A Robust Fault Detection and Prediction Scheme for Nonlinear Discrete Time Input-Output Systems

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Abstract: *Model-based fault detection (MFD) techniques are preferred over hardware based schemes due to low cost and minimal changes to the system when the system states are available. However, one of the major challenges in model based monitoring, diagnosis and prognosis (MDP) approach was to develop a detection and prognosis (DP) scheme in discrete-time in the presence of partial state information since discrete-time schemes are normally preferred for ease of implementation. Therefore, in this paper, we propose a unified fault detection and prediction (FDP) scheme for a nonlinear discrete-time input-output system in the presence of modeling uncertainties when certain states are not available for measurement. A nonlinear estimator with an online tunable approximator and a robust term is introduced to monitor the system. A residual is generated by comparing the output of the system with that of the estimator. A unknown fault is detected when the generated residual exceeds a mathematically derived threshold. Subsequently, the online approximator and the robust terms are initiated. The approximator uses the system input and output measurements while its own parameters are tuned online using a novel update law. Additionally, robustness, sensitivity, and the stability of the fault detection scheme are rigorously examined. The proposed scheme is guaranteed to be asymptotically stable due to the introduction of the robust term and using some mild assumption on the system uncertainty. Subsequently the process of determining the time to failure (TTF) is introduced. Finally, the FDP scheme is simulated on a magnetic suspension system.*

Keywords: Fault detection and prognostics, nonlinear discrete time system, online approximator, Lyapunov stability.

I. INTRODUCTION

Traditionally fault detection and prognostics schemes were developed individually due to lack in understanding of how to learn the fault dynamics. In general, the process of fault detection, prognosis and accommodation consists of: (a) detection deals with determining if a fault has occurred; (b) diagnosis considers the problem of root cause and location of the fault; (c) prognosis deals with the prediction of TTF and (c) accommodation attempts to correct a particular fault, through controller reconfiguration. In particular, prognostic schemes have been found to be vital since the prediction of TTF helps the maintenance personnel to take action in the event of a fault.

From the available fault detection (FD) schemes, the model based FD appear to be most preferred [5, 9] over any hardware based schemes due to reduced cost. In such an approach, a model representative of the nonlinear system behavior is first developed and residuals are obtained by comparing the response of the model with that of the actual system. A fault is detected when the residuals exceed a pre-determined threshold. However, modeling uncertainties can cause performance degradation of the FD scheme rendering false alarms and missed detection thus demanding a robust FD scheme. Quantitative modeling schemes such as state-space [9], parity relations [5] as well as the qualitative schemes such as expert systems [10] have been introduced for linear systems [5, 9, 10] as a robust FD scheme.

With the development of advance nonlinear modeling techniques [8], it is now possible to develop FD schemes for nonlinear systems with nonlinear incipient or abrupt faults [1, 3, 7, 20, 23, 24]. This classification of faults is based on the time profile, where an incipient fault would be a slowly growing whereas an abrupt fault would be suddenly occurring [7]. However, most of the above discussed schemes [5, 9, 7, 10, 20, 24] of FD are for continuous-time systems. There has been limited previous work on FD of discrete time system [1, 3], but has mainly been on sensor or actuator faults, and requires the persistency of excitation (PE) condition to prove the stability of the scheme. It is noted that the development of a FD scheme in discrete-time is difficult due to the stability or convergence. In other words, the first difference of a Lyapunov function is quadratic with respect to the states which makes the detection scheme in discrete-time difficult whereas it is linear in the case of continuous-time systems. Therefore, the authors have recently introduced a robust FD framework for nonlinear discrete-time systems [8] by assuming that all the states are available for measurement and relaxing the requirement of the PE condition. However, availability of all the states means the need for more sensors, which makes the scheme expensive. This is the main focus of this paper.

One of the noted problems in the literature for the above mentioned schemes even for continuous-time systems is the lack of prognostics or TTF determination. One of the earlier works on prognostics [16, 17] assumed a specific degradation model of the system, which is found to be quite limited to the system or material type under consideration. On the other hand, deterministic polynomial and a probabilistic method were developed for prognosis [19, 21] by assuming that only certain parameters affect the fault. The fault dynamics are not being learned online making the prediction inaccurate. Finally, a black box approach using NN was developed in [22] using failure data which is expensive to collect apriori.

By contrast, in this paper, we unify the development of the fault detection and prognostics (FDP) scheme for nonlinear discrete-time input-output systems [7, 20, 24]. Such an approach has not been previously developed either in continuous or discrete time systems [1, 3]. First, a systematic learning methodology and some analytical results for the FDP scheme are introduced for a class of nonlinear discrete time input-output systems by using a robust term and assuming an upper bound on the modeling uncertainties. As a consequence, the proposed FDP scheme guarantees asymptotic stability in contrast to other schemes where a bounded stability [1, 3, 7, 20, 23, 24] is ensured. The proposed FDP scheme could detect nonlinear system faults, which are modeled as a nonlinear function of the input and output variables rather than actuator faults [1, 3]. Subsequently, the TTF is introduced by using the learning methodology.

The main idea behind this methodology, is to monitor the system for any abnormal behavior (which could be due to the faults or modeling uncertainties) utilizing a nonlinear estimator consisting of an online approximator in discrete-time (OLAD) with adjustable parameters and a robust term. Commonly used OLAD models are neural network, fuzzy logic, and spline function. By comparing the output of the estimator and the system output, residuals are generated and compared against a mathematically derived threshold for FD. After the detection of a fault, the OLAD and the robust term are initiated to learn the fault dynamics online. A stable adaptive update law is proposed for tuning the OLAD. Subsequently, the parameter update law is utilized to solve for the TTF. Further, the stability, the sensitivity, and the robustness of the FDP scheme are demonstrated through Lyapunov analysis in the presence of reconstruction errors and unmodeled dynamics. Finally, it is important to note that fault detection schemes and adaptation laws developed in continuous-time [7, 20, 24] cannot be directly applied to nonlinear systems represented in discrete-time.

This paper is organized as follows: In Section II the nonlinear discrete-time input-output system under consideration is explained. In Section III, the fault detection scheme is introduced. In Section IV, the robustness, the sensitivity, and the performance of the fault detection scheme is shown extensively with mathematical proofs by using the Lyapunov theory and in Section V the prognostics scheme is developed. In Section VI, a magnetic suspension system is used to illustrate the fault detection and prognostics scheme. Finally, in Section VII some concluding remarks and some possible future work are given. This paper introduces a fault detection and prediction algorithm in discrete-time and not a fault isolation and accommodation scheme. However, published literature on fault isolation and accommodation could be found elsewhere [10, 20, 26].

II. PROBLEM FORMULATION

The discrete time input-output system under consideration is described by

$$\begin{aligned} x(k+1) &= Ax(k) + \zeta(y(k), u(k)) + \eta(x(k), u(k)) \\ &\quad + \Pi(k-T)\, f(y(k), u(k)) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

where $x \in \Re^n$ is the state vector, $y \in \Re$ is the output, $\zeta, f : \Re \times \Re^m \to \Re^n$, $\eta : \Re^n \times \Re^m \to \Re^n$ are smooth vector fields, $T \geq 0$ is the starting time of the fault, $\zeta(y(k), u(k))$ represents the nominal dynamics of system, $\eta(x(k), u(k))$ is the modeling uncertainty, $f(y(k), u(k))$ is the fault dynamics, and $\Pi(k-T)$, a $n \times n$ square matrix function representing the time profiles of the fault.

A system fault typically changes the parameters of the system or its dynamics which is expressed as a nonlinear function of the output and input. It is important to note that (1) does not address sensor faults. The time profiles of the incipient faults are modeled by [23]

$$\Pi(k-T) = diag(\Omega_1(k-T), \Omega_2(k-T), \ldots, \Omega_n(k-T))$$

where

$$\Omega_i(\tau) = \begin{cases} 0 & \text{if } \tau < 0 \\ 1 - e^{-\kappa_i \tau}, & \text{if } \tau \geq 0 \end{cases} \quad i = 1, 2 \ldots n \tag{2}$$

with $\kappa_i > 0$ is an unknown constant that represents the rate at which the fault in the state x_i occurs. For large values of κ_i, the time profile function $\Omega_i(\tau)$ approaches a step function to model an abrupt fault. In this paper, we address only abrupt faults.

Remark 1: Modeling of faults using time profile is commonly found in the fault detection literature [25], and is used extensively by researchers [1, 3, 7, 20, 23, 24].

Next, throughout this paper, we make the following assumptions.

