A Novel Numerical Corrective Technique to Mass Spring Systems

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This paper presents a numerical corrective technique that allows control of nonlinearity in a mass-spring system (MSS) independent of its spring constants or system topology. The governing equations of MSS in the form of ordinary differential equations or a regular function companied by any boundary or initial condition as known constraints are employed to modify the results. A least-squares algorithm coupled with the finite difference method is used to discretize the basic residual function implemented in this corrective technique. This numerical solution is applicable to both static and dynamic MSS. This technique is easy to implement and has similar accuracy as the equivalent finite element method (FEM) solution to the same system with the solutions are obtained in a fraction of the CPU time required for FEM. The proposed technique can also be used to smooth solutions from other methods such as finite element method or boundary element method (BEM).

Keywords: Mass-spring system, finite difference method, least square algorithm, finite element method, soft tissue mechanics

1. INTRODUCTION

Mass-spring systems (MSS) have been extensively used in mechanical simulation of soft tissues over the past fifteen years [1, 2]. They are also used in modeling of deformable objects for facial animation, animation of artificial animals, cloth modeling and recently in surgical simulation [3-14]. Although, MSS are known as non-time-consuming solvers, it suffers from the lack of accuracy as compared to other numerical techniques such as the finite element method (FEM) [24]. To address this shortcoming, numerical corrective techniques have been developed for the MSS. The approaches used include the inverse dynamics procedure to eliminate super elongation of the springs, the implicit integration method to take large time steps and the heuristic method of handling post-buckling instability for robust modeling [15-18].

Both isotropic and anisotropic elastic materials can be found among the objects to animate. For instance, most soft tissues are strongly anisotropic, due to their fiber structure and/or composite nature. One of the main limitations with the mass-spring systems is that neither isotropic nor anisotropic materials can be generated and controlled easily [19-23].

In this study a new numerical corrective technique to MSS is developed by which the accuracy of MSS results is enhanced by utilizing the material properties and the boundary and/or initial conditions of the system. FORTRAN custom code is used to implement the corrected MSS technique in this study. Application of this new corrective technique is illustrated by three examples. These include the force-displacement of a nonlinear spring, a highly nonlinear displacement of a nonlinear spring and a highly non-linear stress-strain relationship of a polymer hydrogel, a biomaterial that mimics cardiovascular tissue properties [25].

2. METHOD

2.1. Governing Equation for MSS

The internal force of a MSS which is due to the changes of spring length and velocity takes the form of:

$$F_{\text{int}} = [K(\|x\| - x_0) + D(\frac{\ddot{x} \bullet \vec{v}}{|x|})] \frac{\ddot{x}}{|x|}$$  

(1)

Where $\ddot{x}$ and $\vec{v}$ are position and velocity vectors between the two ends of the springs and $|x|$ and $x_0$ are the current and initial length of the spring, respectively. $K$ and $D$ are linear constants of the MSS elements representing the spring and dashpot constants.

Equation (1) can be represented in the following nonlinear form:

$$F_{\text{int}} = K(\cdot) \cdot x_K + D(\cdot) \cdot x_D$$  

(2)

$$x_K = (|x| - x_0) \frac{\ddot{x}}{|x|}$$  

(3)

$$x_D = \frac{\ddot{x} \bullet \vec{v}}{|x|} \frac{\ddot{x}}{|x|}$$  

(4)

Equations (3) and (4) are defined as directional vectors and the nonlinear functions $K(\cdot)$ and $D(\cdot)$ represent spring stiffness and damping function. Newton’s equation of motion can be rearranged in the following form:
\[ F_{\text{int}}(k+1) = \frac{M}{(\Delta t)^2} [\Delta x(k) - \Delta x(k-1)] - F_{\text{ext}} \]  
\[ (5) \]

Where \( \Delta x(k) \) and \( \Delta x(k-1) \) are position changes and \( F_{\text{ext}} \) refers to the external force as boundary conditions. Given the input and output data, nonlinear functions \( K(\cdot) \) and \( D(\cdot) \) can be approximated using, for instance, ordinary differential equation (ODE) numerical solvers or neural network approaches.

