{A}$, where

$$\alpha = \max\left\{\frac{\mu(A)+k_1+k_2}{2}, \gamma\right\}. \tag{3.27}$$

A sufficient condition for (3.25) is (3.18), so the result follows. In the special case that $B \equiv 0$ a very similar argument can be used to show that (3.19) is sufficient for exponential stability. □

Since the matrix $G - C^T GC - \frac{1}{k}C^T H^T HC - \frac{1}{|\mu(A)|-k}B^T B$ is Hermitian, the condition (3.18) (or (3.19) in the case $B \equiv 0$) is easy to check. Also the decay rate is given by α. One measure of the sharpness of a sufficient condition is what it reduces to in the scalar case ($n = 1$), where sharp conditions for stability are known. In particular, when $B = 0$ and A, C are scalars, a necessary and sufficient condition for exponential stability of $T(t)$ is

$$A < 0 \text{ and } |C| < 1. \tag{3.28}$$

When our sufficient conditions (3.8) and (3.18) are applied to this case we obtain exactly (3.28), meaning that our conditions are sharp. However if we apply the stability conditions of [14] and [11] in this case, we see that both (3.2) and (3.3) require $A < 0$ and $|C| < 1/2$. The condition $|C| < 1/2$ may be considered significantly more restrictive than $|C| < 1$, and so by this measure our sufficient conditions are quite satisfactory.

Next we show that our condition (3.18) is sharper than Li's condition (3.2) and independent of Hu and Hu's condition (3.3). We have the following result.

Lemma 3.1 *The sufficient condition* (3.2)*) implies* (3.18) *(and* (3.19) *when* $B \equiv 0$*).*

Proof. Assume that (3.2) is true and that A is not Hermitian (when A is Hermitian we can use Theorem 3.1 and Theorem 3.2). We shall assume that $B \not\equiv 0$ and show that (3.18) holds, and remark that when $B \equiv 0$ a similar argument can be used to show that (3.19) holds. We have that $\mu(A) < 0$, so the matrix G is positive definite and self-adjoint, and

$$|\mu(A)|\,\|x\|_n^2 \le x^T G\overline{x} \le \|A\|\,\|x\|_n^2$$

for all $x \in \mathbb{C}^n$. Also

$$\begin{aligned} x^T C^T G C\overline{x} = (Cx)^T G\overline{(Cx)} &\le \|A\|\,\|Cx\|_n^2 \le \|A\|\,\|C\|^2\|x\|_n^2 \\ x^T B^T B\overline{x} &\le \|B\|^2\|x\|_n^2 \end{aligned}$$

and

$$x^T C^T H^T H C\overline{x} = (HCx)^T\overline{(HCx)} = \|HCx\|_n^2 \le \|H\|^2\|C\|^2\|x\|_n^2$$

for all $x \in \mathbb{C}^n$. Thus to verify (3.18) it is sufficient to show that

$$|\mu(A)| > \|C\|^2\|A\| + \frac{1}{k}\|C\|^2\|H\|^2 + \frac{1}{|\mu(A)| - k}\|B\|^2 \tag{3.29}$$

for some k satisfying $0 < k < |\mu(A)|$. In particular, set

$$k = \frac{|\mu(A)|\,\|C\|\,\|H\|}{\|B\| + \|C\|\|H\|}.$$

Note that $0 < k < |\mu(A)|$, because $k < |\mu(A)|$ as long as $B \not\equiv 0$, which is the case under consideration, and $0 < k$ because A not being Hermitian implies $\|H\| \neq 0$. (Of course, if $\|C\| = 0$, then we are in the retarded case and would instead invoke Theorem 2.1). For this choice of k, (3.29) becomes

$$|\mu(A)| > \|C\|^2\|A\| + \frac{1}{|\mu(A)|}(\|B\| + \|H\|\,\|C\|)^2, \tag{3.30}$$

or equivalently,

$$|\mu(A)|^2 > |\mu(A)|\,\|C\|^2\|A\| + \|B\|^2 + \|H\|^2\|C\|^2 + 2\|B\|\,\|H\|\,\|C\|. \tag{3.31}$$

