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Applications of Digital Geometry to Surface Reconstruction

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Digital geometry is a discipline dealing with geometric properties of digital objects. It provides an adequate mathematical background for new advanced approaches and algorithms for various problems arising in visual computing. The present paper is a brief survey on the applications of digital geometry to surface reconstruction, the latter being an important problem of image analysis and processing.

Keywords: *digital geometry, digital planarity, surface representation, multigrid convergent estimator, polyhedral complexity.*

1. INTRODUCTION

Digital geometry deals with geometric properties of *digital objects* (also called *digital pictures*). These are usually modeled as sets of points with integer coordinates representing the pixels/voxels of the considered digital objects. Digital geometry has its roots in a number of classical mathematical disciplines, such as number theory (since C. F. Gauss), geometry of numbers (since H. Minkowski), graph theory (since L. Euler), and combinatorial topology (since the middle of the 19th century). It has established itself as an independent discipline comparatively recently, in the second half of the 20th century, with the initiation of research in visual computing, including various applied areas such as image analysis and processing, computer vision, computer graphics and, more recently, multimedia technologies. The nature of the used research approaches and the obtained results put digital geometry on the border of applied mathematics and theoretical computer science, as the framework of the performed research is determined by practical applications in mind.

At present, research in digital geometry requires knowledge of a variety of mathematical disciplines, such as number theory, geometry of numbers, classical Euclidean geometry, analytical geometry, affine geometry, projective geometry, algebraic geometry, linear algebra (vector spaces, metric spaces), combinatorial geometry, discrete geometry, tilings and patterns, computational geometry, general topology, combinatorial topology, graph theory, linear programming, integer programming, Diophantine equations, polyhedral combinatorics, lattice polytopes, mathematical morphology, discrete dynamical systems, fractal theory, combinatorics on words, approximation theory, Diophantine approximations, continued fractions, probability theory and mathematical statistics, design and analysis of algorithms, and complexity theory, among others. Knowledge and approaches from the above-listed subjects are used to obtain theoretical results and design algorithms for solving various specific problems. Occasionally, results of digital geometry turn out to be known in different terms in the framework of earlier studies. Overall, however, digital geometry has provided a lot of new results, some of which are not only useful regarding specific practical applications, but also technically sound and deep from mathematical point of view.

Digital geometry is germane with *discrete geometry* that deals with similar matters but from a bit more general perspective (see the topics of Mathematical Subject Classification number 52Cxx). In particular, discrete geometry includes a number of subjects (e.g., ones related to matroid theory) that are not directly connected with computer imagery, and tackles them from more abstract point of view. Instead, digital geometry is closely focused on problems arising from image analysis and processing,computer graphics, computer vision, and related disciplines. Below we list some basic subjects of digital geometry, among others.

Digital topology (topology of digital objects, basic topological invariants, topology of digital curves and surfaces, topology of linear digital objects, classification of digital topologies).

Geometry of digital manifolds (geometry of digital curves and surfaces, digital straightness in 2D and 3D, digital planarity, length and curvature of digital arcs, area and curvature of digital surfaces, digital convexity).

Transformations (axiomatic digital geometry, transformation groups and symmetries, neighborhood-preserving transformations, magnification and demagnification).

Discrete Tomography

Morphologic operations (dilation, erosion, simplification, segmentation, decomposition)

Deformations (topology-preserving deformations, shrinking, thinning; deformations of curves, 3D pictures, and multivalued pictures).

Picture properties (moments, operations on pictures, invariant properties, spatial relations).

For detailed presentation of these and other areas of digital geometry the reader is referred to the recent monograph [41].

Digital geometry is developed with the expectation to provide an adequate theoretical (mathematical) background for new advanced approaches to and algorithms for solving various problems arising in visual computing.

In a recent paper [9] we have discussed the mathematical foundations, motivations, purposes, and basic directions of digital geometry. In the present survey we focus our attention on an important problem of visuall computing, known as *surface reconstruction*. Basically, given a set of voxels obtained, e.g., through digitization of some (usually unknown) real object, one aims to obtain a continuous (e.g., polyhedral) surface that faithfully models the surface of the original real object.

In the next Section 2 we provide an overview of some advantages and disadvantages of discrete and continuous object representations. In Section 3 we briefly present some basic approaches to polyhedrization. In Section 4, we consider more in detail the digital geometry approach. In Section 5 we exhibit relations between digital object polyhedrizationand multigrid convergent estimators. In Section 6 we discuss complexity issues. We conclude with some remarks in Section 7.

2. SURFACE REPRESENTATION

Traditionally objects in computer graphics are represented through their outer surfaces which are modeled in various ways: via meshes of polygons, 3D splines, or operations on curves. This approach, called *continuous*, is based on the classical Euclidean geometry and has a number of advantages (see, e.g., [35]) In particular, geometric transformations such as rotation and scaling can easily be performed applying simple formulas. Continuous representations have compact mathematical models and hence do not require a lot of memory. From these models, it is easy to obtain the area, perimeter, distances, and other object characteristics.