Assumption 1: Initial state of the system is known, i.e., $x(0) = x_0$.

Assumption 2: The state and the inputs are bounded before and after the fault, a standard assumption often made in the literature [7].

Assumption 3: The nominal system is assumed to be observable [24] in some domain of interest.

Assumption 4: The modeling uncertainty is unstructured and bounded [7, 24], i.e.,

$$\|\eta(x(k),u(k))\| \le \eta_0, \ \forall (x,u) \in (\chi \times U)$$

where there exists the compact sets $\chi \subset \Re^n$ and $U \subset \Re^m$, with $\eta_0 \ge 0$ a known constant.

During the past decade, many design schemes so called the robust fault diagnosis schemes have resulted in a variety of tools in continuous-time for dealing with modeling uncertainties [5]. In these robust detection schemes, when the system dynamics change above a predefined threshold, then a fault is declared [7, 20, 24]. On the other hand, another approach [5] attempts to decouple the effects of faults and modeling errors as a way of improving robustness. In the following section, a fault detection scheme is developed by using a mathematically derived threshold and OLAD. Subsequently, the parameter tuning scheme of the OLAD is utilized for prediction.

III. FAULT DETECTION SCHEME

The input-output system with fault under study uses the following nonlinear estimator given by

$$\hat{x}(k+1) = (A-KC)\hat{x}(k) + \zeta(y(k),u(k)) + Ky(k) + \hat{f}(y(k),u(k);\hat{\theta}(k)) - v(k)$$

$$\hat{y}(k) = C\hat{x}(k) \tag{3}$$

with $\hat{x}(0) = x_0$, where $\hat{x} \in \Re^n$ is the estimated state vector, $\hat{y} \in \Re$ is the estimated output, $\hat{f}$ is the OLAD, $\hat{\theta} \in \Re^q$ is a set of adjustable parameters, v is a robust term and would be defined later in the text, and K is a design constant, which is chosen such that $G = A - KC$ has all its eigenvalues within the unit disc. The initial value of the OLAD in (3) is selected such that $\hat{\theta}(0) = \hat{\theta}_0$, so that $\hat{f}(y,u,\hat{\theta}_0) = 0$ for all $y \in Y$ and $u \in U$. Given the initial conditions, the next step involves the development of an adaptive law for the parameter $\hat{\theta}(k)$, so that the OLAD $\hat{f}(y(k),u(k);\hat{\theta}(k))$ reconstructs the fault dynamics $f(y(k), u(k))$. An accurate modeling of the nonlinear discrete-time system would enable us to track any changes in the system dynamics and helps in the development of a robust fault detection algorithm.

Remark 2: Only upon detection of a fault, the OLAD and the robust term are initiated.

During the last few years, several online approximation based models have been studied primarily in continuous-time in the context of intelligent and learning control. In addition to conventional approximation models like polynomials, spline functions etc., various neural networks such as sigmoidal activation functions, radial basis functions, CMAC etc and others such as fuzzy logic systems and wavelets, have emerged. For the OLAD, y and u are considered as the input vectors, $\hat{\theta}(k)$ is the vector of adjustable parameters, and $\hat{f}(y,u;\hat{\theta})$ is the output. In this paper, we consider a general class of sufficiently smooth online approximators, $\hat{f} \in C^\infty$.

Next define the state estimation error as $e = x - \hat{x}$. Also define $e_o = y - C\hat{x}$ as the output estimation error or residual. Under the ideal conditions with no modeling errors, a fault is declared active whenever the output of the online approximator $\hat{f}(y(k),u(k);\hat{\theta}(k))$ and the residual becomes nonzero. An intuitive way of generating robustness with respect to modeling uncertainties is to start the adaptation whenever the residual is above a certain threshold. This can be easily implemented by using a dead-zone operator D[.], which is defined for improving the robustness of the fault detection scheme as

$$D[e_o(k)] = \begin{cases} 0, \text{ if } |e_o(k)| \le \varepsilon \\ e_o(k), \text{if } |e_o(k)| > \varepsilon \end{cases} \tag{4}$$

where $e_o(k)$ is the residual and $\varepsilon > 0$ is a design constant. The dead-zone size ε clearly provides a tradeoff between reducing the possibility of false alarms (robustness) and improving the sensitivity of the faults.

In the next section, ε is derived in terms of the modeling uncertainty bound (η_0), which guarantees robustness in the presence of modeling uncertainty. Based on the estimation model in (3) and the dead-zone in (4), the following parameter update law is proposed for tuning the OLAD

$$\hat{\theta}(k+1) = \hat{\theta}(k) + \alpha Z\ B_0 D[e_o(k)] - \gamma \left\| I - \alpha Z Z^T \right\| \hat{\theta}(k) \tag{5}$$

where $\alpha > 0$ the learning rate or adaptation gain, $0 < \gamma < 1$ is a design parameter, $B_0 \in \Re^n$ is a constant vector, and Z is a $q \times n$ matrix defined as

$$Z = \left[\frac{\partial \hat{f}(y,u;\hat{\theta})}{\partial \hat{\theta}} \right]^T \tag{6}$$

The key advantage of the proposed parameter update law is the relaxation of parameter drift, a phenomenon that may occur with standard adaptive laws in the presence of approximation errors and due to the lack of the persistency of excitation (PE) of input signals. The last term is similar to e-modification in continuous-time adaptive control.

Next we define the robust term as

$$v(k) = \frac{B_1^T \hat{\theta}(k)}{\hat{\theta}^T(k)\ B_1 B_1^T \hat{\theta}(k) + c_m} \tag{7}$$

where $B_1 \in \Re^{q \times n}$ is a constant matrix and its selection is addressed later in the paper and $c_m > 0$ is a design constant. The performance of the parameter update law is shown mathematically by using Lyapunov theory in the next section.

Remark 3: In our earlier work [23], the authors have developed a nonlinear estimator for robust fault detection in dynamic systems with full state feedback. In the case of full state measurement with n states and m inputs, the input to the online approximator will be $(n + m)$ whereas it is $(1 + m)$ for the proposed work. This has a major impact on the online approximator especially for linearly parameterized approximators since for high dimensional input spaces, the number of adjustable parameters needed to achieve a given approximation accuracy increases with the input dimension [2]. Therefore, the use of output sensor data instead of full state vector has obvious practical advantages similar to the case of continuous-time systems.

IV. ANALYTICAL RESULTS

In this section, the robustness, the sensitivity, and the stability of the nonlinear fault detection scheme is rigorously examined. The robustness analysis deals with the investigation of the behavior of the OLAD in the presence of modeling uncertainties prior to the occurrence of any faults. The sensitivity analysis examines the behavior of the OLAD after the occurrence of the fault and characterizes the class of faults that can be detected by the robust fault detection scheme. On the other hand, the stability analysis included in this section deals with the asymptotic convergence of the system signals, even after the fault occurrence.

In an ideal case, where there is no modeling errors and prior to the occurrence of a fault, i.e., $k \in [0, T)$, from (1) and (3), the state estimation error satisfy

$$e(k + 1) = Ge(k) \tag{8}$$

Since G is a stable matrix, hence the stability follows trivially, i.e., $e \to 0$ as $k \to \infty$. Next, in the presence of modeling errors, (8) becomes

$$e(k + 1) = Ge(k) + \eta(x(k), u(k)) \tag{9}$$

To determine an appropriate value for ε, we derive an upper bound for $e_o(k)$ prior to the fault. From (9), we have $e(k) = \sum_{j=0}^{k-1} G^{k-1-j}\eta(x(j),u(j))$. Hence the residual is given by $e_o(k) = C\sum_{j=0}^{k-1} G^{k-1-j}\eta(x(j),u(j))$. Since the matrix G is stable, there exist two positive constants μ and β_C such that (Frobenius norm) $\|G^k\| \le \beta_C\mu^k \le 1$. Therefore by using $\|C\| = 1$ [9], and taking $\beta = \beta_c\mu$, we get $|e_o(k)| \le \beta\eta_o \frac{(1-\mu^k)}{(1-\mu)}$.

Thus we choose the size of the dead-zone $\varepsilon = \frac{\beta\eta_o}{(1-\mu)}$. Next to show the robustness of the proposed scheme (using equations (3), (4), (5), (9)), the following theorem is proposed.

Theorem 1 (Robustness): The robust nonlinear fault detection scheme described by (3), (4), (5) and (9) guarantees that $\hat{f}(y(k),u(k),\hat{\theta}(k)) = 0$, for $k \le T$ prior to the occurrence of the fault.

