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Existence and uniqueness for neutral equations with state dependent delays

A. Gołaszewska, J. Turo*

Department of Mathematics, Gdańsk University of Technology, Narutowicza 11/12, 80-952 Gdańsk, Poland

Abstract. By means of the comparison method we proved an existence and uniqueness theorem for neutral equations with state dependent delays.

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1. Introduction

For any metric spaces U and W we denote by $C(U, W)$ the class of all continuous functions from U to W. Let E be an arbitrary Banach space with the norm $\|\cdot\|$. Let $a > 0$, $r > 0$, and $\mathbb{R}_+ = [0, +\infty)$. For a function $z : [-r, a] \to E$, and $t \in [0, a]$ we define the function $z_t : [-r, 0] \to E$ by $z_t(\tau) = z(t + \tau)$, $\tau \in [-r, 0]$. Given the functions $f : [0, a] \times C([-r, 0], E) \times C([-r, 0], E) \to E$, $\varphi \in C^1([-r, 0], E)$, and $\xi, \eta : [0, a] \times E \to [0, a]$.

We consider the problem

$$x'(t) = f(t, x_{\xi(t,x(t))}, x'_{\eta(t,x(t))}), \quad t \in [0, a], \tag{1.1}$$

$$x(t) = \varphi(t), \quad t \in [-r, 0], \tag{1.2}$$

where $x_{\xi(t,x(t))}$ is the restriction of x to the set $[\xi(t, x(t)) - r, \xi(t, x(t))]$, $t \in [0, a]$, and this restriction is shifted to the set $[-r, 0]$. The same convention is applied to $x'_{\eta(t,x(t))}$.

E-mail addresses: golaszewska@mif.pg.gda.pl (A. Gołaszewska), turo@mif.pg.gda.pl (J. Turo)

*Corresponding author

Put $x'(t) = z(t)$ for $t \in [0, a]$. Then the Cauchy problem (1.1), (1.2) is equivalent to the following problem

$$z(t) = f(t, (Vz)_{\xi(t,(Vz)(t))}, z_{\eta(t,(Vz)(t))}), \quad t \in [0, a], \tag{1.3}$$

$$z(t) = \varphi'(t), \quad t \in [-r, 0], \tag{1.4}$$

where

$$(Vz)(t) = \varphi(0) + \int_0^t z(s)ds. \tag{1.5}$$

Particular cases of equation (1.1) arise as a model for a two - body problem of classical electrodynamics and were studied extensively by Driver [4]-[6].

Neutral equations with state dependent delays have attracted the attentions of several authors in recent years [1], [4] - [13], [16].

Using the Banach fixed point theorem we can prove the existence and uniqueness theorem for problem (1.1), (1.2) (see Remark 2.3). Unfortunately this method involves strong conditions concerning the function f. This condition can be slightly weakened if it is supposed more on the functions ξ, and η.

In this paper we prove by using the comparison method an existence and uniqueness result for (1.1), (1.2) under conditions involving some relation between the Lipschitz constants of the function f, and the estimations imposed on the functions ξ, η (see Remark 3.1). A general formulation of the comparison method can be found in [20]. This method has been used in various versions and under various assumptions on given functions for different problems concerning ordinary or partial differential equations, integral differential equations, functional differential or functional integral equations, and general functional equations in some abstract spaces (see [2], [3], and [14]-[20]).

2. Assumptions and lemmas

We define

$$(Lg)(t) = l(t)g(\beta(t)), \qquad \text{and} \qquad (Kg)(t) = k(t) \int_0^{\alpha(t)} g(s)ds,$$

where $t \in [0, a]$, $g, k, l \in C([0, a], \mathbb{R}_+)$, and $\alpha, \beta \in C([0, a], [0, a])$. Put

$$L^0 = J, \qquad \text{and} \qquad L^n = LL^{n-1}, \qquad n = 1, 2, \ldots,$$

where J denotes the identity operator in $C([0, a], \mathbb{R})$. We can write

$$(L^n g)(t) = l_n(t)g(\beta_n(t)),$$

where

$$\beta_0(t) = t, \qquad \beta_{n+1}(t) = \beta(\beta_n(t)), \qquad n = 0, 1, \ldots, \qquad t \in [0, a],$$

$$l_0(t) = 1, \qquad l_{n+1}(t) = l(t)l_n(\beta(t)), \qquad n = 0, 1, \ldots, \qquad t \in [0, a].$$

Let us define

$$Mg = \sum_{n=0}^{+\infty} L^n g$$

with the pointwise convergence of the series in $[0, a]$. We need the following lemmas.

