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Population Models With Delay in Dynamic Environment

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Abstract. We consider the Getz type harvesting model with a delay

$$\frac{dN}{dt} = N(t)\left[\frac{r(t)}{1+\left(\frac{N(g(t))}{K(t)}\right)^{\gamma}} - m(t)\right] - \lambda(t)N(t).$$

For this non-autonomous delayed differential equation we study the existence of global solutions of the initial value problem, extinction and persistence conditions, and the existence of periodic solutions.

AMS Subject Classifications: 34K13, 37N25, 92D25

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1. Introduction and Preliminaries

Consider the following standard differential equation which is widely used in Population Dynamics [5,7,13]

$$\frac{dN}{dt} = [\beta(t,N) - m(t,N)]N - \lambda(t)N, \tag{1.1}$$

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where N = N(t) is the population biomass, $\beta(t, N)$ is the per-capita fecundity rate, $m(t, N)$ is the per-capita mortality rate, and $\lambda(t)$ is the harvesting rate per-capita. A classical logistic harvesting model [7] has the following form

$$\frac{dN}{dt} = r\left[1 - \frac{N(t)}{K}\right] - Y(t).$$

Here $Y(t)$ is a function of captures. It is a well-known fact [5,13] that the canonical logistic model in some cases produces artificially complex dynamics, therefore it would be reasonable to get away from the specific logistic form in studying population dynamics and use more general classes of growth models [1-4]. For example, in order to drop an unnatural symmetry of the logistic curve, recently we have considered [2] the modified delay logistic form of Pella and Tomlinson or Richards' growth delay equation

$$\frac{dN}{dt} = r\left[1 - \left(\frac{N(g(t))}{K}\right)^{\gamma}\right].$$

In equation (1.1) let $\beta(t, N)$ be a Hill's type function

$$\beta(t, N) = \frac{r}{1 + (N/K)^{\gamma}},$$

where $r > 0, K > 0$. Parameter $\gamma > 0$ is referred to by W. Getz [8] as the "abruptness" parameter.

Generally, models with the delay in the reproduction term recognize that for real organisms it takes time to develop from newborns to reproductively active adults. If we take into account that delay and assume that $m(t, N) = m(t)$, then we have the following time-lag model based on equation (1.1)

$$\frac{dN}{dt} = N(t)\left[\frac{r(t)}{1 + \left(\frac{N(g(t))}{K(t)}\right)^{\gamma}} - m(t)\right] - \lambda(t)N(t). \tag{1.2}$$

Here $t \geq 0$, $r(t) > 0$ is a fecundity factor, $m(t) > 0$ is a mortality factor, $\lambda(t) > 0$ is a harvesting factor, $K(t) > 0$ is a carrying capacity, $g(t)$ is the time to develop from newborns to reproductively active adults, $0 < g(t) \leq t$. For $\lambda(t) \equiv 0$, $r(t) \equiv r$, $g(t) = t$, $m(t) \equiv m$, $K(t) \equiv K$, equation (1.2) is the Getz type differential equation [8] that describes dynamics of the marine population in a stable environment.

The paper is organized as follows. In the next section we consider proportional harvesting and obtain the explicit conditions for the existence of a unique positive solution of equation (1.2). In Section 3 sufficient conditions for the existence of positive periodic solutions of equation (1.2) were obtained.

2. Proportional Harvesting in Dynamic Environment

If we denote $M(t) = m(t) - \lambda(t)$, then equation (1.2) has the following form

$$\frac{dN}{dt} = N(t)\left[\frac{r(t)}{1+\left(\frac{N(g(t))}{K(t)}\right)^{\gamma}}\right] - M(t)N(t) \tag{2.1}$$

with the initial function and the initial value

$$N(t) = \varphi(t), t < 0, N(0) = N_0 \tag{2.2}$$

under the following conditions:

(a1) $\gamma > 0$;

(a2) $r(t)$, $M(t)$, $K(t)$ are continuous on $[0, \infty)$ functions, $r(t) > 0$, $M(t) \geq m > 0$, $K(t) \geq k > 0$;

(a3) $g(t)$ is a continuous function, $g(t) \leq t$;

(a4) $\varphi : (-\infty, 0) \to \mathbb{R}$ is a continuous bounded function, $\varphi(t) \geq 0, N_0 > 0$.