On the other hand, the main computer system components (the processor, memory, raster display, and keyboard) have a discrete nature. Thus, it is logical to represent the objects in *discrete* form too, e.g., as a set of 3D integer points contained in the object. This approach is very appropriate for sampled data, obtained through various types of scanners or measurements, as well as for porous objects and amorphous phenomena. Although discrete representations usually require more memory than their continuous counterparts, the visualization performance is insensitive to scene complexity and direct rendering can be performed. Certain Boolean and set operations such as intersection, union, and difference can easily be performed, which is not the case in continuous representation.

Clearly, each of these representations has certain advantages that are not available in the other one. Thus, it is an important task to develop techniques for correct conversion from continuous to discrete representation and vice versa. For example, a large amount of discrete medical information obtained by a scanner may need to be represented in a continuous (e.g., polyhedral) form for efficient storage. Alternatively, later on the continuous representation of that medical image may need to be converted back into a discrete form for the purposes of high quality visualization. Further discussion on the possible benefits of each form of representation is given next.

2.1 Digitization

Some basic advantages of discrete representations have been mentioned above. Because of these, related discrete models attract growing interest among a broad group of researchers. In addition to the major applications in medical imaging, one can list many in 3D image processing (e.g., time-varying 2D images), biology (e.g., confocal microscopy), geology (e.g., seismic measurement), synthetic volume visualization (e.g., in studies of galaxy, fluid dynamics, and molecular structures in chemistry), CAD (e.g., solid modeling), robot mapping using laser range finders, 3D animation, 3D simulation (e.g., instrumentation simulation), as well as applications merging empirical and synthetic images. Therefore, having very efficient and precise methods for digitization of continuous object is paramount.

Some methods for efficient digitization of polyhedral surfaces have been reviewed in [9]. In the rest of this paper we focus our attention on the continuous form of surface representation, more specifically, on polyhedrization of digital objects.

2.2 Polyhedrization

Let M be an arbitrary discrete set of integer points representing a real three-dimensional object S. M can be an empirical 3D medical image of S obtained using tomography scan of a human scalp, or a digital image of some live tissue obtained by magnetic resonance imaging techniques, or a synthetic image representing a medical instrument (such as an injection needle) or any technical devise (e.g., an aircraft). The problem of interest is to obtain a polyhedral representation of M. Because of its rich set of applications in solid modeling, computer vision, and computer graphics, this problem has been extensively studied by a large number of researchers that have applied diverse approaches (see Section 3). A recent monograph [25] summarizes some of the works based mainly on computational geometric techniques. An important requirement for such a representation is to preserve the topology of the original real object. Topological properties—such as connectivity, separability, and genus—play an important role since these are the most primitive object features to which the human visual system is well-adapted. This may be vital for the purposes of faultless simulations as well as from a point of view of the theory of knowledge representation (see Section 3).

Another issue is the optimality of the obtained representation. For instance, for the purposes of economic encoding one can require that the number of linear constraints defining the representation be minimal or as close to the minimal as possible. To certify such a closeness means that upper and lower bounds on the algorithm performance are to be established (see Section 6).

3. BASIC APPROACHES TO POLYHEDRIZATION

Given a digital image $M \subset \mathbb{Z}^3$, one looks for a (possibly, non-convex) polyhedron P, which is called a *continuous reconstruction* of M. We will also say that P *encloses* M. The problem of finding such a polyhedron P will be referred to as *polyhedrization* of M.

In recent years digital object polyhedrization is attracting an increasing interest, mainly driven by its rich set of applications. A substantial body of literature has been developed on the subject. A brief survey on the basic approaches and results is given next.

3.1 The Grid-Based Approach

A classical approach for generating an enclosing polyhedron P for a given set $M \subset \mathbb{Z}^3$ is the *grid-based approach*. The most popular method belonging to this class of methods is the Marching Cubes algorithm (MC)[48]. Informally, for each "border" voxel of M this algorithm generates a triangular patch of a surface contained in the voxel. Then all these patches are "glued" together thus forming a triangulated surface as a boundary of P.

Despite of its popularity, the MC algorithm has a number of shortcomings. Often, the number of the triangular facets of the obtained polyhedrization is very large, comparable with the number of the integer points of M. Another problem may be caused by some ambiguities in the polygon linking in the process of obtaining the polyhedral representation. As a result, P may contain small holes [26, 53]. Many authors have studied the problem and have proposed approaches to guarantee that the obtained polyhedrization is "topologically sound" (see, e.g., [17, 47, 26, 29, 45, 52, 53, 68]). However, the latter only means that the result is a true manifold. It does not necessarily ensure that the actual topology of the original real object S is preserved. The first result that provides such a guarantee for a certain class of 3D objects has been obtained recently. It is discussed in Section 3.3.