Lemma 2.1. *Suppose that functions $k, l, h \in C([0, a], \mathbb{R}_+)$ are nondecreasing, α, $\beta \in C([0, a], [0, a])$ are nondecreasing, $\alpha(t)$, $\beta(t) \in [0, t]$, and*

$$(Mh)(t) < +\infty, \qquad \bar{s}(t) = M(k\alpha)(t) < +\infty, \qquad t \in [0, a],$$

and

$$\sup_{t\in(0,a]} \frac{\bar{s}(t)}{t} < +\infty,$$

Then

(i) there exists $\bar{g} \in C([0, a], \mathbb{R}_+)$ which is a nondecreasing, and unique solution of the equation

$$g = MKg + Mh \tag{2.1}$$

in the class $P([0, a], \mathbb{R}_+)$ of upper semicontinuous functions defined on $[0, a]$;

(ii) the function $\bar{g}$ is a nondecreasing, and unique solution of the equation

$$g = Kg + Lg + h \tag{2.2}$$

in the class

$$P([0, a], \mathbb{R}_+, \bar{g}) = \{g \in P([0, a], \mathbb{R}_+) : \ \|g\|_\star < +\infty, \},$$

where

$$\|g\|_\star = \inf \{c \in \mathbb{R}_+ : \ g(t) \leqslant c\bar{g}(t), t \in [0, a]\};$$

(iii) the function $g = 0$ is the unique solution of the inequality

$$g \leqslant Kg + Lg \tag{2.3}$$

in the class $P([0, a], \mathbb{R}_+, \bar{g})$.

Proof. At first we prove (i). It is quite clear that the solution of equation (2.1) can be considered in the class $C([0, a], \mathbb{R}_+)$. Put

$$\|g\|_\chi = \sup_{t\in[0,a]} \exp(-\chi t)g(t), \quad g \in C([0, a], \mathbb{R}_+),$$

with $\chi > \Lambda = \sup\limits_{t\in(0,a]} \frac{\bar{s}(t)}{t}$.

Now we can prove that the operator MK is a contraction i. e. $\|MK\|_\chi < 1$. Indeed, from the inequality $\exp(\varepsilon t) - 1 \leqslant \varepsilon \exp(t)$ for $\varepsilon \in [0,1]$, $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
\|MKg\|_\chi &\leqslant \sup_{t\in[0,a]} \exp(-\chi t) \sum_{n=0}^{\infty} l_n(t)k(\beta_n(t)) \int_0^{\alpha(\beta_n(t))} g(s)ds \\
&\leqslant \sup_{t\in[0,a]} \exp(-\chi t) \sum_{n=0}^{+\infty} l_n(t)k(\beta_n(t)) \int_0^{\alpha(\beta_n(t))} [g(s)\exp(-\chi s)]\exp(\chi s)ds \\
&\leqslant \|g\|_\chi \sup_{t\in[0,a]} \exp(-\chi t) \sum_{n=0}^{+\infty} l_n(t)k(\beta_n(t)) \int_0^{\alpha(\beta_n(t))} \exp(\chi s)ds \\
&\leqslant \frac{\|g\|_\chi}{\chi} \sup_{t\in[0,a]} \exp(-\chi t) \sum_{n=0}^{+\infty} l_n(t)k(\beta_n(t))[\exp(\chi\alpha(\beta_n(t))) - 1] \\
&\leqslant \frac{\|g\|_\chi}{\chi} \sup_{t\in(0,a]} \exp(-\chi t) \sum_{n=0}^{+\infty} l_n(t)k(\beta_n(t)) \left[\exp(\chi \frac{\alpha(\beta_n(t))}{t} t) - 1\right] \\
&\leqslant \frac{\|g\|_\chi}{\chi} \sup_{t\in(0,a]} \exp(-\chi t) \sum_{n=0}^{+\infty} l_n(t)k(\beta_n(t))\alpha(\beta_n(t))\frac{1}{t}\exp(\chi t) \\
&\leqslant \frac{\|g\|_\chi}{\chi} \sup_{t\in(0,a]} \frac{\bar{s}(t)}{t} \\
&\leqslant \frac{\Lambda}{\chi}\|g\|_\chi.
\end{aligned}$$