Proof: Let us assume that there exists a time k_r, $0 < k_r < T$, such that $|e_o(k)| < \varepsilon$ for $k < k_r$ and

$$|e_o(k_r)| = \varepsilon = \frac{\beta\eta_0}{(1-\mu)} \tag{10}$$

It is could be seen that the parameter $\hat{\theta}(k)$ has not adopted in the time interval $[0, k_r)$ by using (5) and the continuity of $e_o(k)$ [24]. Hence in the time interval $[0, k_r)$ the state estimation error $e(k)$ satisfies

$$e(k + 1) = Ge(k) + \eta(x(k), u(k)) \tag{11}$$

Therefore, in the interval $[0, k_r)$, the residual or the output estimation error is given by

$$e_0(k) = Ce(k) = C\left[\sum_{j=0}^{k-1} G^{k-1-j}\eta(x(j),u(j))\right]$$

By using $\|C\| = 1$, $\left\|\sum_{j=0}^{k-1} G^{k-1-j}\right\| \le \frac{\beta(1-\mu^k)}{(1-\mu)}$ and $\|\eta(x(k),u(k))\| \le \eta_0$, we get $|e_o(k)| \le \beta\eta_o \frac{(1-\mu^k)}{(1-\mu)} = \varepsilon(1-\mu^k)$.

Hence, $|e_o(k)| \le \varepsilon(1-\mu^k)$ for all $k \in [0, k_r)$ Thus by using the continuity of $e_o(k)$ we obtain that $|e_o(k_r)| < \varepsilon$, which contradicts our assumption in (10). In other words, the residual remains within the dead-zone and the output of the OLAD remains zero.

Remark 4: The proof of the theorem is quite analogous to the continuous-time case [24].

Next after the occurrence of the fault at $k \ge T$, by using equations (3) and (4), the state estimation error satisfies

$$\begin{aligned} e(k + 1) &= Ge(k) + \eta(x(k), u(k)) + \Pi(k - T) f(y(k), u(k)) \\ &\quad - \hat{f}(y(k),u(k);\hat{\theta}(k)) + v(k) \\ &= Ge(k) + \eta(x(k),u(k)) + \Pi(k-T)\hat{f}(y(k),u(k),\theta) \\ &\quad - \hat{f}(y(k),u(k);\hat{\theta}(k)) + \varepsilon(k) + v(k) \end{aligned}$$

where the approximation error is given by $\varepsilon(k) = \Pi(k-T)[f(y(k),u(k)) - \hat{f}(y(k),u(k),\theta)]$ and θ is an optimal value chosen such that it minimizes the L_2 norm distance between $\hat{f}(y,u;\hat{\theta})$ and $f(y, u)$ for all (y, u) in some compact domain $y \times U$. Also θ is constrained to a compact set $w \subset \Re^q$. Based on the smooth assumptions on $\hat{f}(y,u,\hat{\theta})$ [7], further, the above defined error equation can be expressed as

$$e(k+1) = Ge(k) + \eta(x(k),u(k)) - [I - \Pi(k-T)]\hat{f}(y(k),u(k),\theta)$$

$$+\frac{\partial \hat{f}(y,u;\hat{\theta})}{\partial \hat{\theta}}(\hat{\theta}-\theta)+\Delta(y,u;\hat{\theta},\theta)+\varepsilon(k)+v(k) \qquad (12)$$

where $\Delta(y,u;\hat{\theta},\theta)=-\hat{f}(y,u;\hat{\theta})+\hat{f}(y,u,\theta)-\frac{\partial \hat{f}(y,u;\hat{\theta})}{\partial \hat{\theta}}(\hat{\theta}-\theta)$ with $\Delta(y,u;\hat{\theta},\theta)$ represents the higher order terms of the Taylor series expansion of $\hat{f}(y,u;\hat{\theta})$ w.r.t to $\hat{\theta}$. Let $\tilde{\theta}=\theta-\hat{\theta}$ is the parameter estimation error, denote $\omega(k)=\Delta(y,u;\hat{\theta},\theta)-[I-\Pi(k-T)]\hat{f}(y(k),u(k),\theta)+\eta(x(k),u(k))+\varepsilon(k)$, and $\Psi_1(k)=Z^T(k)\tilde{\theta}(k)$, then the error equation (12) becomes

$$e(k+1)=Ge(k)+\Psi_1(k)+\omega(k)+v(k)$$

Now using the definition of the robust term from (7), we get

$$e(k+1)=Ge(k)+\Psi_1(k)+\omega(k)+\frac{B_1^T\hat{\theta}(k)}{\hat{\theta}^T(k)B_1B_1^T\hat{\theta}(k)+c_m}$$

Add and subtract $\frac{\left(B_1^T\theta-C_1\right)}{\hat{\theta}^T(k)B_1B_1^T\hat{\theta}(k)+c_m}$ in the above equation, where $C_1\in\Re^n$ is a constant vector, to get

$$e(k+1)=Ge(k)+\Psi_1(k)+\omega(k)-\Psi_2(k)+\frac{\left(B_1^T\theta-C_1\right)}{\hat{\theta}^T(k)B_1B_1^T\hat{\theta}(k)+c_m} \qquad (13)$$

where $\Psi_2(k)=\frac{\left(B_1^T\tilde{\theta}(k)-C_1\right)}{\hat{\theta}^T(k)B_1B_1^T\hat{\theta}(k)+c_m}$. Next we consider the sensitivity of the proposed fault detection scheme. The class of detectable fault is given by the sensitivity theorem and is shown below; this theorem is obtained under the worst-case detectable conditions [9]

Theorem 2 (Sensitivity): For some $k_d>0$, if the fault dynamics $f(y(k),u(k))$ satisfies the following inequality

$$\left|\sum_{j=T}^{T+k_d-1}CG^{(T+k_d-1-j)}f(y(j),u(j))\right|\geq(1+\beta_c)\varepsilon \qquad (14)$$

Then the residual is given by $|e_o(T+k_d)|\geq\varepsilon$.

Proof: The state estimation error in the presence of a fault and prior to the OLAD adaptation is given by

$$e(k+1)=Ge(k)+\eta(x,u)+f(y,u)$$

Therefore for $k>0$, the residual is given by

$$e_o(T+k)=CG^ke(T)+\sum_{j=T}^{T+k-1}CG^{(T+k-1-j)}\eta(x(j),u(j))$$

$$+\sum_{j=T}^{T+k-1}CG^{(T+k-1-j)}f(y(j),u(j))$$

Using $|e_o(T)|<\frac{\beta\eta_0}{(1-\mu)}$, $\left\|G^k\right\|\leq\beta_c\mu^k$ and the triangle inequality, we obtain

$$|e_0(T+k)|\geq-\frac{\beta\eta_0\beta_c\mu^k}{(1-\mu)}-\beta\eta_0\frac{(1-\mu^k)}{(1-\mu)}$$

$$+\left|\sum_{j=T}^{T+k-1}CG^{(T+k-1-j)}f(y(j),u(j))\right|$$

$$\geq-\varepsilon\left[\beta_c\mu^k+(1-\mu^k)\right]+\left|\sum_{j=T}^{T+k-1}CG^{(T+k-1-j)}f(y(j),u(j))\right|$$

Using $\|C\|=1$, $\left\|G^k\right\|\leq\beta_c\mu^k\leq1$ and taking $k=0$, we obtain $\beta_c\leq1$. If $\beta_c=1$, $\mu^k\leq1$ and also if there exists a time $k_d>0$ and if the condition in (14) is satisfied then it can be concluded that $|e_0(T+k_d)|\geq\varepsilon$. This theorem shows that the OLAD would start adapting, if $|e_0(T+k_d)|\geq\varepsilon$ and hence the output of the OLAD ($\hat{f}(y,u;\hat{\theta})$) becomes non-zero.

Remark 5: The above theorem characterizes the class of faults that are detectable by the robust nonlinear discrete-time fault detection scheme. Note that the left-hand side of (14) represents the fault function. Intuitively the sensitivity theorem states that if the magnitude of the fault function after some time k_d becomes greater than $(1+\beta_c)\varepsilon$, then such faults can be detected under worst-case detectability conditions. In other words, similar to the continuous-time case, the inequality (14) is a sufficient (but not necessary) condition for activating adaptation of the OLAD in the presence of any modeling uncertainty satisfying Assumption 4.

One of the most important parameters in fault detection is the time interval between the occurrence of a fault and the detection of the fault which is referred to as fault detection time. The sensitivity theorem not only characterizes the class of faults but it also provides a measure of the detection time. In other words, the smallest k_d for which the inequality (14) holds is equal to the detection time under the worst case detectability conditions. Hence, k_d represents the maximum detection time over all allowable scenarios of modeling uncertainties.