Let us mention that a recent work [6] proposes a grid-based algorithm for computing a polyhedral surface with the same topology as S. However, that algorithm assumes that the boundary of S is the zero-level set of an available implicit function g: $\mathbb{R}^3 \to \mathbb{R}$ of class C^2. Such a function is not necessarily known, in general.

3.2 Voronoi-Based Algorithms

Another approach to polyhedrization is based on the traditional computational geometry techniques [1, 25]. The basic idea is to define a Voronoi diagram whose vertices are the points of M and then to select a subset of triangles of the Voronoi diagram defining a triangular mesh. To get acquainted with the concept of a Voronoi diagram and its use in computational geometry the reader is referred to [55]. It is proved that if the set of integer points M satisfies a certain *sampling density restriction* then the obtained triangular mesh is guaranteed to be homeomorphic to the boundary of the digitized original real set S, that is, the topological properties of the object are preserved.

The main weakness of Voronoi-based algorithms is that in practice it is not so easy to assure the conditions under which such guarantee exists [15]. Thus, for larger sets M the algorithms of this type (e.g., [1, 25]) are basically non-applicable.

Similar to the grid-based algorithm from [6], there are also Voronoi-based topology preserving algorithms which assume that the boundary of S is the zero-level set of an available implicit function g: $\mathbb{R}^3 \to \mathbb{R}$ of class C^1 [7,16]. However, the function g may not be always known, in general.

3.3 Topology Preserving Methods

Topology deals with the invariance of fundamental object features such as connectivity and separability. As already mentioned, these are among the most primitive object characteristics to which the human visual system seems to be well-adapted. Since humans do not have direct access to spatial properties of real objects, the latter are usually represented as bounded subsets of the Euclidean space R^3. From point of view of the theory of knowledge representation, this relates two different representations of real world objects: discrete and continuous.

Two of the first books in computer vision deal with the relation between the continuous object and its digital images obtained by a digitization process. Pavlidis [54] and Serra [61] proved independently in 1982 that an r-regular continuous 2D set S and the continuous analog of the digital image of S are homeomorphic. (Recall that a set $A \subseteq \mathbb{R}^n$ is called *r-regular* if for each point x from the boundary of A there exist two tangent open balls of radius r at x such that one lies entirely in A and the other entirely in its complement.) An analogous result in the 3D case remained an open question for over 20 years. The problem was solved only recently [64]. A further task is seen in considering more general cases, e.g., when the original set is not r-regular.

4. ALGORITHMS BASED ON DIGITAL PLANARITY PROPERTIES

In this section we consider more in detail another class of algorithms that exploit properties of digital planarity and based on them routines for digital plane segment recognition

[8, 13, 23, 38, 39, 42, 62, 63, 67, 69]. Comprehensive surveys on digital planarity and related matters are available in [12, 20]. A short introduction to the matter is given next.

4.1 Digital Planarity

Similar to classical geometry, linear objects play a central role in digital geometry. Theoretical research on digital planarity is naturally driven by important practical applications in image analysis, pattern recognition and volume modeling. In this section we review some basic algorithms for digital plane recognition, digital surface segmentation, and digital polyhedra generation. Before that, let us recall one of the available several equivalent definitions of a digital plane.

Definition 1. A set $D_{a,b,c,\mu,\omega} = \{(i, j, k) \in \mathbb{Z}^3 : \mu \le ai + bj + ck < \mu + \omega\}$ is called a *digital plane* with *normal* $\mathbf{n} = (a, b, c)$, *intercept* μ, and *thickness* ω.

If $\omega = \max\{|a|, |b|, |c|\}$, then $D_{a,b,c,\mu,\omega}$ is called a *naive plane*, that is the thinnest hole-free digital plane. A digital plane with $\omega = |a| + |b| + |c|$ is called *standard*. A *digital plane segment* (DPS) is a connected portion of a digital plane. One can define lower (resp. upper) supporting points that determine the lower (resp. upper) supporting continuous planes defining a digital plane (see Figure 1). The preimage of a DPS, *S*, is the set of planes whose digitizations contain *S*. It appears to be the solution of a system of linear inequalities with unknowns α_1, α_2, and β. Thus, it is a convex polyhedron (possibly empty).

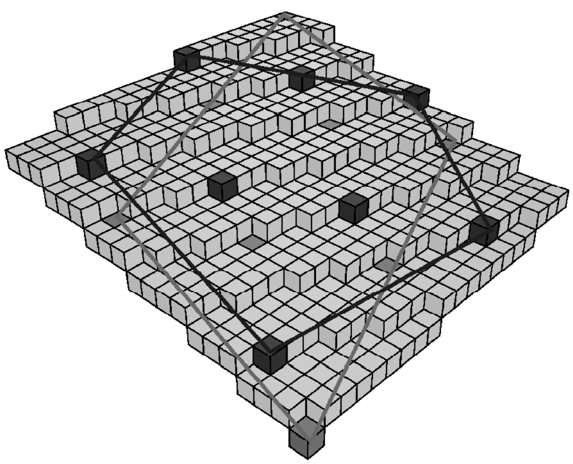

Figure 1: Illustration of a subset of a digital plane $D_{7,17,57,0,57}$ with its lower and upper convex hulls on the supporting planes

4.2 DPS Recognition

DPS recognition and digital surface segmentation are fundamental problems in image analysis. Table 1 lists different algorithms and their computational costs. All complexity bounds are given with respect to the number *n* of grid points in *S*.