In order to verify (3.31) notice that (3.2) implies

$$|\mu(A)| > |\mu(A)|\,\|C\| + \|A\|\,\|C\| + \|B\|,$$

so it follows that

$$\begin{aligned} |\mu(A)|^2 &> |\mu(A)|^2\|C\|^2 + \|A\|^2\|C\|^2 + \|B\|^2 \\ &\quad +2\|A\|\,\|B\|\,\|C\| + 2|\mu(A)|\,\|A\|\,\|C\|^2 + 2|\mu(A)|\,\|B\|\,\|C\| \\ &> |\mu(A)|^2\|C\|^2 + \|H\|^2\|C\|^2 + \|B\|^2 \\ &\quad +2\|H\|\,\|B\|\,\|C\| + 2|\mu(A)|\,\|A\|\,\|C\|^2 + 2|\mu(A)|\,\|B\|\,\|C\| \\ &\ge \|H\|^2\|C\|^2 + \|B\|^2 + 2\|H\|\,\|B\|\,\|C\| + |\mu(A)|\,\|A\|\,\|C\|^2 \end{aligned}$$

where we used that $\|H\| \leq \|A\|$. Thus (3.31) holds and the result follows. $\square$

Thus our condition is sharper than Li's condition. Next we show by a numerical example that our condition is independent of Hu and Hu's condition.

Example When $B \equiv 0$ the condition of Hu and Hu reduces to

$$\mu(A) + \frac{\|CA\|}{1 - \|C\|} < 0 \text{ and } \|C\| < 1. \tag{3.32}$$

We used a simple Matlab routine to generate random matrices and check the conditions (3.19) and (3.32). First consider the matrices

$$\begin{aligned} A &= \begin{bmatrix} -1.9415 & 0.8606 & 1.9198 \\ 1.6590 & -1.8770 & 1.7767 \\ -4.3995 & -1.7607 & -4.7607 \end{bmatrix} \\ C &= \begin{bmatrix} -0.0761 & -0.0083 & -0.0406 \\ 0.0729 & -0.0470 & -0.0176 \\ -0.0801 & -0.0802 & 0.0140 \end{bmatrix}. \end{aligned}$$

We see that $\mu(A) \approx -0.4643$, $\|C\| \approx 0.1373$ and

$$\mu(A) + \frac{\|CA\|}{1 - \|C\|} \approx -0.0108 < 0,$$

so (3.32) is satisfied. However the eigenvalues of $G - C^T GC - \frac{1}{|\mu(A)|} C^T H^T HC$ are approximately 5.2151, 2.7887, and −0.0548, so (3.19) is not satisfied. Thus the condition of Hu and Hu is satisfied while our condition is not satisfied. On the other hand, consider the matrices

$$\begin{aligned} A &= \begin{bmatrix} -10.6449 & 1.8236 & -0.9194 \\ -0.4264 & -8.3802 & -0.8692 \\ 2.1469 & -4.1161 & -3.0405 \end{bmatrix} \\ C &= \begin{bmatrix} 0.0970 & 0.1448 & -0.1442 \\ 0.0527 & 0.0979 & 0.0775 \\ -0.0383 & -0.0619 & -0.0728 \end{bmatrix}. \end{aligned}$$

We get $\mu(A) \approx -2.0461$, and the eigenvalues of $G - C^T GC - \frac{1}{|\mu(A)|} C^T H^T HC$ are approximately 11.0611, 8.5926, and 1.5996, so (3.19) is satisfied. However, $\|CA\| \approx 1.8356$ and $\|C\| \approx 0.2319$, so

$$\mu(A) + \frac{\|CA\|}{1 - \|C\|} \approx 0.3436 \not< 0,$$

and (3.32) is not satisfied in the 2 norm. It turns out that (3.32) is not satisfied in the 1 or ∞ norms either, since $\mu_\infty(A) \approx 3.2225$, and $\mu_1(A) \approx -1.2519$, $\|CA\|_1 \approx 2.2349$,

$\|C\|_1 \approx 0.3046$, so

$$\mu_1(A) + \frac{\|CA\|_1}{1 - \|C\|_1} \approx 1.9619 \not< 0.$$

Thus our condition is satisfied while the condition of Hu and Hu is not satisfied.

