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Mathematical Models of Fuzzy PID Controllers with Multi-fuzzy Sets

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Abstract: *Mathematical model for fuzzy PID controllers employing* N_1 (≥ 3) *number of symmetric fuzzy sets for the input variable 'displacement',* N_2 (≥ 3) *number of symmetric fuzzy sets for the input variable 'velocity',* N_3 (≥ 3) *number of symmetric fuzzy sets for the input variable 'acceleration' and* $(N_1 + N_2 + N_3 - 2)$ *number of symmetric fuzzy sets for the output variable is revealed in this paper. The basic components used to derive this model are symmetric trapezoidal membership functions for fuzzification of inputs and output, algebraic product triangular norm, bounded sum triangular conorm and Mamdani minimum inference method for the evaluation of the control rules, and center of sums (COS) method for defuzzification. Properties of such a model are investigated. Mathematical model with* N_1 (≥ 3) *,*N_2 (≥ 3) *and* N_3 (≥ 3) *number of asymmetric fuzzy sets for the three input variables and* $N_1 + N_2 + N_3 - 2$ *number of asymmetric output fuzzy sets is also presented. The fuzzy PID controller model derived via symmetric fuzzy sets becomes a special case of the mathematical model obtained with asymmetric fuzzy sets. Finally, to demonstrate the effectiveness of the fuzzy PID controllers, some numerical examples along with their simulation results are included.*

Keywords: *PID control, Fuzzy control, Mathematical model.*

1. INTRODUCTION

A fuzzy PID controller structure (configuration 1 in Figure 1) which retains the characteristics similar to the conventional PID controller was proposed [2]. Moreover, in order to improve further the performance of the fuzzy controller, a parameter adaptive method was introduced to tune the parameters of the fuzzy controller on line.

In [3], fuzzy PID elements were proposed and different fuzzy PID structures were constructed. Expressions for the outputs of fuzzy PID elements were deduced based on linear like and nonlinear like fuzzy logic controllers. Using these expressions, apparent linear and apparent nonlinear fuzzy PID gains were deduced while considering two levels of tuning. The fuzzy PID structures were evaluated in terms of two levels of tuning. A quantitative model for fuzzy PID control, consisting of a nonlinear relay and a nonlinear PID controller, was developed [5] for mathematical analysis and gain design. Under certain approximations, this nonlinear model was found to have PID nature around the equilibrium state. The connection between the scaling gains and the control actions was expressed in an explicit mathematical form by directly comparing the proposed fuzzy PID control with the conventional PID control. This theoretical analysis revealed that fuzzy PID had led to more damping and hence less oscillations than did its conventional counterpart.

A fuzzy PID controller comprising fuzzy P, fuzzy I, and fuzzy D controllers in parallel has been proposed [6]. Fuzzy P and fuzzy I controllers have been implemented in incremental form while fuzzy D controller has been realized in position form. A fuzzy inference algorithm has been developed [7] to produce a closed-form solution of a three-input fuzzy PID system using Zadeh-Mamdani's min-max-gravity fuzzy reasoning. An input transformation technique has been proposed to reduce the number of input conditions required in defining the fuzzy output. It has been shown that with this technique the solution can be represented using a minimum number of nonlinear expressions.

Recently, optimal fuzzy reasoning technique [8] has been proposed and integrated with a PID control structure for better robust control. This fuzzy PID controller has been analyzed quantitatively and compared with other existing fuzzy PID control methods to show its improved robustness.

It is evident from the literature that

- fuzzy PID controllers of configuration 1 (Figure 1) have been developed using asymmetric triangular input fuzzy sets [2], and symmetric triangular input and output fuzzy sets [8].
- fuzzy PID controllers of different configurations have been studied [3, 5, 6] using symmetric triangular input and output fuzzy sets.
- fuzzy PID controllers of different configurations have been investigated [7] using asymmetric triangular input fuzzy sets, symmetric triangular output fuzzy sets, and unevenly distributed singleton output fuzzy sets.