Definition 2.1 *A function $N : \mathbb{R} \to \mathbb{R}$ with continuous derivative is called a solution of problem (2.1)-(2.2), if it satisfies equation (2.1) for all $t \in [0, \infty)$ and equalities (2.2) for $t \leq 0$.*

If t_0 is the first point, where the solution $N(t)$ of (2.1)-(2.2) vanishes, i.e., $N(t_0) = 0$, then we consider the only positive solutions of the problem (2.1)-(2.2) on the interval $[0, t_0)$.

Theorem 2.1 *Suppose (a1)-(a4) hold and*

(1) $\inf_{t\geq 0}\left(\frac{r(t)}{M(t)} - 1\right) > 0,$

(2) $\sup_{t\geq 0}\left(\frac{r(t)}{M(t)} - 1\right) < \infty,$

(3) $\sup_{t\geq 0}\int_{g(t)}^{t}[r(s) - M(s)]ds < \infty,$

(4) $\sup_{t\geq 0}\int_{g(t)}^{t} M(s)ds < \infty.$

Then problem (2.1)-(2.2) has on $[0, \infty)$ a unique positive solution $N(t)$ such that

$$\min\left\{N_0, \inf_{t\geq 0} K(t)\left(\frac{r(t)}{M(t)} - 1\right)^{\frac{1}{\gamma}} \exp\left[-\sup_{t\geq 0}\int_{g(t)}^{t} M(s)ds\right]\right\} \leq N(t) \tag{2.3}$$

and

$$N(t) \leq \max\left\{N_0, \sup_{t\geq 0} K(t)\left(\frac{r(t)}{M(t)}-1\right)^{\frac{1}{\gamma}} \exp\left[\sup_{t\geq 0}\int_{g(t)}^{t}[r(s)-M(s)]ds\right]\right\}. \quad (2.4)$$

Proof. The existence of the unique local solution is a consequence of well-known results for nonlinear delay differential equations (see, for example, [9,10,13]). Clearly

$$N(t) = N_0 \exp\left\{\int_0^t \left[\frac{r(s)}{1+\left(\frac{N(g(s))}{K(s)}\right)^{\gamma}} - M(s)\right] ds\right\},$$

hence the local solution (2.1)-(2.2) is positive. If $[0,\alpha)$ is the maximal interval of the existence of this solution, where $\lim_{t\to a^-} N(t) = +\infty$ for $\alpha < \infty$.

Eq. (2.1) implies that

$$\frac{dN}{dt} \leq [r(t) - M(t)]N(t),$$

therefore

$$N(t) \leq N_0 \exp\left\{\int_0^t [r(s)-M(s)]ds\right\}.$$

Then the local solution $N(t)$ is bounded on the maximal interval $[0,\alpha)$ of the existence of this solution, and we have a contradiction. Hence the maximal interval of the existence of $N(t)$ is $[0,\infty)$ and the global solution is positive. Therefore we have to prove only inequalities (2.3)-(2.4).

Suppose that $\frac{dN}{dt} > 0$ for any $t > 0$. Then (2.1) implies that

$$\frac{r(t)}{1+\left(\frac{N(g(t))}{K(t)}\right)^{\gamma}} - M(t) > 0$$

or

$$N(g(t)) < K(t)\left(\frac{r(t)}{M(t)}-1\right)^{\frac{1}{\gamma}},$$

and we have $\frac{dN}{dt} \leq N(t)(r(t)-M(t))$. Finally

$$\begin{aligned} N(t) &\leq N(g(t))\exp\left\{\int_{g(t)}^{t}[r(s)-M(s)]ds\right\} \\ &\leq K(t)\left(\frac{r(t)}{M(t)}-1\right)^{\frac{1}{\gamma}}\exp\left\{\int_{g(t)}^{t}[r(s)-M(s)]ds\right\}. \end{aligned}$$

The latter inequality proves (2.4) for $\frac{dN}{dt} > 0$. If $\frac{dN}{dt} < 0$ for any $t > 0$ inequality (2.4) is evident.