In [65] a DPS recognition algorithm based on convex hull separability is proposed. The recognition of DPSs in grid adjacency models is discussed in [66, 40] (recognition by least-square optimization), and [50, 55, 67, 14] (linear programming when the dimension is fixed). [24] proposes an approach based on tests for existence of lower and upper supporting planes for a given set of points.

[28] suggests a recognition method for DPSs by converting the problem into a system of n^2 linear inequalities, where *n* is the cardinality of the given set of points. The system is solved by the Fourier elimination algorithm. One can also apply the CDD algorithm[1] for solving systems of linear inequalities by successive intersection of half-spaces defined by inequalities [30]. A very efficient incremental algorithm based on a similar approach is proposed in [42]. It also provides a polyhedrization of a given digital surface.

Typical timing results for the last three algorithms are shown in Figure 2, using a polyhedrized digital ellipsoid at grid resolutions ranging from 10 to 100.

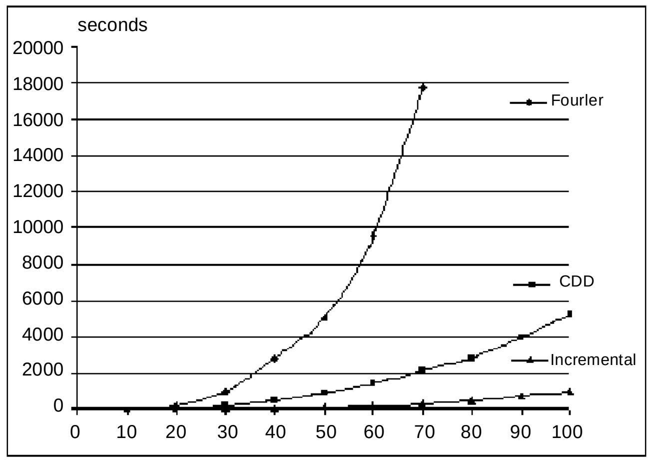

Figure 2: Running times of three DPS recognition algorithms on a PIII 450 running Linux

4.3 Polyhedrization Algorithms

Digital planarity based algorithms are all of greedy type. They augment "digital facets"—portions of digital planes—of the input set *M*. The augmentation process results in a "digital polyhedron" whose facets are "digital space polygons" that are portions of digital planes. Digital facets are subsequently transformed into polygons that constitute the facets of the resulting polyhedron.

Specifically, one of the most efficient algorithms [42] takes advantage of certain geometric properties of digital planes and repeatedly updates a list of supporting planes. The set of points is accepted as a DPS iff the final list of planes is non-empty. The updating step is time-efficient.

One can perform a breadth-first search of the face graph to agglomerate the faces into DPSs. Figure 3 illustrates results of the agglomeration process for a digitized sphere and for an ellipsoid with semi-axes 20, 16, and 12. Faces that have the same gray level belong to the same DPS. The respective numbers of faces of the digital surfaces of the sphere and ellipsoid are 7,584 and 4,744, respectively. The numbers of DPSs are 285 and 197; the average sizes of these DPSs are 27 and 24 faces.

Table 1
Algorithms for DPS Recognition

Reference	*Description*	*Complexity*	*Comments*
Kim 1984 [37]	Detection of a supporting plane	$O(n^4)$	Based on an incorrect theorem
Megiddo 1984 [50]	Linear programming	$O(n)$	
Preparata & Shamos 1985 [55]	Linear programming	$O(n \log n)$	Provides the complete preimage
Kim & Stojmenović 1991 [39]	Detection of a supporting plane	$O(n^2 \log n)$	Optimized [37], also based on an incorrect theorem
Stojmenović & Tosić 1991 [65]	Convex hull separability	$O(n \log n)$	
Veelaert 1994 [66]	Evenness property	$O(n^2)$	Rectang. DPS
Debled-Renesson & Reveillès 1994 [23]	Arithmetic structure	n.a.	Rectang. DPS
Reveillès 1995 [56]	Arithmetic geometry	$O(n)$	Rectang. DPS
Vittone & Chassery 2000 [67]	Linear programming and Farey series	$O(n^3 \log n)$	Preimage computation with arithmetic solutions
Klette & Sun 2001 [42]	Comb. procedure	n.a.	
Buzer 2002 [14]	Linear programming for DPS recognition	$O(n)$	On-line algorithm
Gérard *et al.* 2005 [33]	Convex hull analysis	$O(n')$	Fast algorithm in practice