Next the stability and performance of the fault detection scheme is examined. For the following results, it is taken that $|e_o(k)|>\varepsilon$. For a gradient-based tuning updates used in a fault detection scheme [1, 3] which cannot exactly reconstruct certain unknown parameters because of the presence of unmodeled nonlinearities or approximation errors, cannot be guaranteed to yield bounded estimates. Then the PE condition is required to guarantee boundedness of the parameter estimates. However, it is very difficult to guarantee or verify the PE. In the next theorem, improved parameter tuning schemes for the fault detection scheme is presented so that PE is not required.

Theorem 3 (Stability): (PE condition not required) let the initial conditions for the nonlinear estimator is bounded in a compact set $S \subset \Re^n$. In the event of a fault, the fault detection scheme guarantees robust stability in the presence of modeling and approximation errors, such that $e_o(k)$ is locally asymptotically stable and $\tilde{\theta}(k)$ is bounded.

Proof: Consider a Lyapunov candidate as

$$V = \frac{1}{5}e_o^2(k) + \frac{1}{3\alpha}[\tilde{\theta}^T(k)\tilde{\theta}(k)]$$

The first difference is given by

$$\Delta V = \underbrace{\frac{1}{5}e_o^2(k+1) - e_o^2(k)}_{\Delta V_1} + \underbrace{\frac{1}{3\alpha}[\tilde{\theta}^T(k+1)\tilde{\theta}(k+1) - \tilde{\theta}^T(k)\tilde{\theta}(k)]}_{\Delta V_2}$$

Consider the first term (ΔV_1) in the first difference ΔV and substituting $e_O = y - C\hat{x} = Cx - C\hat{x} = Ce$, using the error equation (13), and applying the Cauchy-Schwarz inequality $(a_1 + a_2 + \ldots + a_n)^T . (a_1 + a_2 + \ldots + a_n) \le$ $\le n.(a_1^T a_1 + a_2^T a_2 + \ldots + a_n^T a_n)$ gives us

$$\Delta V_1 \le ((e^T(k)G^TC^T)^2 + (\Psi_1^T(k)C^T)^2 + (\omega^T(k)C^T)^2 + (\Psi_2^T(k)C^T)^2$$

$$+\left(\frac{\left(B_1^T\theta - C_1\right)^T C^T}{\left(\hat{\theta}^T(k)B_1B_1^T\hat{\theta}(k) + c_m\right)^2}\right)^2\Bigg] - \frac{1}{5}\left(Ce(k)\right)^2 \qquad (15)$$

Next, considering the second term (ΔV_2) in the first difference of the Lyapunov function ΔV

$$\Delta V_2 = \frac{1}{3\alpha}[\tilde{\theta}^T(k+1)\tilde{\theta}(k+1) - \tilde{\theta}^T(k)\tilde{\theta}(k)]$$

by using the parameter update law (5), applying the dead-zone operator in (4), and $\tilde{\theta} = \theta - \hat{\theta}$, one obtains

$$\Delta V_2 = \frac{1}{3\alpha}\Big\{\Big[(I - \gamma\|I - \alpha ZZ^T\|I)\tilde{\theta}(k) - \alpha Z\,B_0 e_o(k)$$

$$+\gamma\|I - \alpha ZZ^T\|\theta\Big]^T$$

$$\times\Big[(I - \gamma\|I - \alpha ZZ^T\|I)\tilde{\theta}(k) - \alpha Z B_0 e_o(k) + \gamma\|I - \alpha ZZ^T\|\theta\Big]$$

$$-\tilde{\theta}^T(k)\tilde{\theta}(k)\Big\}$$

Applying the Cauchy-Schwarz inequality $(a_1 + a_2 + \ldots + a_n)^T . (a_1 + a_2 + \ldots + a_n) \le n.(a_1^T a_1 + a_2^T a_2 + \ldots + a_n^T a_n)$ in the above equation gives us

$$\Delta V_2 \le \frac{1}{3\alpha}\Big\{\Big[3\tilde{\theta}^T(k)(I - \gamma\|I - \alpha ZZ^T\|I)(I - \gamma\|I - \alpha ZZ^T\|I)\tilde{\theta}(k)$$

$$+3\alpha^2 B_0^T Z^T e_o(k) Z B_0 e_o(k) + 3\gamma^2\|I - \alpha ZZ^T\|^2\theta^T\theta\Big] - \tilde{\theta}^T(k)\tilde{\theta}(k)\Big\}$$

In the above equation, performing some mathematical manipulations would result in the following equation

$$\Delta V_2 \le \frac{2}{3\alpha}\tilde{\theta}^T(k)\tilde{\theta}(k) - \frac{2\gamma}{\alpha}\|I - \alpha ZZ^T\|\tilde{\theta}^T(k)\tilde{\theta}(k) + \frac{\gamma^2}{\alpha}\|I - \alpha ZZ^T\|^2\tilde{\theta}^T(k)\tilde{\theta}(k)$$

$$+\alpha B_0^T Z^T Ce(k) Z B_0 Ce(k) + \frac{\gamma^2}{\alpha}\|I - \alpha ZZ^T\|^2\theta^T\theta \qquad (16)$$

Combining ΔV_1 from (15) and ΔV_2 from (16) results in the following equation

$$\Delta V \le ((e^T(k)G^TC^T)^2 + (\Psi_1^T(k)C^T)^2 + (\omega^T(k)C^T)^2 + (\Psi_2^T(k)C^T)^2$$

$$+\left(\frac{\left(B_1^T\theta - C_1\right)^T C^T}{\left(\hat{\theta}^T(k)B_1B_1^T\hat{\theta}(k) + c_m\right)^2}\right)^2\Bigg] - \frac{1}{5}\left(Ce(k)\right)^2 +$$

$$\frac{2}{3\alpha}\tilde{\theta}^T(k)\tilde{\theta}(k) - \frac{2\gamma}{\alpha}\|I - \alpha ZZ^T\|\tilde{\theta}^T(k)\tilde{\theta}(k) + \frac{\gamma^2}{\alpha}\|I - \alpha ZZ^T\|^2\tilde{\theta}^T(k)\tilde{\theta}(k)$$

$$+\alpha B_0^T Z^T Ce(k) Z B_0 Ce(k) + \frac{\gamma^2}{\alpha}\|I - \alpha ZZ^T\|^2\theta^T\theta \qquad (17)$$

Next, we introduce the following Lemma

Lemma 1: The term $\omega(k)$ in (17) comprising of the approximation error and the basis function of the OLAD, is assumed to be upper bounded by a smooth nonlinear function of state estimation and parameter estimation errors [6, 11]

$$(\omega^T(k)C^T)^2 \le \varepsilon_M = \beta_0 + \beta_1\|e(k)\|^2 + \beta_2\|\tilde{\theta}(k)\|^2$$

$$+\beta_3\|e(k)\|\|\tilde{\theta}(k)\|$$

where β_0, β_1, β_2 and β_3 are computable positive constants.

Proof: Use some standard norm inequalities, Assumption 1, and the fact that the reconstruction error can be expanded as a function of the residual error and error in adaptive estimation parameters. The steps follow similar to the case in continuous-time in proving the boundedness for a NN controller [15].

Then taking the Frobenius norm and using lemma 1, equation (17) could be rewritten as

$$\Delta V \le -\left(\frac{1}{5} - G_{\max}^2 - \alpha B_{0\max}^2 Z_{\max}^2 - \beta_1 - \beta_4)\|e(k)\|^2\right)$$

$$-\left(\frac{2\gamma}{\alpha}\|I - \alpha ZZ^T\| - Z_{\max}^2 - 2B_{1\max}^2 - \frac{2}{3\alpha} - \frac{\gamma^2}{\alpha}\|I - \alpha ZZ^T\|^2 - \beta_2 - \beta_4\right).$$

$$\|\tilde{\theta}(k)\|^2 + B_{1\max}^2\theta_{\max}^2 - 2B_{1_{\min}}\theta_{\min}C_{1_{\min}} + 2C_{1\max}^2$$

$$+\frac{\gamma^2}{\alpha}\|I - \alpha ZZ^T\|^2\theta_{\max}^2 + \beta_0 \qquad (18)$$

where $\theta_{\min} \le \|\theta\| \le \theta_{\max}$, $Z_{\min} \le \|Z\| \le Z_{\max}$, and $\beta_4 = (\beta_3/2)$.