4. Conclusion

We have obtained some new results for exponential stability of differential-difference systems of retarded- and neutral-type linear delay equations. Our approach makes use of linear semigroup theory and an appropriate renorming of the underlying state space. We use our new norms to obtain sufficient conditions for exponential stability of the corresponding solution semigroups. Then we compare and contrast them with some known sufficient conditions, and some improvements over existing conditions are noted. Along this line we also obtain estimates of the exponential decay rate of the solution semigroup, estimates which are usually not available.

References

[1] R.K. Brayton and R.A. Willoughby, On the numerical integration of a symmetric system of difference-differential equations of neutral type, Journal of Mathematical Analysis and Applications, v.1 (1967), 182-189.

[2] J. A. Burns, T. L. Herdman and H. W. Stech, Linear functional differential equations as semigroups on product spaces, SIAM J. Math. Anal., v.14 (1983), 98-116.

[3] D.Q. Cao and P. He, Stability criteria of linear neutral systems with a single delay, Applied Mathematics and Computation, v.148 (2004), 135-143.

[4] C.A. Desoer and M. Vidyasagar, Feedback Systems: Input-Output Properties, Academic Press, (1975).

[5] R. H. Fabiano, Stability preserving spline approximations for scalar functional differential equations, Computers Math. Applic., v.29 (1995), 87-94.

[6] R.H. Fabiano and J. Turi, Preservation of stability under approximation for a neutral FDE, Dynamics of Continuous, Discrete and Impulsive Systems, v.5 (1999), 351-364.

[7] M. I. Gil', Stability of Finite and Infinite Dimensional Systems, Kluwer, (1998).

[8] K. Gu, V.L. Kharitonov and J. Chen, Stability of Time-Delay Systems, Birkhäuser, Control Engineering, (2003).

[9] J.K. Hale and E.F. Infante and F.P. Tsen, Stability in linear delay equations, Journal of Mathematical Analysis and Applications, v.105 (1985), 533-555.

[10] J.K. Hale and S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer-Verlag, Applied Mathematics and Computation, v.99 (1993).

[11] G. Hu and G. Hu, Some simple stability criteria for stability of neutral delay-differential systems, Applied Mathematics and Computation, v.80 (1996), 257-271.

[12] G. Hu and G. Hu and B. Cahlon, Algebraic criteria for stability of linear neutral systems with a single delay, Journal of Computational and Applied Mathematics, v.135 (2001), 125-133.

[13] V. B. Kolmanovskii, Applications of differential inequalities for stability of some functional differential equations, Nonlinear Analysis, TMA, v.25 (1995), 1017-1028.

[14] L.M. Li, Stability of linear neutral delay-differential systems, Bull. Austral. Math. Soc., v.38 (1988), 339-344.

[15] T. Mori, Criteria for asymptotic stability of linear time-delay systems, IEEE Trans. Automatic Control, v.30 (1985), 158-161.

[16] S-I. Niculescu, Delay Effects on Stability, Springer-Verlag, Lecture Notes in Control and Information Sciences, v.269 (2001).

[17] J.H. Park and S. Won, A note on stability of neutral delay-differential systems, Journal of the Franklin Institute, v.336 (1999), 543-548.

[18] U. Stroinski, Delay-independent stability criteria for neutral differential equations, Differential Integral Equations, v.7 (1994), 593-599.