Hence it follows that $\|MK\|_\chi < 1$. Now the assertion (i) follows from the Banach fixed point theorem.

Now we prove (ii). At first we show that any solution of equation (2.1) is a solution of equation (2.2). Indeed, if $\bar{g}$ is a solution of (2.1), then from the equality $LMg = Mg - g$ we get

$$\begin{aligned}
K\bar{g} + L\bar{g} + h &= K\bar{g} + L(MK\bar{g} + Mh) + h \\
&= K\bar{g} + LMK\bar{g} + LMh + h \\
&= K\bar{g} + MK\bar{g} - K\bar{g} + Mh - h + h \\
&= MK\bar{g} + Mh \\
&= \bar{g}.
\end{aligned}$$

We observe that for any solution $\bar{g}$ of equation (2.1)

$$L^n\bar{g} = L^nMK\bar{g} + L^nMh = \sum_{i=n}^{+\infty} L^iK\bar{g} + \sum_{i=n}^{+\infty} L^ih,$$

hence we get

$$L^n \bar{g} \to 0 \quad \text{if} \quad n \to +\infty.$$

If $\tilde{g} \in P([0,a], \mathbb{R}_+, \bar{g})$ is a solution of equation (2.2), then by induction we obtain easily the following

$$\tilde{g} = \sum_{i=0}^{n-1} L^i K \tilde{g} + \sum_{i=0}^{n-1} L^i h + L^n \tilde{g}, \quad n = 1, 2, \dots \tag{2.4}$$

Since $\tilde{g} \in P([0,a], \mathbb{R}_+, \bar{g})$, then for some $c \geqslant 0$ we have $0 \leqslant \tilde{g} \leqslant c\bar{g}$, now according to $L^n \tilde{g} \leqslant c L^n \bar{g}$, we infer $L^n \tilde{g} \to 0$ if $n \to +\infty$. If we let $n \to +\infty$ in relation (2.4) we get $\tilde{g} = MK\tilde{g} + Mh$ i.e. $\tilde{g}$ is the solution of (2.1), but this equation has only the solution $\bar{g}$, thus $\tilde{g} = \bar{g}$, and (ii) is proved.

Finally we prove (iii). If $g \in P([0,a], \mathbb{R}_+, \bar{g})$ is the solution of inequality (2.3) then by induction we get

$$g \leqslant \sum_{i=0}^{n-1} L^i K g + L^n g, \quad n = 1, 2, \dots$$

We have for some $c \in \mathbb{R}_+$, $g \leqslant c\bar{g}$. From here we find that g satisfies the inequality $g \leqslant MKg$.

Because of $\|MK\|_\chi < 1$ we get that $g = 0$ is the unique solution of (2.3) in the class $C([0,a], \mathbb{R}_+)$ with the norm $\|\cdot\|_\chi$. Thus $g = 0$ is the unique solution of (2.3) in the class with the supremum norm. Lemma is proved. □

Remark 2.1. *If assumptions of Lemma 2.1 are satisfied for $\bar{h} \in C([0,a], \mathbb{R}_+)$, where $\bar{h}(t) \leqslant h(t)$, $t \in [0,a]$, then the suitable solution $\tilde{g}$ of equation (2.1) with $\bar{h}$ instead of h established in Lemma 2.1, is the unique solution of the equation (2.2) with h replaced by $\bar{h}$ in the class $P([0,a], \mathbb{R}_+, \bar{g})$.*

This fact follows immediately from the part (ii) of the proof of Lemma 2.1.