Taking $B_{1_{\min}} = \dfrac{B_{1\max}^2\theta_{\max}^2 + 2C_{1\max}^2 + \dfrac{\gamma^2}{\alpha}\|I-\alpha ZZ^T\|^2\theta_{\max}^2 + \beta_0}{2\theta_{\min}C_{1_{\min}}}$.

Using this definition in (18) results in the following equation

$$\Delta V \le -(\frac{1}{5} - G_{\max}^2 - \alpha B_{0\max}^2 Z_{\max}^2 - \beta_1 - \beta_4)\|e(k)\|^2$$

$$-\Big(\frac{2\gamma}{\alpha}\|I-\alpha ZZ^T\| - Z_{\max}^2 - 2B_{1\max}^2 - \frac{2}{3\alpha} - \frac{\gamma^2}{\alpha}\|I-\alpha ZZ^T\|^2 - \beta_2$$

$$-\beta_4\Big)\|\tilde{\theta}(k)\|^2$$

Hence in the above equation, $\Delta V < 0$ if we choose the following gains

$$B_{0_{\max}} = \frac{G_{\max}}{\sqrt{\alpha}Z_{\max}}, G_{\max} \le \sqrt{\frac{\frac{1}{5} - \beta_1 - \beta_4}{2}}, \beta_1 + \beta_4 < \frac{1}{5},$$

$$a = \frac{1 \mp \sqrt{\frac{1}{3} - \alpha(Z_{\max}^2 + 2B_{1\max}^2 + \beta_2 + \beta_4)}}{\|I-\alpha ZZ^T\|},$$

$$\min(a) \le \gamma \le \frac{1}{\|I-\alpha ZZ^T\|}, \text{ and } \alpha < 1.$$

Thus as long as the first difference $\Delta V < 0$ which indicates that the error signals are stable in the sense of Lyapunov. Additionally, in absence of measurement noise, $e_0(k) = Ce(k)$, hence $e_0(k)$ and $\tilde{\theta}(k)$ are bounded, provided $e_0(k_0)$ and $\tilde{\theta}(k_0)$ are bounded in a set S. Hence $e_0(k)$ and $\tilde{\theta}(k)$ converges asymptotically to zero.

Remark 6: From the above theorem, it is observed that by using the robust term and the lemma on the approximation error, we proved local asymptotic stability of the closed loop system.

Next we propose stability without using the robust term and also removing the lemma 1, thus we present the following corollary. In this corollary, we show that the FD scheme is only semi-globally uniformly ultimately bounded (SGUUB). Thus (13) without the robust term could be written as

$$e(k+1) = Ge(k) + \Psi(k) + \omega(k) \quad (19)$$

where $\Psi(k) = Z^T(k)\tilde{\theta}(k)$ and $\omega(k) = \Delta(y,u;\hat{\theta},\theta) - [I - \Pi(k-T)]\hat{f}(y(k),u(k),\theta) + \eta(x(k), u(k)) + \varepsilon(k)$. Next the corollary on the stability is presented.

Corollary 1: Consider the hypothesis given in Theorem 3 with the robust term being removed. In the presence of bounded uncertainties and reconstruction or approximation errors, the output estimation error or residual $e_o(k)$ and the parameter estimation error $\hat{\theta}(k)$ are SGUUB.

Proof: Consider a Lyapunov candidate as

$$V = \frac{1}{3}e_o^2(k) + \frac{1}{3\alpha}[\tilde{\theta}^T(k)\tilde{\theta}(k)]$$

The first difference is given by

$$\Delta V = \underbrace{\frac{1}{3}e_o^2(k+1) - e_o^2(k)}_{\Delta V_1} + \underbrace{\frac{1}{3\alpha}[\tilde{\theta}^T(k+1)\tilde{\theta}(k+1) - \tilde{\theta}^T(k)\tilde{\theta}(k)]}_{\Delta V_2}$$

Consider the first term (ΔV_1) in the first difference ΔV and substituting $e_o = y - C\hat{x} = Cx - C\hat{x} = Ce$, using the error equation (19), applying the Cauchy-Schwarz inequality $(a_1 + a_2 + \ldots + a_n)^T \cdot (a_1 + a_2 + \ldots + a_n) \le$ $\le n.(a_1^Ta_1 + a_2^Ta_2 + \ldots + a_n^Ta_n)$ in the above equation gives us

$$\Delta V_1 \le \frac{2}{3}\big(CGe(k)\big)^2 + \big(C\Psi(k)\big)^2 + \big(C\omega(k)\big)^2 \quad (20)$$

Next, considering the second term (ΔV_2) in the first difference of the Lyapunov function ΔV, we get

$$\Delta V_2 = \frac{1}{\alpha}[\tilde{\theta}^T(k+1)\tilde{\theta}(k+1) - \tilde{\theta}^T(k)\tilde{\theta}(k)]$$

by using the parameter update law (5), applying the dead-zone operator in (4), and $\tilde{\theta} = \theta - \hat{\theta}$, one obtains

$$\Delta V_2 = \frac{1}{3\alpha}\Big\{\Big[(I - \gamma\|I-\alpha ZZ^T\|I)\tilde{\theta}(k) - \alpha Z\,\mathrm{B}_0 e_o(k)$$

$$+\gamma\|I-\alpha ZZ^T\|\theta\Big]^T$$

$$\times\Big[(I - \gamma\|I-\alpha ZZ^T\|I)\tilde{\theta}(k) - \alpha Z\mathrm{B}_0 e_o(k) + \gamma\|I-\alpha ZZ^T\|\theta\Big]$$

$$-\tilde{\theta}^T(k)\tilde{\theta}(k)\Big\}$$

Applying the Cauchy-Schwarz inequality $((a_1 + a_2 + \ldots + a_n)^T \cdot (a_1 + a_2 + \ldots + a_n) \le n.(a_1^Ta_1 + a_2^Ta_2 + \ldots + a_n^Ta_n)$ in the above equation gives us

$$\Delta V_2 \le \frac{1}{3\alpha}\Big\{\Big[3\tilde{\theta}^T(k)(I - \gamma\|I-\alpha ZZ^T\|I)(I - \gamma\|I-\alpha ZZ^T\|I)\tilde{\theta}(k)$$

$$+3\alpha^2\mathrm{B}_0^TZ^Te_o(k)Z\mathrm{B}_0e_o(k) + 3\gamma^2\|I-\alpha ZZ^T\|^2\theta^T\theta\Big] - \tilde{\theta}^T(k)\tilde{\theta}(k)\Big\}$$

In the above equation, performing some mathematical manipulations would result in the following equation

$$\Delta V_2 \le \frac{2}{3\alpha}\tilde{\theta}^T(k)\tilde{\theta}(k) - \frac{2\gamma}{\alpha}\|I-\alpha ZZ^T\|\tilde{\theta}^T(k)\tilde{\theta}(k) + \frac{\gamma^2}{\alpha}\|I-\alpha ZZ^T\|^2\tilde{\theta}^T(k)\tilde{\theta}(k)$$

$$+\alpha B_0^T Z^T Ce(k) Z B_0 Ce(k) + \frac{\gamma^2}{\alpha} \| I - \alpha Z Z^T \|^2 \theta^T \theta \quad (21)$$

Combining ΔV_1 from (20) and ΔV_2 from (21) results in the following equation

$$\Delta V \le \frac{2}{3}\left(CGe(k)\right)^2 + \left(C\Psi(k)\right)^2 + \left(C\omega(k)\right)^2 + \frac{2}{3\alpha}\tilde{\theta}^T(k)\tilde{\theta}(k)$$

$$-\frac{2\gamma}{\alpha}\| I - \alpha Z Z^T \| \tilde{\theta}^T(k)\tilde{\theta}(k) + \frac{\gamma^2}{\alpha}\| I - \alpha Z Z^T \|^2 \tilde{\theta}^T(k)\tilde{\theta}(k)$$

$$+\alpha B_0^T Z^T Ce(k) Z B_0 Ce(k) + \frac{\gamma^2}{\alpha}\| I - \alpha Z Z^T \|^2 \theta^T \theta$$