Suppose now that $\{t_n\}$ is the sequence of the points where the function $N(t)$ has the local maximum. Then $\frac{dN}{dt}(t_n) = 0$, hence

$$N(g(t_n)) = K(t_n)\left(\frac{r(t_n)}{M(t_n)} - 1\right)^{\frac{1}{\gamma}}.$$

In the interval $[g(t_n), t_n]$ we have

$$\frac{dN}{dt} \le N(t)[r(t) - M(t)],$$

then

$$N(t) \le N(g(t_n)) \exp\left\{\int_{g(t_n)}^{t} [r(s) - M(s)]ds\right\}.$$

Hence

$$N(t_n) \le N(g(t_n)) \exp\left\{\int_{g(t_n)}^{t_n} [r(s) - M(s)]ds\right\}.$$

Inequality (2.4) is satisfied since $\sup_{t\ge 0} N(t) = \sup_n N(t_n)$.

Suppose now that $\frac{dN}{dt} < 0$ for any $t > 0$. Then

$$\frac{r(t)}{1 + \left(\frac{N(g(t))}{K(t)}\right)^{\gamma}} - M(t) < 0$$

and

$$N(g(t)) > K(t)\left(\frac{r(t)}{M(t)} - 1)\right)^{\frac{1}{\gamma}}.$$

Equation (2.1) implies $\frac{dN}{dt} \ge -M(t)N(t)$. Therefore

$$\begin{aligned} N(t) &\ge N(g(t)) \exp\left\{-\int_{g(t)}^{t} [r(s) - M(s)]ds]\right\} \\ &\ge K(t)\left(\frac{r(t)}{M(t)} - 1\right)^{\frac{1}{\gamma}} \exp\left\{-\int_{g(t)}^{t} [r(s) - M(s)]ds\right\}. \end{aligned}$$

Hence inequality (2.3) has been proven for $\frac{dN}{dt} < 0$.
If $\frac{dN}{dt} > 0$ for any $t > 0$, then inequality (2.3) is evident.
Suppose now that $\{\tau_n\}$ is the sequence of the points where the function $N(t)$ has the local minimum. Then $\frac{dN}{dt}(\tau_n) = 0$ and

$$N(g(\tau_n)) = K(\tau_n)\left(\frac{r(\tau_n)}{M(\tau_n)} - 1\right)^{\frac{1}{\gamma}}.$$

In the interval $[g(\tau_n), \tau_n]$ we have

$$\frac{dN}{dt} \ge -M(t)N(t),$$

therefore

$$N(t) \geq N(g(\tau_n)) \exp\left\{-\int_{g(\tau_n)}^{t} M(s)ds\right\}$$

and

$$N(\tau_n) \geq N(g(\tau_n)) \exp\left\{-\int_{g(\tau_n)}^{\tau_n} M(s)ds\right\}.$$

Since $\inf_{t\geq 0} N(t) = \inf_n N(\tau_n)$, inequality (2.3) is true in this case, therefore we proved Theorem 2.1. $\square$

3. Periodic Proportional Harvesting in a Dynamic Environment

Periodic harvesting is used frequently as a tool by fishery managers to protect a stock during a spawning season for a defined period of time [5-7,12]. Consider

$$\frac{dN}{dt} = N(t)\left[\frac{r(t)}{1+\left(\frac{N(g(t))}{K(t)}\right)^{\gamma}} - m(t)\right] - \lambda(t)N(t). \qquad (3.1)$$

Now we assume that r, K, m and λ are all positive continuous T-periodic functions. Function $g(t)$ is defined as $g(t) = t - \theta(t)$, where $\theta(t)$ is a T-periodic function. If fish reproduce primarily in the spring of the year, the functions $r(t)$ and $m(t)$ might be modeled by some periodic functions, such as

$$r(t) = r_0 + A_r \cos(2\pi(t - 0.25)),$$

where $0 < A_r \leq r_0$,

$$m(t) = m_0 + A_m \cos(2\pi(t - 0.25)).$$

If the food supply peaks each year in the fall, then

$$K(t) = K_0 + A_k \cos(2\pi(t - 0.75)),$$

where $0 < A_k \leq K_0$. To introduce a periodic harvesting rate we used

$$\lambda(t) = 0.5 \sin\frac{\pi(t - n - t_{start})}{H}$$

if $n + t_{start} < t < n + t_{start} + H, n = 0, 1, 2, ..,$ and $\lambda(t) = 0$ otherwise, where H is the harvesting time, t_{start} is the harvest starting time within one year, n is the number of years, e.g., if $H = 0.25$ and $t_{start} = 0.25$ (harvest in the summer season only).