### 2.2. Corrective Technique to MSS

MSS is governed by a set of ODE as mentioned above. There is also additional information in the form of boundary or initial conditions that can also be formulated. These additional information accompanied together with the governing equations can be expressed as:

\[ R(u_i, \zeta_j, \eta_k) = \sum_i \frac{1}{2}(u_i - u_i)^2 + \sum_j \zeta_j F(u) + \sum_k \eta_k T(u) \]
\[ (6) \]

Where \( R \) is the residual function to the MSS, \( j \) is the number of internal points of the domain to be corrected and \( \zeta_j \) and \( \eta_k \) are the Lagrange multipliers. The certain quality to be corrected is displacement \( \tilde{u}_i(x) \) available in \( i \) points. The corrected values of the \( \tilde{u}_i(x) \) are considered as \( u_i(x) \). The governing equation of the MSS is \( F(u) = 0 \) where order of \( F \) is either 0 (static) and 2 (time-dependent). Also the additional information from the system such as boundary and initial conditions is \( T(u) = 0 \) and \( T \) is a known function of \( u \). Equation 6 is solved for \( u \) by minimizing the residual function \( R \).

It should be noted \( i \) and \( j \) have different ranges, \( i \) refers to all points of the domain including boundary points and additional constraints, \( j \) refers only to the internal points of the domain and \( k \) is the number of constraints where \( T(u_j) \) is defined. The residual function \( R \) should be minimized on the entire domain by taking the first derivative of the function \( R \) with respect to \( u \) as such:

\[ \frac{\partial R}{\partial u} = \sum_i \partial (u - u_i)^2 + \sum_j \zeta_j \frac{\partial F(u)}{\partial u} + \sum_k \eta_k \frac{\partial T(u)}{\partial u} = 0 \]
\[ (7) \]

Also the first derivative of the function \( R \) with respect to \( \zeta_j \) and \( \eta_k \) delivers the following complementary set of equations:

\[ F(u) = 0 \]
\[ T(u) = 0 \]
\[ (8) \]
\[ (9) \]

The function \( F \) can be replaced by the equivalent finite difference forms calculated for the points \( j \). Equations (6) and (7) are now reduced to a set of algebraic equations, with \( u \) being the only variable. These sets of equations can be solved by any relevant numerical technique as they are in the form of linear or non-linear algebraic equations. The order of function \( F \) which is called \( r \) defines the number of connected points, which are to be solved in each step. When order of \( F \) is 2 (time-dependent), three consecutive values: \( u_{j-1}, u_j \) and \( u_{j+1} \) are defined and consequently \( j = i - 2 \) as \( j = i - r \), because equation (8) and (9) can only be written for the internal points of the domain.

Equations (6) can also be written in matrix form, as such:

\[ R = \frac{1}{2}(u - u)^2 + (Fu)^T \psi \]
\[ (10) \]

where:

\[ \psi = \begin{bmatrix} \zeta \\ \eta \end{bmatrix} \]
\[ (11) \]

and eventually the matrix form of the error function \( R \) takes the form of:

\[ \begin{bmatrix} I & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ \zeta \end{bmatrix} = \begin{bmatrix} \bar{u} \\ 0 \end{bmatrix} \]
\[ (12) \]

where \( I \) is the unit diagonal matrix and \( B \) is the matrix resulting from the additional equations (8) and (9), which is of the order \( i + j + k \). For ease of calculations, equation (10) can be broken down to two lower systems of the order \( i \) and \( j + k \). The new partitioning of equation (10) takes the matrix form, as such:

\[ Iu + B^T \psi = \bar{u} \]
\[ (13) \]

\[ Bu = 0 \]
\[ (14) \]

Equation (13) can be multiplied by \( B \):

\[ BLu + BB^T \psi = B \bar{u} \]
\[ (15) \]