Applying Frobenius norm in the above equation gives us

$$\Delta V \le -\left(\frac{1}{3} - G_{\max}^2 - \alpha B_{0\max}^2 Z_{\max}^2\right)\| e(k) \|^2$$

$$-\left(\frac{2\gamma}{\alpha}\| I - \alpha Z Z^T \| - Z_{\max}^2 - \frac{2}{3\alpha} - \frac{\gamma^2}{\alpha}\| I - \alpha Z Z^T \|^2\right)\left\|\tilde{\theta}(k)\right\|^2 + D_M^2$$

where $D_M^2 = \omega_{\max}^2 + \frac{\gamma^2}{\alpha}\| I - \alpha Z Z^T \|^2 \theta_{\max}^2$, $\|\omega(k)\| \le \omega_{\max}$. Then $\Delta V \le 0$ as long as the following conditions hold

$$\| e(k) \| \ge \frac{D_M}{\sqrt{\left(\frac{1}{3} - G_{\max}^2 - \alpha B_{0\max}^2 Z_{\max}^2\right)}} \text{ or}$$

$$\| \tilde{\theta}(k) \| \ge \frac{D_M}{\sqrt{\left(\frac{2\gamma}{\alpha}\| I - \alpha Z Z^T \| - Z_{\max}^2 - \frac{2}{3\alpha} - \frac{\gamma^2}{\alpha}\| I - \alpha Z Z^T \|^2\right)}}$$

also $B_{0\max} = \frac{G_{\max}}{\sqrt{\alpha} Z_{\max}}$, $G_{\max} \le 0.408$, $b = \frac{1 \mp \sqrt{\frac{1}{3} - \alpha Z_{\max}^2}}{\| I - \alpha Z Z^T \|}$,

$$\min(b) \le \gamma \le \frac{1}{\| I - \alpha Z Z^T \|},\ Z_{\max} < 0.577, \text{ and } \alpha < 1. \quad (22)$$

Therefore, $\Delta V \le 0$ and it can be concluded that the residual or output estimation error $e_o(k)$ and the parameter estimation error $\hat{\theta}(k)$ are SGUUB.

Remarks 7: It is important to note that in the above two theorems (Theorem 3 and Corollary 1) the requirement of the PE condition and certainty equivalence (CE) assumption are relaxed for the adaptive estimator, in contrast to standard work in discrete-time adaptive control [13]. In the latter, two separate Lyapunov functions are considered to show the bound on the state estimation error and the parameter estimation error [13, 23]. By contrast in our proof, the residual, $e_o(k)$ and the parameter estimation errors $\hat{\theta}(k)$ are combined in one Lyapunov function. Hence the proof is exceedingly complex due to the presence of several different variables. However, it obviates the need for the CE assumption and it allows parameter-tuning algorithms to be derived during the proof, not selected a priori in an ad hoc manner.

Remark 8: The parameter updating rule (5) is a nonstandard scheme that was derived from Lyapunov analysis and does include an extra term referred to as discrete-time ε-mod [13], which is normally used to provide robustness due to the coupling in the proof between the residual and the parameter estimation error terms. The Lyapunov proof shows that the term is necessary. Unless the term is utilized, the time to failure cannot be derived.

In this section we presented the robustness, sensitivity, and the stability of the proposed FD scheme. Additionally, two different stability results were obtained, i.e., asymptotic stability and SGUUB under certain conditions. In the next section, we would introduce a new method of predicting TTF.

V. PREDICTION SCHEME

The interest of most modern industrial maintenance is to predict impending faults and alert the concerned maintenance personal by predicting the TTF so that the failing component or system can be replaced thus avoiding any catastrophic failure. The prognosis scheme will help out in this regard so that costs can be controlled due to failures. Though it is usually difficult to predict failure, TTF can be approximately obtained by predicting time to limit, In other words, systems parameters are monitored with fault and the TTF is obtained by projecting the time at which the value of the parameters reach their maximum limit usually set by a designer. The maximum limit could be the value up to which the system could perform it's intend task or operation safely. In general for most physical systems, the system parameters could be related to physical parameters. Hence in the event of a fault, the parameters may tend to increase or decrease depending on the fault characteristics.

To predict the TTF by using the parameter update law in (5), we propose the following theorem. In this theorem, we show that an explicit mathematical formula could be derived to predict the TTF. Before proceeding any further, we make the following assumption.

Assumption 5: The parameter $\hat{\theta}(k)$ is an estimate of the actual system parameter.

Remark 9: This assumption is satisfied when a system can be expressed as linear in the unknown parameters (LIP). For example in a mass damper system or civil infrastructure such as a bridge, the mass, damping and spring constants can be expressed as unknown parameters. Hence in the event of a fault, we assume that system parameters change and tend to reach their limits defined by the designer. When any one of the parameters exceeds its limit, it is considered unsafe to operate. TTF will be defined as the time that the first parameter reaches its maximum limit. Here the TTF analysis can be done with lower limits as well.

Theorem 4 (Time to failure): Assume that the parameter update law can be treated time invariant during the time interval k and $k+1$ and consider system (1) can be expressed as LIP, the TTF for the i^{th} system parameter could be iteratively determined by solving

$$k_{f_i} = \frac{\left|\log\left(\left|\frac{\left(\gamma\|I-\alpha zz^T\|\theta_{i_{\max}} - \alpha\sum_{j=1}^{n} z_{ij}B_{0_j}e_o\right)}{\left(\gamma\|I-\alpha zz^T\|\theta_{i_0} - \alpha\sum_{j=1}^{n} z_{ij}B_{0_j}e_o\right)}\right|\right)\right|}{|\log(1-\gamma\|I-\alpha zz^T\|)|} + k_{o_i} \tag{23}$$

where k_{fi} is the TTF, k_{0_i} is the time instant when the prediction starts (starts at k_d and incremented with time), $\theta_{i_{\max}}$ is the maximum value of the system parameter, and θ_{i_0} is the value of the system parameter at the time instant k_{0_i}.

Remark 10: The mathematical equation (23) is derived for the i^{th} system parameter. In general for a given system, the TTF would be $k_{ft} = \min(k_{fi})$, $i = 1, 2, \ldots l$, where l the number of system parameters. This also implies that for a fault that is occurring in the system, the TTF is obtained as the time that the first parameter reaches its limit.

Proof: In general for any system satisfying *Assumption* 5, the maximum value of the system parameter in the event of a fault is determined via physical limitation. Hence we take $\hat{\theta}_i(k_{f_i}) = \theta_{i_{\max}}$. Note that the equation (23) holds only in the time interval $k \in [k_d, k_f]$ when the residual and other terms are held constant at each k. Thus the values of Z and e_0 are known and would be held fixed for the k^{th} time instant. Under the assumption, the parameter update law shown in (5) could be written as

$$\hat{\theta}(m+1) = (I - \gamma\|I - \alpha ZZ^T\| I)\hat{\theta}(m) + \alpha Z\ B_0 e_0$$

where we use m as the time index to simplify the understanding of the theorem, and the above defined equation could be written as

$$\bar{x}(m+1) = \bar{A}.\bar{x}(m) + \bar{B}.\bar{u} \tag{24}$$

where $\bar{x}(m+1) = \hat{\theta}(m+1)$, $\bar{A} = (I - \gamma\|I - \alpha ZZ^T\| I)$ is a diagonal matrix, $\bar{x}(m) = \hat{\theta}(m)$, and $\bar{B} = \alpha$, and $\bar{u} = Z\ B_0 e_0$. Since the above defined $\bar{A}$ matrix is diagonal, (24) could be written as

$$\bar{x}_i(m+1) = \bar{a}_{ii}\bar{x}_i(m) + \bar{b}_i\bar{u}_i \tag{25}$$

where $\bar{a}_{ii} = 1 - \gamma\|I - \alpha ZZ^T\|$, $\bar{b}_i = \alpha$, and $\bar{u}_i = \sum_{j=1}^{n} z_{ij}B_{0_j}e_0$ with the elements of input being constant between the time instant k and $k+1$.