Denote $M(t) = m(t) - \lambda(t)$. Equation (3.1) takes the following form

$$\frac{dN}{dt} = N(t)\left[\frac{r(t)}{1+\left(\frac{N(t-\theta(t))}{K(t)}\right)^{\gamma}} - M(t)\right]. \qquad (3.2)$$

Theorem 3.1 *Let $M(t), K(t)$, $\theta(t)$ and $r(t)$ be T-periodic functions. Suppose (a1)-(a4) hold and $M(t) \geq m > 0, r(t) - M(t) \geq r > 0$ for some positive constants m and r. If at least one of the following conditions hold:*

(b1) $\inf_{t\geq 0} \left[\frac{r(t)}{M(t)} - 1\right] K^{\gamma}(t) > 1$

(b2) $\sup_{t\geq 0} \left[\frac{r(t)}{M(t)} - 1\right] K^{\gamma}(t) < 1$

then equation (3.2) has at least one periodic positive solution $N_0(t)$.

To prove this result we will use the following Lemma [14].

Lemma 3.1 *Consider the delay differential equation*

$$\frac{dN}{dt} = \pm N(t)G(t, N(t - \theta(t, N(t)))), \tag{3.3}$$

where functions G and $\theta \in C(R^2, R)$. We also assume that G and θ are T-periodic functions with respect to the first argument. Suppose that there exist constants B, α and $\beta > 0$ such that:

(I) when $|x| < B$, the inequality $|G(t, e^x)| \leq \beta$ holds uniformly for $t \in R$, and when $|x| > B$, the inequality $xG(t, e^x) > 0$ holds uniformly for $t \in \mathbb{R}$.

(II) one of the following conditions holds:

(i) when $x < -B$, then $G(t, e^x) > -\alpha$ holds uniformly for $t \in \mathbb{R}$

(ii) when $x > B$, $G(t, e^x) < \alpha$ holds uniformly for $t \in \mathbb{R}$.

Then equation (3.3) has at least one positive T-periodic solution.

Proof of Theorem 2. Denote

$$G(t, u) = M(t) - \frac{r(t)}{1 + \left(\frac{u}{K(t)}\right)^{\gamma}},$$

then

$$G(t, e^x) = M(t) - \frac{r(t)}{1 + \left(\frac{e^x}{K(t)}\right)^{\gamma}}.$$

We have

$$|G(t, e^x)| \leq M(t) + r(t) \leq \sup_{t\geq 0}[M(t) + r(t)] = \beta.$$

Hence for every $B > 0$ the first part of statement (I) of Lemma 1 holds for $|x| < B$. Now we have to prove that for $|x| > B$ the inequality $xG(t, e^x) > 0$ is true.

Suppose (b1) holds. The inequality $G(t, e^x) > 0$ is equivalent to

$$x > \frac{1}{\gamma} \ln \left[\frac{r(t)}{M(t)} - 1 \right] K^\gamma(t).$$

Denote

$$B = \frac{1}{\gamma} \sup_{t \geq 0} \ln \left[\frac{r(t)}{M(t)} - 1 \right] K^\gamma(t).$$

Condition (b1) implies that $B > 0$. We also have $B < \infty$. Hence for $x > B$ we have $xG(t, e^x) > 0$. Let $x < -B$. The inequality $G(t, e^x) < 0$ is equivalent to

$$x < \frac{1}{\gamma} \ln \left[\frac{r(t)}{M(t)} - 1 \right] K^\gamma(t).$$

This inequality is satisfied by (b1):

$$\ln \left[\frac{r(t)}{M(t)} - 1 \right] K^\gamma(t) > 0.$$

Therefore it follows that $xG(t, e^x) > 0$.

To check the first condition of part (II) of Lemma 1 we have

$$G(t, e^x) = M(t) - \frac{r(t)}{1 + \left(\frac{e^x}{K(t)} \right)^\gamma} \geq M(t) - r(t) > 0.$$

Hence for every positive x and α we have $G(t, e^x) > -\alpha$. Then first part of condition (II) of Lemma 1 holds, therefore equation (3.2) has at least one positive T-periodic solution. If the second condition (b2) is satisfied then the proof of the Theorem 3.1 is similar. $\square$

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