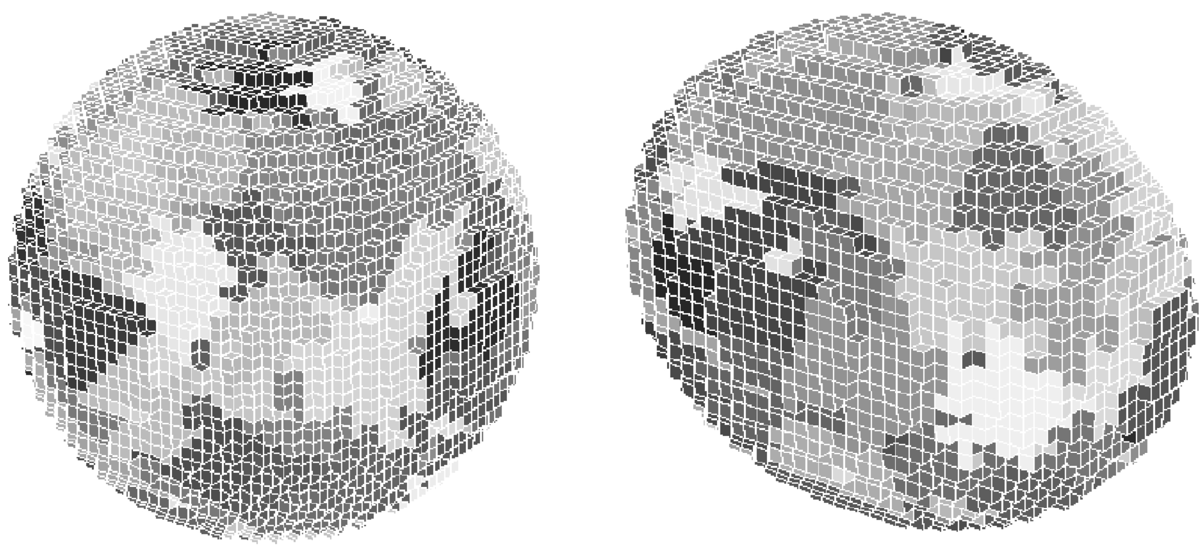

Figure 3: Agglomeration into DPSs of the faces of a sphere and an ellipsoid (grid resolution *h* = 40)

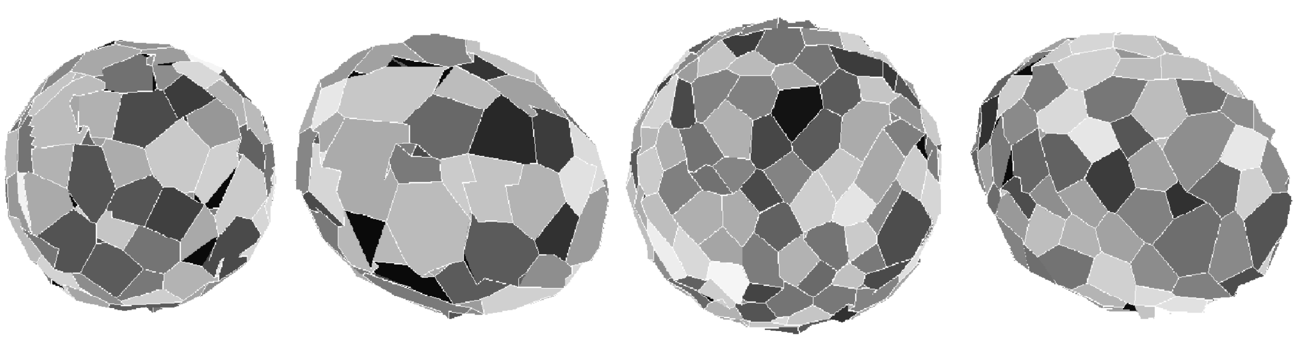

Figure 4: From left to right: A polyhedrized sphere and ellipsoid; The polyhedrized sphere and ellipsoid where the breadth-first search depth is restricted to 7

To complete the polyhedrization process, one sets all the face vertices that are incident to at least three of the DPSs to be vertices of the polyhedron. The first two subfigures of Figure 4 show the final polyhedra for the sphere and ellipsoid. Note that the surfaces of these polyhedra are not always hole-free.

Restricting the depth of the breadth-first search changes the polyhedrization from global to local and results in "more uniform" polyhedra. The last two subfigures of Figure 4 show results when the depth is restricted to 7. The number of small DPSs is reduced and the sizes of the DPSs are more evenly distributed. The respective numbers of DPSs are 282 and 180 and their average sizes are 27 and 26. Note that these are nearly the same as in the unrestricted case.

The output of Klette-Sun's algorithm is not, in general, a valid polyhedron but like a *patchwork* of planar segments. It is desirable to obtain a polyhedron with the following reversibility property: the polyhedron digitization coincides with the originally given set of grid points. Some recent algorithms [21, 22] combine ideas from Marching Cubes and digital planarity approaches. They appear to be reversible and attempt to achieve a smaller number of polygonal facets by employing linear programming techniques [21].