Solving (25) to determine TTF using [4], we get

$$\bar{x}_i(m) = \bar{a}_{ii}^{(m-m_0)}\bar{x}_i(m_0) + \sum_{j=m_0+1}^{m} \bar{b}_i\bar{a}_{ii}^{(m-j)}\bar{u}_i \tag{26}$$

Since at a given instance k, u_i is time-invariant in (26), thus the above equation becomes

$$\bar{x}_i(m) = \bar{a}_{ii}^{(m-m_0)}\bar{x}_i(m_0) + \bar{b}_i\bar{u}_i \sum_{j=m_0+1}^{m} \bar{a}_{ii}^{(m-m_0)}$$

Now using results of geometric series, the above equation could be written as

$$\bar{x}_i(m) = \bar{a}_{ii}^{(m-m_0)}\bar{x}_i(m_0) + \bar{b}_i\bar{u}_i\left(\frac{1-\bar{a}_{ii}^{m-m_0}}{1-\bar{a}_{ii}}\right)$$

After performing some simple mathematical manipulation, one obtains

$$\bar{a}_{ii}^{m-m_0} = \frac{[\bar{x}_i(m)(1-\bar{a}_{ii}) - \bar{b}_i\bar{u}_i]}{[\bar{x}_i(m_0)(1-\bar{a}_{ii}) - \bar{b}_i\bar{u}_i]}$$

Since $0 < \bar{a}_{ii} < 1$, take absolute value and logarithm on both sides and apply again the absolute operator to get

$$m = \frac{\left|\log\left(\left|\frac{\bar{x}_i(m)(1-\bar{a}_{ii}) - \bar{b}_i\bar{u}_i}{\bar{x}_i(m_0)(1-\bar{a}_{ii}) - \bar{b}_i\bar{u}_i}\right|\right)\right|}{|\log(\bar{a}_{ii})|} + m_0$$

Next we take $m = k_{f_i}$, and $m_0 = k_{0_i}$. Additionally, we have

$\bar{x}_i(m) = \bar{x}_i(k_{f_i}) = \theta_{i_{\max}}$, $\bar{x}_i(m_0) = \bar{x}_i(k_{0_i}) = \theta_{i_0}$, and we know that $\bar{a}_{ii} = 1 - \gamma\|I - \alpha ZZ^T\|$, $\bar{b}_i = \alpha$, and $\bar{u}_i = \sum_{j=1}^{n} z_{ij}B_0 e_0$. Thus, we get equation (23). Hence completes the proof.

After fault detection, (23) is utilized iteratively to obtain TTF in the time interval $k \in [k_d, k_f]$. To better understand the idea of updating the TTF, refer to the flowchart in Figure 1. From the flowchart, upon detecting the fault, at each time instance, $z(k)$, $\hat{\theta}(k)$ and $e_0(k)$ are calculated. Then TTF is estimated by using (23), as the parameter $\hat{\theta}(k) \to \theta_{\max}$ as $k \to k_f$. This iterative procedure allows one to accurately assess the TTF at every time instant more accurately when compared to probabilistic methods [21], where the change in the direction of the fault parameter is not known.

Next, the performance of the developed FDP scheme is simulated onto an application. The details of the simulation are given in the next section.

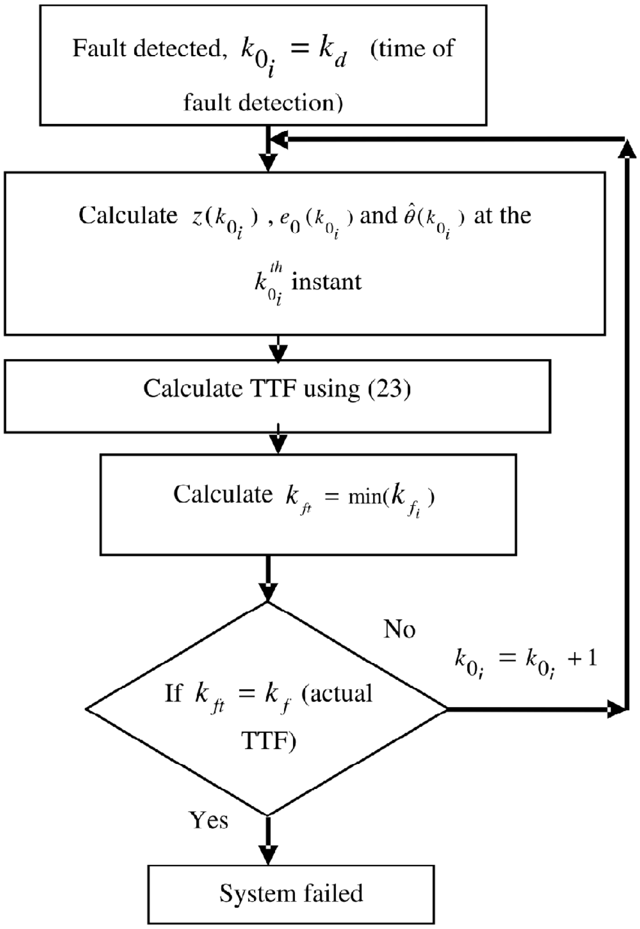

Figure 1: Procedure to Iteratively Update the TTF

VI. SIMULATION RESULTS

In this section the FDP scheme is simulated with a magnetic suspension system. The performance of the FDP scheme is shown with and without system uncertainty and measurement noise. The learning capability of the OLAD is also presented for the chosen example.

Fault Detection Scheme

To begin with, first we analyze the performance of the fault detection scheme. A simplified discrete time state space representation of a magnetic suspension system is given below [14]

$$x_1(k+1) = T_S x_2(k) + x_1(k) + \eta_1(x(k), u(k))$$

$$x_2(k+1) = T_S \left\{ \frac{1}{m}\left(-k_1 x_2(k) + 9.8 + f(y(k)) + F\right) \right\} + x_2(k)$$

$$y(k) = x_2(k) \tag{27}$$

where x_1 and x_2 are the system states, F is the input for the system in (27) and for the estimator in (28) which is taken as F = 5 sin (kT_S). A fault induced by changing the coil resistance in a nonlinear fashion by simply adding it to the system in (27) using $f(y(k))$. The following nonlinear estimator is used to study the system described in (27)

$$\hat{x}_1(k+1) = T_S x_2(k) + x_1(k) + \mathrm{a}_1 \hat{x}_2(k) + \mathrm{a}_3 \hat{x}_1(k) + \mathrm{a}_2 x_2(k)$$

$$\hat{x}_2(k+1) = T_S \left\{ \frac{1}{m}(-k_1 x_2(k) + 9.8 + F) \right\} + x_2(k) + \hat{f}(y(k), \hat{\theta}(k))$$

$$+ a_4 \hat{x}_2(k) + a_5 \hat{x}_1(k) + a_6 x_2(k)$$

$$\hat{y}(k) = \hat{x}_2(k) \tag{28}$$

where $\hat{x}_1$ and $\hat{x}_2$ are the estimated states of the system in (27), and $\hat{f}(y(k), \hat{\theta}(k))$ is the OLAD. For this simulation, the OLAD is chosen to be a single layer sigmoid function network with sixteen neurons, and the initial weights of the network ($\hat{\theta}$) are chosen randomly. The system is simulated with an abrupt fault that occurs at $T = 30$ seconds and is given by

$$\Pi(k - T)\, f(y\,(k)) = \{5 \sin(0.01\, y(k)), \text{ if } k \geq 15, \text{ else } 0 \text{ if } k \leq 15\}$$

The parameter values for the actual system (27) and the estimator (28) are taken as follows m = 1, k_1 = 0.5, a_1 = 0.0005, a_2 = 0.00005, a_3 = 0.009, a_4 = –0.5, a_5 = 0.000005, $a_3 = 0.5$ $x_1(0) = 0$, $x_2(0) = 0$, $\hat{x}_1(0) = 0$, $\hat{x}_2(0) = 0$, and T_S = 0.01. In this simulation we present two different scenarios, where in the first scenario, it is assumed that no system uncertainty (i.e., $\eta_1(x(k), u(k)) = 0$) is present with no measurement noise and in the second scenario, a fixed system uncertainty and a measurement noise of Gaussian type is considered. For both the scenarios, to tune the OLAD, the parameter update law (5) is employed. The learning rate and the design constant in (5) are taken randomly as $\alpha = 0.03$ and $\gamma = 0.001$ respectively. The simulation results for the first scenario are shown in Figs. 2 and 3. Figure 2 shows the absolute value of the residual under normal operation wherein the residual appears to be zero. However, during a fault, this residual will increase above zero indicating the presence of a fault and by initiating the OLAD.

Figure 3 shows the evolution of the fault term and the OLAD response. From this figure, it could be observed that the chosen OLAD learns the occurring fault dynamics

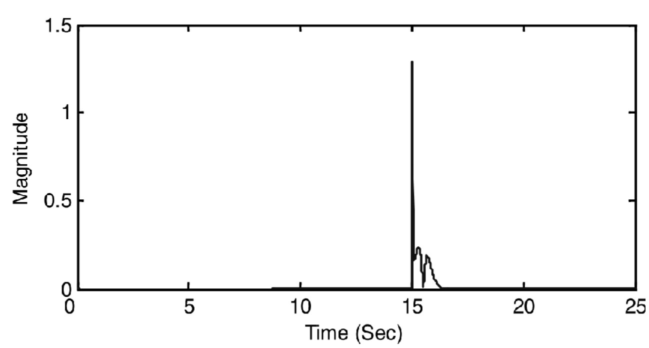

Figure 2: Absolute Value of the Residual

satisfactorily. Such online fault estimates are useful for fault isolation. To study the robustness of the scheme, we introduce a fixed system uncertainty, i.e., $\eta_1(x(k), u(k)) = 0.5$ and a measurement noise of Gaussian type with a maximum amplitude of 0.02.