In the space $C([-r,0], E)$ we define the norm

$$\|v\|_0 = \sup_{\tau \in [-r,0]} \|v(\tau)\|,$$

where $v \in C([-r,0], E)$. We write

$$B([-r,a], \bar{g}) = \left\{ u \in C([-r,a], E) : \ u\,|_{[-r,0]} = \varphi', \ \|u(t)\| \leqslant \bar{g}(t), \ t \in [0,a] \right\},$$

where $\bar{g}$ is defined in Lemma 2.1.

Assumption H_1**.** Suppose that

(i) there exist nondecreasing functions $\bar{k}, \bar{l}, \sigma, \delta : [0,a] \to \mathbb{R}_+$, and $\bar{\alpha}, \bar{\beta} : [0,a] \to [0,a]$, such that $\bar{\alpha}(t)$, $\bar{\beta}(t) \in [0,t]$, and

$$\|f(t,u,v) - f(t,\bar{u},\bar{v})\| \leqslant \bar{k}(t)\|u - \bar{u}\|_0 + \bar{l}(t)\|v - \bar{v}\|_0,$$

$$\xi(t,y) \leqslant \bar{\alpha}(t), \quad \eta(t,y) \leqslant \bar{\beta}(t),$$
$$|\xi(t,y)-\xi(t,\bar{y})| \leqslant \sigma(t)\|y-\bar{y}\|, \quad |\eta(t,y)-\eta(t,\bar{y})| \leqslant \delta(t)\|y-\bar{y}\|$$

for $(t,u,v),(t,\bar{u},\bar{v}) \in [0,a]\times C([-r,0],E)\times C([-r,0],E), y,\bar{y}\in E$;

(ii) $\varphi \in C^1([0,a],E)$ and $\|\varphi'(\tau)\| \leqslant \bar{g}(0)$ for $\tau \in [-r,0]$.

The following estimation is a consequence of the assumption H_1:

$$\|f(t,u,v)\| \leqslant \bar{k}(t)\|u\|_0 + \bar{l}(t)\|v\|_0 + \gamma(t),$$

where $(t,u,v)\in[0,a]\times C([-r,0],E)\times C([-r,0],E)$, and $\gamma(t) = \sup_{s\in[0,t]} \|f(s,\theta,\theta)\|$, and θ denotes the zero in the space $C([-r,0],E)$. We define the operator $\mathcal{F}$ as follows

$$\begin{aligned} \mathcal{F}[z](t) &= f(t,(Vz)_{\xi(t,(Vz)(t))}, z_{\eta(t,(Vz)(t))}), \quad t\in[0,a], \\ \mathcal{F}[z](t) &= \varphi'(t), \quad t\in[-r,0], \end{aligned}$$

where V is given by (1.5).

Lemma 2.2. *If Assumption H_1, and assumptions of Lemma 2.1 are satisfied with $\alpha(t)=\bar{\alpha}(t)$, $\beta(t)=\bar{\beta}(t)$, $l(t)=\bar{l}(t)$, $k(t)=\bar{k}(t)$, $h(t)=\gamma(t)+\bar{k}(t)\|\varphi(0)\|$, and let $\bar{g}$ be the corresponding solution of (2.2), then*

$$\mathcal{F}: B([-r,a],\bar{g}) \to B([-r,a],\bar{g}),$$

Proof. Let $v\in B([-r,a],\bar{g})$, and $w(t)=\mathcal{F}[v](t)$. Then for $t\in[0,a]$ we have

$$\begin{aligned} \|w(t)\| &= \|f(t,(Vv)_{\xi(t,(Vv)(t))}, v_{\eta(t,(Vv)(t))})\| \\ &\leqslant \bar{k}(t)\|(Vv)_{\xi(t,(Vv)(t))}\|_0 + \bar{l}(t)\|v_{\eta(t,(Vv)(t))}\|_0 + \gamma(t) \\ &\leqslant \bar{k}(t)\int_0^{\bar{\alpha}(t)} \bar{g}(s)ds + \bar{l}(t)\bar{g}(\bar{\beta}(t)) + \bar{k}(t)\|\varphi(0)\| + \gamma(t) \\ &= \bar{g}(t). \end{aligned}$$