The main idea of the algorithm from [22] is to simplify the polyhedron obtained by the Marching-Cubes (MC) algorithm [48], using information about the digital surface segmentation. Recall that the MC considers local grid point configurations to replace them by small triangles composing the global isosurface. With a reference to [43], the obtained triangulated surface is a combinatorial manifold. In other words, the surface is closed, hole-free and without self-crossing. Furthermore, the object boundary quantization of this polyhedron is exactly the input binary object. See the first two subfigures of Figure 5.

The output of the algorithm is a digital polyhedron such that a large facet is associated to each recognized DPS. The facets of the polyhedron are stitched together by strips of triangles. These triangles are called *non-homogeneous* in [22] because their three vertices do not belong to the same digital plane. The obtained polyhedron is a combinatorial manifold and possesses the reversibility property. See the last two subfigures of Figure 5.

For more details on the presented problems and algorithms we refer to the recent survey on digital planarity [12].

The practical computational efficiency and performance of some of the described algorithms is satisfactory. However, similar to the Marching Cubes algorithms, the obtained polyhedral surface is not always a manifold surface and, in general, these algorithms are not topology preserving [42]. Also, due to the specificity of the augmentation processes and the very nature of the underlying discrete structure, the appearance of the obtained polyhedron may be non-satisfactory regarding some applications. Thus sometimes the obtained polygonal facets may be non-convex (Figures 4, 5) or may significantly differ in size (Figure 5); see [12] for more details and illustrations. Bounds on these algorithms' performance are not available. Coping with the above problems is seen as a challenging further task.

5. MULTIGRID CONVERGENT ESTIMATORS AND POLYHEDRIZATION

Polyhedrization is also useful for the purposes of geometric approximation of digital surfaces and estimation of their properties. Estimating geometric features (properties) of digital objects without any knowledge of the corresponding continuous shape is a classical problem in image analysis and pattern recognition. Some of the estimated geometric properties—such as area, perimeter, and moments—are global, while others—such as tangent, normal, and curvature—are local. A desired property of an estimator is to converge to the actual value of the real (continuous) object as the digitization resolution increases. (Roughly speaking, this interprets the possible improvement one can gain as a result of improvement of technology.) Problems of this kind are related to some important questions in number theory and therefore have long ago attracted the attention of mathematicians like C. F. Gauss [32] and R. Lipschitz [46]. Some estimators (known as "moments") admit physical interpretations, e.g. in terms of total mass or inertia of an object. More details and an extensive list of references are available in the recent monograph [41].

A general scheme for comparing measurements made on digital pictures with the actual measurement on the preimage in the Euclidean space has been suggested by J. Serra [61]. Specifically, let $\mathbb{F}$ be a family of sets $S \subseteq \mathbb{R}^n$, and let $dig_h(S)$ denote a digitization of S on a grid of resolution h. (*Grid resolution h* is the inverse of the *grid constant* that refers to the number of grid elements per unit of distance

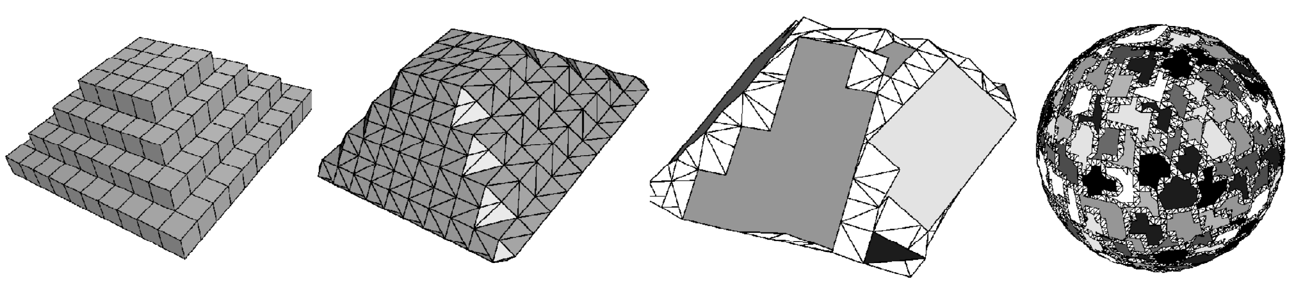

Figure 5: ***From left to right:*** **A {0, 1}-binary object; A Marching-Cubes surface obtained with an iso-level in (0, 1); Final result on the object from the left; Result on a sphere of radius 25**

without specifying the physical size of the unit.) Assume that a property Q (e.g., area, volume, or curvature) is defined for all $S \in \mathbb{F}$. An estimator E_Q is *multigrid convergent* for $\mathbb{F}$ and dig_h iff for any $S \in \mathbb{F}$ there is a grid resolution $h_S > 0$, such that the estimated value $E_Q(dig_h(S))$ is de.ned for any grid resolution $h > h_S$ and $|EQ(dig_h(S)) - Q(S)| \leq k(h)$, where k is a function defined on $\mathbb{R}$ that takes only positive real values and converges to zero as h tends to infinity. The function k specifies the speed of convergence of the estimator.