Using equation (15) the Lagrange multipliers matrix can be computed as follows:

\[ \psi = SB \bar{u} \]
\[ (16) \]

and,

\[ u = Y \bar{u} \]
\[ (17) \]

where \( Y \) is defined as:

\[ Y = I - B^T S \]
\[ (18) \]

which is considered the corrective matrix where matrix \( S \) is defined as:

\[ S = (BB^T)^{-1} \]
\[ (19) \]

The order of matrix \( S \) is \((j + k) \times (j + k)\) and the correction matrix \( Y \) is essentially a square matrix of order
The correction matrix is unique for a given MSS and once it is solved it can be retained as a modulus for the MSS. It can also be proven that the Lagrange multipliers $\zeta_j$ and $\eta_j$ contributes to correction of the results and therefore, they can be considered as correction factors as well.

Now, the set of equations (6) to (19) can appropriately interpret the MSS and can be solved for $u$ and the multipliers $\zeta_j$ and $\eta_j$. The solution is unique since it refers only to the corresponding set of governing equations and selected constraints of the MSS.

3. RESULTS

3.1. Example 1 – Time Independent Application
(A non-linear MSS)

**Part I:** The force-displacement of a nonlinear spring ($f = ku^2$, where $f$, $u$ and $k$ refer to force, displacement and the spring constant respectively) under large deformation is considered. A linear spring is used for uncorrected results and is modified through the proposed technique. For this example, the governing equation is not a differential equation, and however, it is a second order polynomial. The functions $F$ and $T$, take the form of:

$$F(u) = f - ku^2 = 0 \quad (20)$$
$$T(u) = 0 \quad (21)$$

where $k$ is the spring constant which is set to 1 for this example. The minimized residual function can be written as such:

$$\frac{\partial R}{\partial u} = (\bar{u} - u) + 2ku \quad (22)$$

where the $B_{m \times n}$ matrix is written as ($m = n = 2$):

$$B = \begin{bmatrix} 2k & 0 \\ 0 & 2k \end{bmatrix} u \quad (23)$$

Given $B$ matrix (equation 23), equation 13 is solved for the corrective matrix $Y$. The $Y$ matrix is now available for the uncorrected results. The exact solutions to this simple model for the linear and the used nonlinear spring constant are in the form of linear and parabolic functions, respectively. The uncorrected, exact and modified results in company with the Lagrange multipliers are shown in Figure 1.

The trial function as uncorrected results for this simple system is linear. The corrected results are close to the result of the FEM simulation, with a deviation of less than 10%.

**Part II:** The solution to the Example 1 can be extended to a relatively complicated MSS. A rectangular polymeric plate ($5 \times 15$ cm$^2$) made of 10% polyvinyl alcohol (PVA) hydrogel is considered. The mechanical properties of the sample in the form of stress-strain relationship has been determined previously using a uniaxial tensile machine and are expressed as follows [25, 27, 28]:

$$\sigma = -0.05923 + 0.0611e^{2.2347\epsilon} \quad (24)$$

where $\sigma$ and $\mu$ are stress and strain, respectively and the Poisson’s ration of the used PVA sample was assumed to be 0.5 (incompressible).

The stress-strain behavior shows a highly non-linear behaviour similar to that of the porcine aortic root [25]. An iterative modified Newton-Raphson method to MSS and a validated finite element solver for soft material considering material nonlinearity has been reported previously [26]. In this finite element code hyperelastic isotropic elements have been employed along with a classical Mooney-Rivlin soft material model. The corrective technique to the results from MSS has also been calculated. The results of uncorrected MSS, modified MSS and nonlinear FEM are shown in Figure 2.

The modified MSS gives results that are close to FEM, with a percentage error of less than 5%. The corrected mass-spring approach is more accurate here than in the solution provided in part 1 as the trial function in this example is already nonlinear. Also in this example, the accuracy approaches the existing finite element solution at 1/20 of the CPU time on a Pentium IV with a CPU speed of 2.4 GHz and 512 MB RAM.