Therefore $\|w(t)\| \leqslant \bar{g}(t)$ for $t\in[0,a]$. Hence it follows that $w\in B([-r,a],\bar{g})$, and the lemma is proved. □

Assumption H_2. Suppose that

(i) there exist $p,b,d\in\mathbb{R}_+$, such that

$$\|f(t,u,v)-f(\bar{t},u,v)\| \leqslant p|t-\bar{t}|,$$

$$|\xi(t,y)-\xi(\bar{t},y)| \leqslant b|t-\bar{t}|, \quad \text{and} \quad |\eta(t,y)-\eta(\bar{t},y)| \leqslant d|t-\bar{t}|$$

for $\|v\|_0 \leqslant \rho = \bar{g}(a)$, $\|u\|_0 \leqslant \bar{\rho} = \bar{\alpha}(a)\rho + \|\varphi(0)\|$, and $\|y\| \leqslant \tilde{\rho} = a\rho + \|\varphi(0)\|$,

(ii) the compatibility condition

$$\varphi'(0_-) = f(0, \varphi, \varphi'),$$

is satisfied, where $\varphi'(0_-)$ denotes the left hand derivative of the function φ at the point $t = 0$.

Put

$$A = p + \bar{k}(a)\bar{g}(a)[b + \bar{g}(a)\sigma(a)], \quad \text{and} \quad B = \bar{l}(a)[d + \bar{g}(a)\delta(a)].$$

We introduce the following class of functions

$$D([-r,a],\bar{g},\lambda) = \{z \in B([-r,a],\bar{g}) : \|z(t) - z(\bar{t})\| \leqslant \lambda|t - \bar{t}|,\ t,\ \bar{t} \in [0,a]\},$$

where the constant λ is fixed, and it satisfies the condition $\lambda \geqslant A[1 - B]^{-1}$.

Lemma 2.3. *If Assumption H_2, and assumptions of Lemma 2.2 are satisfied, and if $B < 1$, then the operator $\mathcal{F}$ maps $D([-r,a],\bar{g},\lambda)$ into itself.*

Proof. Let $z \in D([-r,a],\bar{g},\lambda)$. It follows from Lemma 2.2, that $\mathcal{F}[z] \in B([-r,a],\bar{g})$. Now we have

$$\begin{aligned} \|\mathcal{F}[z](t) - \mathcal{F}[z](\bar{t})\| &\leqslant p|t-\bar{t}| + \bar{k}(t)\|(Vz)_{\xi(t,(Vz)(t))} - (Vz)_{\xi(\bar{t},(Vz)(\bar{t}))}\|_0 \\ &\quad + \bar{l}(t)\|z_{\eta(t,(Vz)(t))} - z_{\eta(\bar{t},(Vz)(\bar{t}))}\|_0 \\ &\leqslant p|t-\bar{t}| + \bar{k}(t)\rho|\xi(t,(Vz)(t)) - \xi(\bar{t},(Vz)(\bar{t}))| \\ &\quad + \bar{l}(t)\lambda|\eta(t,(Vz)(t)) - \eta(\bar{t},(Vz)(\bar{t}))| \\ &\leqslant p|t-\bar{t}| + \bar{k}(t)\rho[b|t-\bar{t}| + \sigma(t)\|(Vz)(t) - (Vz)(\bar{t})\|] \\ &\quad + \bar{l}(t)\lambda[d|t-\bar{t}| + \delta(t)\|(Vz)(t) - (Vz)(\bar{t})\|] \\ &\leqslant (A + B\lambda)|t-\bar{t}| \leqslant \lambda|t-\bar{t}| \end{aligned}$$

for $t,\ \bar{t} \in [0,a]$. Hence it follows that $\mathcal{F}[z] \in D([-r,a],\bar{g},\lambda)$, and the proof is complete. □

Remark 2.2. *If $E = \mathbb{R}^n$, and the Assumptions of Lemma 2.3 are satisfied, then the problem (1.3), (1.4) has at least one solution $\bar{z} \in D([-r,0],\bar{g},\lambda)$.*

We see at once that the continuous operator $\mathcal{F}$ maps the bounded, closed, and convex set $D([-r,0],\bar{g},\lambda)$ into its compact subset $\mathcal{F}[D([-r,a],\bar{g},\lambda)]$. Hence, and from the Schauder fixed - point theorem it follows that $\mathcal{F}$ has at least one fixed point.