Despite a lot of work in the field many problems are still open. For instance, several convergent estimators of global properties have been designed while the existence of ones for most of the local properties is still an open question.

Moreover, there has been very little work that extends these results to higher dimensions. While a lot of estimators exist for properties of digital curves (see the extensive bibliography at the end of Ch. 10 of [41]), the study of the problem in higher dimensions is still in a very initial stage (see, e.g., [36, 42, 44, 69]).

Note that some basic methods for obtaining estimators of digital curve properties are based on finding an appropriate polygonization of the curve [41]. Thus, on the basis of the obtained results on digital volume polyhedrization, new reasonable definitions of area, normal, tangent lines/planes, and curvature of (appropriately defined) digital surfaces are to be sought and algorithms for their efficient computation are to be designed.

6. COMPLEXITY ISSUES

Given a polyhedron $P \subset \mathbb{R}^n$, the number of its i-dimensional facets (i-facets, for short) is denoted by $f_i(P)$, $0 \leq i \leq n$. Usually, the 2-facets of P are required to be convex polygons, as two adjacent polygons may be co-planar. Their number $f_2(P)$ is desired to be as *small as possible*. Note that the latter is a "soft" constraint. It describes the quality of the polyhedron that may not be related to the real object.

Let us define *polyhedral complexity* of M as $PC(M) = \min_P \{f_2(P) : P$ is an enclosing polyhedron for $M\}$. If $f_2(P) = PC(M)$, P is *minimally enclosing* for M. Although there has always been an evidence that finding a minimally enclosing polyhedron is computationally hard, obtaining a formal proof was an open problem for a long time. It was very recently proved that the following version of the problem is strongly NP-hard [10]:

Optimal Discrete Volume Polyhedrization (OptDVP):
Instance: A set $M \subset \mathbb{Z}^3$ and a bound $\beta \in \mathbb{Z}_+$.
Problem: Decide if there is a polyhedron P, such that $M = P_Z$ and with no more than β facets that are all convex polygons some of which may be co-planar.

The NP-hardness of the optimization version of discrete volume polyhedrization suggests to look for efficient approximation algorithms with guaranteed bounds on their *performance*, i.e., ones showing how far the obtained solution is from the optimal one. One can also study the computational complexity of finding a minimally enclosing polyhedron as well as the polyhedral complexity for certain interesting classes of digital objects. The intrinsic complexity of the problem makes it a true challenge to researchers in computational mathematics and theoretical computer science. It explains the overwhelming usage of greedy algorithms, that, as a rule, are not accompanied by rigorous performance estimations. Note that this is not the case in two dimensions where an optimal solution can be found in linear time [27].

6.1 Results on Polyhedral Complexity

The proposed study of the polyhedral complexity of a given discrete set of points is germane with studies in the theory of lattice polytopes and polyhedral combinatorics [2, 3, 4, 5, 18, 19, 34, 49, 51, 57, 58, 59] and integer and linear programming (see, e.g., [60]). The case when the digitized set S is convex is the only non-trivial special case where theoretical bounds are available for the number of facets of a minimally enclosing polytope. Here an enclosing polytope for $M = S_Z$ is provided by the convex hull of M that can be computed in time $O(|M|^2)$ [55]. The following results are valid in arbitrary dimension n. Let $S \in C(D)$ where $C(D)$ is the family of convex bodies with C^2 boundary and radius of curvature at every point and every direction between $1/D$ and D, $D \geq 1$. Under this conditions, the following upper bound for the number of facets of $P = conv(S_Z)$ holds [5]: For every $n \geq 2$ there are constants $c_1(n)$ and $c_2(n)$ such that for all $k \in \{0, 1, \ldots, n-1\}$, $c_1(n) d^{n\frac{n-1}{n+1}} \leq f_k(P) \leq c_2(n) d^{n\frac{n-1}{n+1}}$, where $d = diam(P)$ is sufficiently large. In the particular case of interest $n = 3$ we have $c_1 d^{3/2} \leq f_k(P) \leq c_2 d^{3/2}$ for some positive constants c_1 and c_2. The second of the above two inequalities provides an upper bound $O(d^{3/2})$ for the number of facets of a minimally enclosing polytope. By using results from the theory of lattice polytopes and integer programming, the following lower bound have been recently obtained: $f_{n-1}(P^*) \geq c(n) d_S^{\frac{n(n-1)}{(n+1)\lceil n/2 \rceil}} \Big/ \log^{\frac{n-1}{\lceil n/2 \rceil}} d_S$, where P^* is a minimal enclosing polytope for M [10, 11]. This last result implies an upper bound on the performance of the convex hull algorithm. In particular, for $n = 3$, it follows that $\frac{f_2(P)}{f_2(P^*)} \leq \alpha_0 \log^2 d_S$ for some constant $\alpha_0 \in \mathbb{Z}_+$. Thus for a convex set S, the convex hull algorithm finds an enclosing polytope for M with a guaranteed performance. Moreover, a digitization scheme is presented in [11] so that the convex hull polyhedrization is reversible.