3.2. Example 2 – Time-dependent Application

A vibrating spring with a linear spring constant is considered. The exact results, which are in the form of a...
sinusoidal function, are randomly manipulated by a factor of more than 20% to examine the robustness of the proposed numerical technique for a time-dependent example. The equations (8) and (9) take the form of:

\[ F(u) = \ddot{u} - ku = 0 \]  \hspace{1cm} (25)

\[ T(u) = 0 \]  \hspace{1cm} (26)

where \( \ddot{u} \) is the second derivative of \( u \) with respect to time and \( k \) is the spring constant. The minimized residual function takes the form of:

\[ \frac{\partial R}{\partial u} = (\bar{u} - u) + \frac{\partial}{\partial u}(\ddot{u} - ku) = 0 \]  \hspace{1cm} (27)

The expansion of the equation (26) together with the implementation of finite difference to the equation (25) gives the following set of equations:

\[ (\bar{u}_0 - u_0) + \hat{\zeta}_1 = 0 \]  \hspace{1cm} (28)

\[ (\bar{u}_i - u_i) + (-2 + \hat{k})\hat{\zeta}_i + \hat{\zeta}_2 = 0 \]  \hspace{1cm} (29)

\[ \ldots \]

\[ (\bar{u}_j - u_j) + \hat{\zeta}_{j-1} + (-2 + \hat{k})\hat{\zeta}_j + \hat{\zeta}_{j+1} = 0 \]  \hspace{1cm} (30)

\[ \ldots \]

\[ (u_n - u_n) + \hat{\zeta}_{n-1} = 0 \]  \hspace{1cm} (31)

where \( \hat{\zeta}_i = \frac{\hat{\zeta}}{(\Delta t)^2} \) and \( \hat{k} = k_j (\Delta t)^2 \) and the \( B_{m,n} \) matrix for this case takes the form of \( m = 6, n = 4 \):

\[
B = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-2 + k & 1 & 0 & 0 \\
1 & -2 + k & 1 & 0 \\
0 & 1 & -2 + k & 1 \\
0 & 0 & 1 & -2 + k \\
0 & 0 & 0 & 1
\end{bmatrix}
\]  \hspace{1cm} (32)

The \( Y \) matrix is now calculated for the current MSS. The results of this solution and the exact solution for \( k=1 \) are presented in Figure 3 including the Lagrange multipliers.

Figure 2: (Right) the nonlinear FEM solution, (Left) the regular MSS solution for a 10% PVA rectangular plate \((5 \times 15 \text{ cm})^2\) for a deformation starting from 15% to 100%.

Figure 3: Motion of the vibrating spring for \( k=1 \), manipulated and modified results.

Although the manipulated data are designed to hold more than 20% error, the discrepancy between corrected and exact results are smaller.

The proposed technique can also be implemented on regressing other parameters of a MSS (for instance, for the spring constant used in example 2). For this case, the residual function \( R \) is the same except a new variable \( k \) is added up. An additional equation can be obtained where the residual function, \( R \), is minimized with respect to \( k \). This equation takes the form of: \( \sum_j \mu \psi = 0 \), which
is non-linear and can be solved iteratively using the Newton-Raphson algorithm or similar approaches. The calculated stiffness of the spring was $k = 1.2$ for uncorrected results.

4. CONCLUSION

A corrective numerical technique to modify the MSS results independent of the system topology or spring constants has been developed. Illustrated examples of both static and dynamic systems indicate the successful implementation of this approach to MSS and corrections of more than 80% is achievable. In the dynamic example of the vibrating spring, the corrective technique was able to improve the results to within 10% of the FEM results. In summary, the corrective numerical technique presented here is of general applicability to both static and dynamic applications of MSS. Since the proposed technique is computationally efficient and easy to implement, it can also be employed to smooth solutions obtained using other numerical techniques such as finite element, boundary element or finite difference methods to enhance the precision of the solutions.

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REFERENCES