For an arbitrary Banach space we have the following result.

Remark 2.3. *If assumptions of Lemma 2.3 are satisfied, and $q < 1$, where*

$$q = a\left\{\bar{k}(a)[\rho\sigma(a) + 1] + \lambda\bar{l}(a)\delta(a)\right\} + \bar{l}(a),$$

then problem (1.3), (1.4) has a unique solution in $D([-r,a],\bar{g},\lambda)$.

It is obvious that under these assumptions the operator $\mathcal{F}$ is a contraction in the space $D([-r,0],\bar{g},\lambda)$. The assertion of this remark follows from the Banach fixed - point theorem.

We shall relax this restrictive condition.

3. The main theorem

For the function $v \in C([-r,0],E)$ we define the function $\omega v : [-r,a] \to E$ by

$$\begin{aligned} (\omega v)(t) &= v(t), \qquad t \in [-r,0], \\ (\omega v)(t) &= v(0), \qquad t \in [0,a]. \end{aligned}$$

Let us define the sequence $\{z_n\}$, where z_0 is an arbitrary function from the space $B([-r,a],\bar{g})$, by relations
(i) $z_0(t) = (\omega\varphi')(t)$ for $t \in [-r,a]$,
(ii) if $z_n : [-r,a] \to E$ is given then

$$\begin{aligned} z_{n+1}(t) &= \mathcal{F}[z_n](t) \quad \text{for} \quad t \in [0,a], \\ z_{n+1}(t) &= \varphi'(t) \quad \text{for} \quad t \in [-r,0]. \end{aligned}$$

To prove the convergence of the sequence $\{z_n\}$ we define the sequence $\{g_n\}$ as follows

$$\begin{aligned} g_{n+1} &= Kg_n + Lg_n, \qquad n = 0,1,\ldots, \\ g_0 &= \bar{g}, \end{aligned}$$

where $\bar{g}$ is a solution of equation (2.2) with functions k, l, α, β, and h given by

$$\begin{cases} k(t) = \bar{k}(t)[1+\rho\sigma(t)] + \lambda\bar{l}(t)\delta(t), \\ l(t) = \bar{l}(t), \\ \alpha(t) = t, \\ \beta(t) = \bar{\beta}(t), \\ h(t) = \max\limits_{s\in[0,t]} \|\mathcal{F}[\omega\varphi'](s) - \varphi'(0)\|, \end{cases} \tag{3.1}$$

and $t \in [0,a]$. By induction, we can prove the following lemma (see [18]).

Lemma 3.1. *Suppose that assumptions of Lemma 2.1 are satisfied with functions k, l, α, β, h given by relations (3.1). Then*

$$0 \leqslant g_{n+1} \leqslant g_n \leqslant \bar{g}, \qquad n = 0,1\ldots,$$

and

$$\lim_{n\to+\infty} g_n(t) = 0 \quad \textit{uniformly on} \quad [0,a].$$

Theorem 3.1. *If Assumptions H_1, H_2, and assumptions of Lemma 2.1 are satisfied for functions k, l, α, β, and h defined by relations (3.1) then there exists the only one solution $\bar{z} \in D([-r,a],\bar{g},\lambda)$ of the problem (1.3), (1.4). The sequence $\{z_n\}$ is convergent to $\bar{z}$ uniformly on $[0,a]$, and the following estimations*

$$\|\bar{z}(t) - z_n(t)\| \leqslant g_n(t), \qquad n = 0,1,\ldots, \qquad t \in [0,a], \tag{3.2}$$

hold.