The polyhedral complexity of some interesting classes of digital objects and related issues have been studied also experimentally [20]. It may be of interest to have information about the number of vertices and facets of the convex hull of portions of digital planes. Figures 6 and 7 illustrate some results obtained for two extreme cases: digital

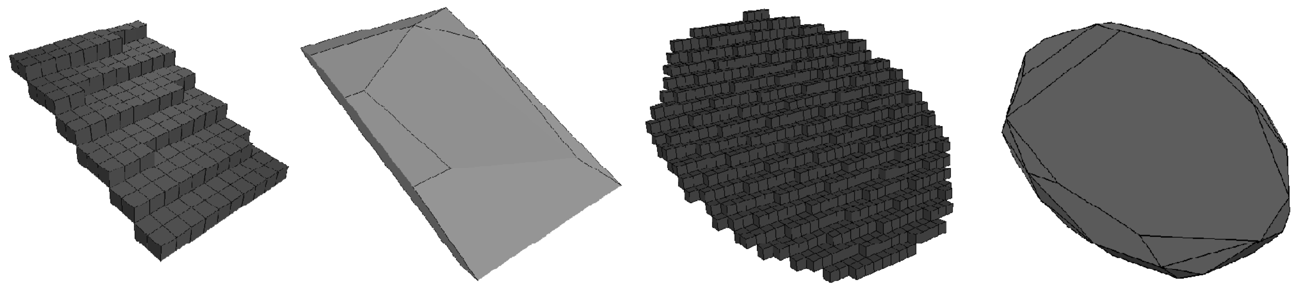

Figure 6: ***From left to right:*** **a digital space rectangle and its convex hull; a digital space disc and its convex hull**

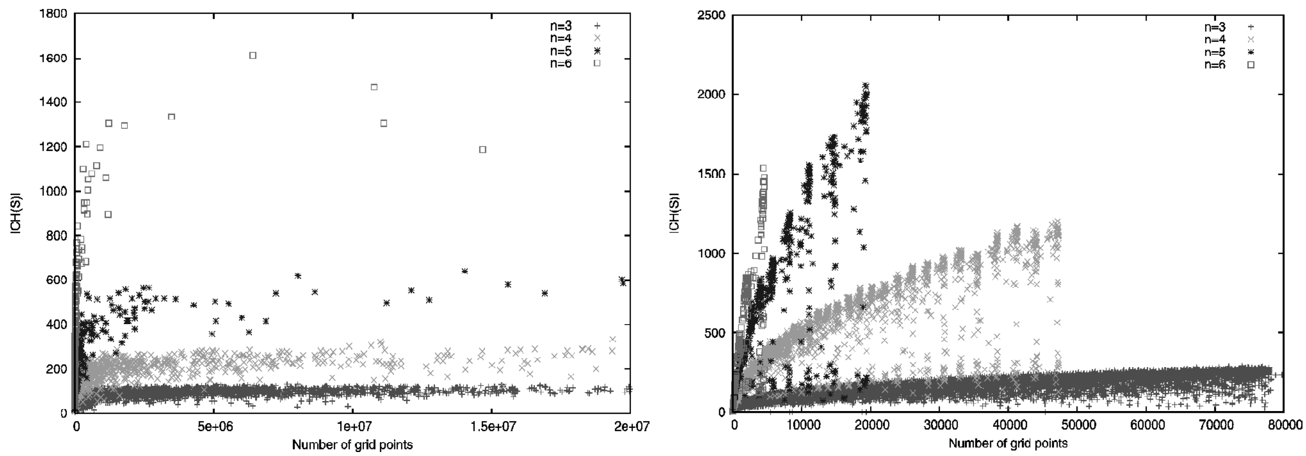

Figure 7: Number of vertices of the convex hull *CH*(*S*) of a digital (hyper) rectangle and a digital (hyper) disc *S* (left/ right, respectively). Dimensions vary from 3 to 6. The digital objects are randomly generated. The x-axis counts the number of voxels in the object

(hyper)rectangle and digital (hyper)disc. One can see that for the case of dimension three the number of vertices increases quite slowly with the increase of the object size. This suggests that the average performance of certain algorithms may be significantly better than the theoretical worst-case bounds.

7. CONCLUDING REMARKS

In this paper we considered the problem of digital object polyhedrization. In particular, we reviewed some common approaches to the problem and demonstrated the potential usefulness of digital geometry to such kind of problems. Relations to multigrid convergence estimators as well as complexity issues have been outlined. Some general shortcomings of the existing methods have also been mentioned. The latter can appear as a starting point for future research in the area.

The paper is an extended version of [9] and is based on a Keynote talk given by the first author at the International Conference CompIMAGE'06–Computational Modelling of Objects Represented in Images: Fundamentals, Methods and Applications, Coimbra, Portugal, October 21-22, 2006.

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