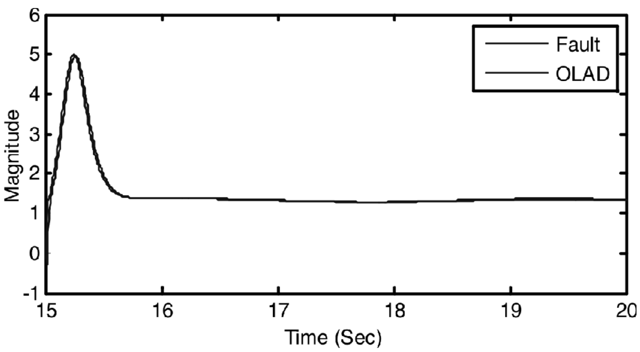

Figure 3: Evolution of the Actual Fault Term $f(y)$ and OLAD response $\hat{f}(y,\hat{\theta})$

The simulation results for this scenario are shown in Figs. 4 and 5 wherein the absolute value of the residual is illustrated in Fig. 4 and due to the presence of the modeling uncertainty, to improve robustness, a threshold is introduced. A fixed threshold of 0.1 is considered as observed in Fig. 4. The threshold is chosen based on the procedure developed in Section IV, where $\varepsilon = \frac{\beta\eta_0}{(1-\mu)}$ and solving this equation using $\eta_0 = 0.5$, $\mu = 0.9$ and $\beta_C = 0.2$, to get $\beta = 0.18$ and $\varepsilon = 0.1$. A fault is detected when the residual exceeds the threshold, which is verified as seen in Fig. 4.

Figure 5 shows the performance of the OLAD during the fault in the presence of the system uncertainty and the measurement noise. Additionally from the figure, it could be seen that the learning of the fault dynamics by the OLAD appears to be highly satisfactory. An important point to be considered here is the selection of the design parameters, size and OLAD activation functions were kept unchanged from the previous simulation. Hence even in the presence of the uncertainty and noise, the performance of the fault detection scheme is not compromised.

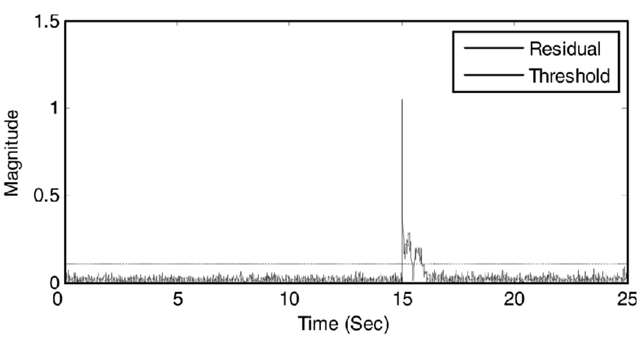

Figure 4: Absolute Value of the Residual and the Fault Detection Threshold

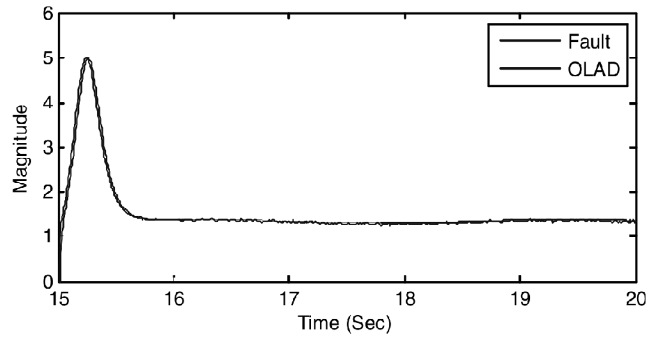

Figure 5: Evolution of the Fault $f(y)$ and OLAD Response $\hat{f}(y,\hat{\theta})$ in the Presence of the System Uncertainty and the Measurement Noise

Thus from the above simulation results, the robustness and the performance of the proposed fault detection scheme, and its learning capabilities of the OLAD were demonstrated. The scheme is able to learn online any type of unknown nonlinear faults, which is an inherent advantage. Although in this simulation, the system considered having abrupt faults, but still the fault detection scheme would be able to capture a wide range of fault conditions, which is evident from the mathematical results as seen in the previous section. This makes the OLAD based approach better than other quantitative or qualitative based methods [5, 10]. Next we illustrate the working of the prognostics scheme, where we assume the same type of fault, i.e., nonlinear change in coil resistance.

Prediction Scheme

For this simulation, a change in coil resistance in the form $f(y(k)) = 5\sin(0.01y(k))$ is considered at the 10^{th} second of operation in (1) and the prognostics scheme is now demonstrated. By using the procedure outlined in Section V, we determine the TTF. The spring constant (k_1) is considered to be unknown. Next, the parameter update law (5) is utilized to estimate the unknown system parameter. The learning rate and the design constant in (5) are chosen as $\alpha = 0.35$, $\gamma = 0.0011$ respectively. The estimated system parameter is compared with the actual system parameter by defining a maximum acceptable limit (usually using safety limit) as shown in Fig. 6. As the fault continues to grow, the actual parameter tends to increase approaching the maximum defined parameter threshold value of 30. This value was chosen randomly to demonstrate the working of the proposed prediction scheme.

From the procedure outlined in the flowchart in Fig. 1, the TTF is estimated at each time instant after the occurrence of the fault and is shown in Fig. 7. From the figure, after the first prediction of TTF, for few seconds the prediction seems to increase, this could possibly be due to the random selection of the gains of the parameter update law in (5) which needs some time to converge. However the prediction of TTF improves as the scheme learns the change in the system

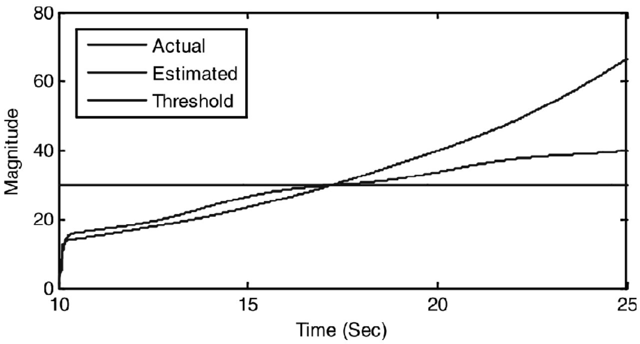

Figure 6: Comparison between the Estimated and the Actual System Parameter, and also Shown the Safe Threshold

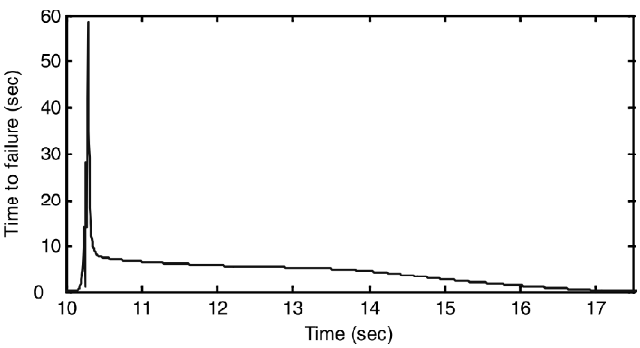

Figure 7: Prediction of TTF After the Occurrence of the Fault

dynamics and converges to the actual time of failure of 17.27 seconds. This could also be observed in Fig. 7, where the TTF decreases as the system parameter approaches the threshold.

Hence with the chosen example, the working of the FDP scheme was illustrated. The simulation results show promising performance of the proposed FDP scheme. Additionally, the robustness of the scheme was also studied by introducing uncertainty and measurement noise in the simulation results.

VII. CONCLUSIONS AND FUTURE WORK

In this paper, we have shown a FDP algorithm for nonlinear discrete time system with input and output measurements. The scheme was developed based on the assumptions that the states and the input being bounded before and after the fault. The scheme also addressed the prediction of TTF. Further more it is assumed that not all the states of the system are available for measurement. A detailed mathematical analysis and the simulation results show the robustness and performance of the proposed FDP scheme. Further based on the proofs, it was seen that the proposed scheme could be used as a robust FDP scheme for nonlinear discrete time input-output systems. Future work involves with developing fault isolation and fault accommodation techniques for a nonlinear discrete time input-output systems.

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