Proof. First we note that from assumptions of this theorem it follows that the Assumptions of Lemmas 2.2 and 2.3 are satisfied. Hence $z_n \in D([-r,a],\bar{g},\lambda)$. Now we prove the estimations

$$\|z_n(t) - z_0(t)\| \leqslant \bar{g}(t), \quad n = 0,1,\ldots, \quad t \in [0,a], \tag{3.3}$$

and

$$\|z_{n+k}(t) - z_n(t)\| \leqslant g_n(t), \quad n,k = 0,1,\ldots, \quad t \in [0,a]. \tag{3.4}$$

Estimate (3.3) is obvious for $n = 0$. Assume that estimate (3.3) holds for a certain $n > 0$. Then for $n+1$ we have

$$\begin{aligned}
\|z_{n+1}(t) - z_0(t)\| &\leqslant \|\mathcal{F}[z_n](t) - \mathcal{F}[z_0](t)\| + \|\mathcal{F}[z_0](t) - z_0(t)\| \\
&\leqslant \bar{k}(t)\|(Vz_n)_{\xi(t,(Vz_n)(t))} - (Vz_0)_{\xi(t,(Vz_0)(t))}\|_0 \\
&\quad + \bar{l}(t)\|(z_n)_{\eta(t,(Vz_n)(t))} - (z_0)_{\eta(t,(Vz_0)(t))}\|_0 + h(t) \\
&\leqslant \left\{\bar{k}(t)[1+\rho\sigma(t)] + \bar{l}(t)\lambda\delta(t)\right\}\int_0^t \bar{g}(s)ds + \bar{l}(t)\bar{g}(\bar{\beta}(t)) + h(t) \\
&= \bar{g}(t),
\end{aligned}$$

so the estimate (3.3) holds for $n = 0,1,\ldots$, $t \in [0,a]$. In the same manner we can prove the estimate (3.4). It follows from Lemma 3.1, that the sequence $\{z_n\}$ is convergent to the solution $\bar{z}$ of the problem (1.3), (1.4). It is obvious, that $\bar{z} \in D([-r,a],\bar{g},\lambda)$. Letting $k \to +\infty$ in the estimate (3.4) we get the estimate (3.2) holds.

To prove uniqueness we assume that $\tilde{z} \in D([-r,a],\bar{g},\lambda)$ is another solution of the problem (1.3), (1.4). Let

$$w(t) = \max_{s\in[0,t]} \|\tilde{z}(s) - \bar{z}(s)\|.$$

Now we have

$$\begin{aligned}
w(t) &\leqslant \max_{s\in[0,t]} \bar{k}(s)\|(V\tilde{z})_{\xi(s,(V\tilde{z})(s))} - (V\bar{z})_{\xi(s,(V\bar{z})(s))}\|_0 \\
&\quad + \max_{s\in[0,t]} \bar{l}(s)\|\tilde{z}_{\eta(s,(V\tilde{z})(s))} - \bar{z}_{\eta(s,(V\bar{z})(s))}\|_0 \\
&\leqslant \max_{s\in[0,t]} \left\{\bar{k}(s)[1+\rho\sigma(s)] + \lambda\delta(s)\bar{l}(s)\right\}\int_0^s \|\tilde{z} - \bar{z}\|_\tau d\tau + \max_{s\in[0,t]} \bar{l}(s)\|\tilde{z} - \bar{z}\|_s \\
&\leqslant (Kw)(t) + (Lw)(t).
\end{aligned}$$

Therefore w is a solution of the inequality (2.3), thus $w(t) = 0$, and $\tilde{z}(t) = \bar{z}(t)$ for $t \in [0,a]$. The proof is finished. □

Remark 3.1. *If Assumptions H_1, and H_2 are satisfied, and if functions $k,l,h \in C([0,a],\mathbb{R}_+)$, $\alpha,\beta \in C([0,a],[0,a])$ are defined by relations (3.1), and there exist $\tilde{l},\tilde{k} \in \mathbb{R}_+$, $\tilde{\beta} \in [0,1]$, such that $l(t) \leqslant \tilde{l}$, $k(t) \leqslant \tilde{k}$, $\beta(t) \leqslant \tilde{\beta}t$, and $h(t) \leqslant Ht^\mu$ for a certain $H,\mu \in \mathbb{R}_+$, then the assertion of Theorem 3.1 holds, if $\tilde{l}\tilde{\beta}^\mu < 1$, $B < 1$, and $A[1-B]^{-1} \leqslant \lambda$.*

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