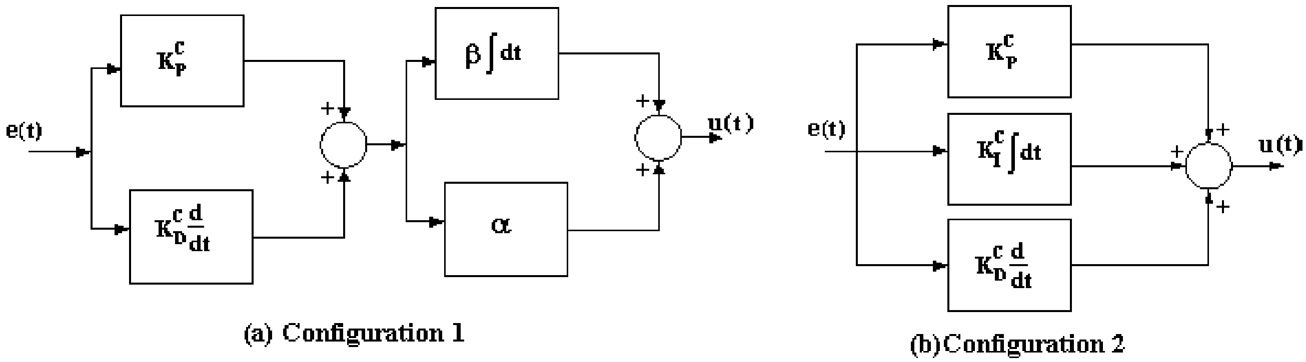

Figure 1: Fuzzy PID Controller Configurations

- fuzzy PID controller of configuration 2, shown in Figure 1, is not yet studied with multiple, asymmetric input and output fuzzy sets.

Therefore, the objectives of this paper are: (i) to derive mathematical model of fuzzy PID controller by employing symmetric triangular membership functions (N_1 for displacement, N_2 for velocity, N_3 for acceleration, and $N_1 + N_2 + N_3 - 2$ for incremental control), linear control rules, algebraic product triangular norm, bounded sum triangular conorm, Mamdani minimum inference, and COS defuzzification, (ii) to investigate the properties of this controller, (iii) to repeat (i) to obtain fuzzy PID controller model by employing asymmetric trapezoidal/triangular membership functions with the assumption that the membership sum of two neighbouring fuzzy sets is equal to unity, and (iv) to demonstrate the superiority of fuzzy controller over the conventional controller through simulation study on some examples.

This paper is organized as follows: The next section describes the principal components of a typical fuzzy PID controller. Section 3 presents mathematical model of the fuzzy PID controller with symmetric fuzzy sets. In Section 4 properties of fuzzy PID controller model are studied. Section 5 presents mathematical models of fuzzy PID controller with asymmetric triangular and trapezoidal fuzzy sets. Section 6 includes simulation results while the last section considers concluding remarks.

2. COMPONENTS OF A FUZZY THREE-TERM CONTROLLER

The principal structure of a fuzzy three-term (PID) controller is shown in Figure 2 which consists of the components such as scaling factors, fuzzification and defuzzification modules, rule base and inference engine. Components of the fuzzy controller are discussed in the following sections.

2.1. Scaling Factors

Normalization is the process of mapping physical values of actual inputs and outputs of the controller into a normalized domain. N_d, N_v, N_a and $N_{\Delta u}$ are the normalization factors for the inputs d, v, a, and the output Δu respectively. Denormalization maps the normalized output value into its physical output domain. $N_{\Delta u}^{-1}$ is the reciprocal of $N_{\Delta u}$, called denormalization factor. These scaling factors play a role similar to that of the gain coefficients K_P^d, K_I^d and K_D^d in a conventional PID controller.

2.2. Fuzzification Module

Let the number of fuzzy sets on normalized input variables "displacement $d_N(kT)$", "velocity $v_N(kT)$" and "acceleration $a_N(kT)$" be N_1, N_2 and N_3 respectively. Assume that there are J_1 number of fuzzy sets on negative displacement (J_2 on negative velocity, J_3 on negative acceleration), one fuzzy set for zero displacement (velocity or acceleration) and J_1 number of fuzzy sets on positive displacement (J_2 on positive velocity, J_3 on positive acceleration). Therefore, there is a total of

$$N = 2J + 1 \geq 3 \tag{1}$$

number of fuzzy sets on each normalized input variable, where N is N_1 (on displacement), N_2 (on velocity) and N_3 (on acceleration), and J is J_1 (for displacement), J_2 (for velocity) and J_3 (for acceleration). The fuzzy sets on each normalized input variable are as shown below:

$$\{X_{-J}, X_{-(J-1)}, \ldots, X_{-1}, X_0, X_1, \ldots, X_\lambda, \ldots, X_{J-1}, X_J\} \tag{2}$$

where X is D (for displacement), V (for velocity) and A (for acceleration) and λ is i (for displacement), j (for velocity) and n (for acceleration).The membership functions corresponding to members in Eq. (2) are considered as

$$\{\mu_{-J}(x_N), \mu_{-(J-1)}(x_N), \ldots, \mu_{-1}(x_N), \mu_0(x_N), \mu_1(x_N), \ldots, \mu_\lambda(x_N), \ldots, \mu_{J-1}(x_N), \mu_J(x_N)\} \tag{3}$$

where x_N is d_N (for displacement), v_N (for velocity) and a_N (for acceleration). The membership function $\mu_\lambda(x_N)$ corresponding to the input fuzzy set X_λ in Eq. (2) is defined as follows:

For $\lambda = -(J-1), -(J-2), \ldots, (J-2), (J-1)$

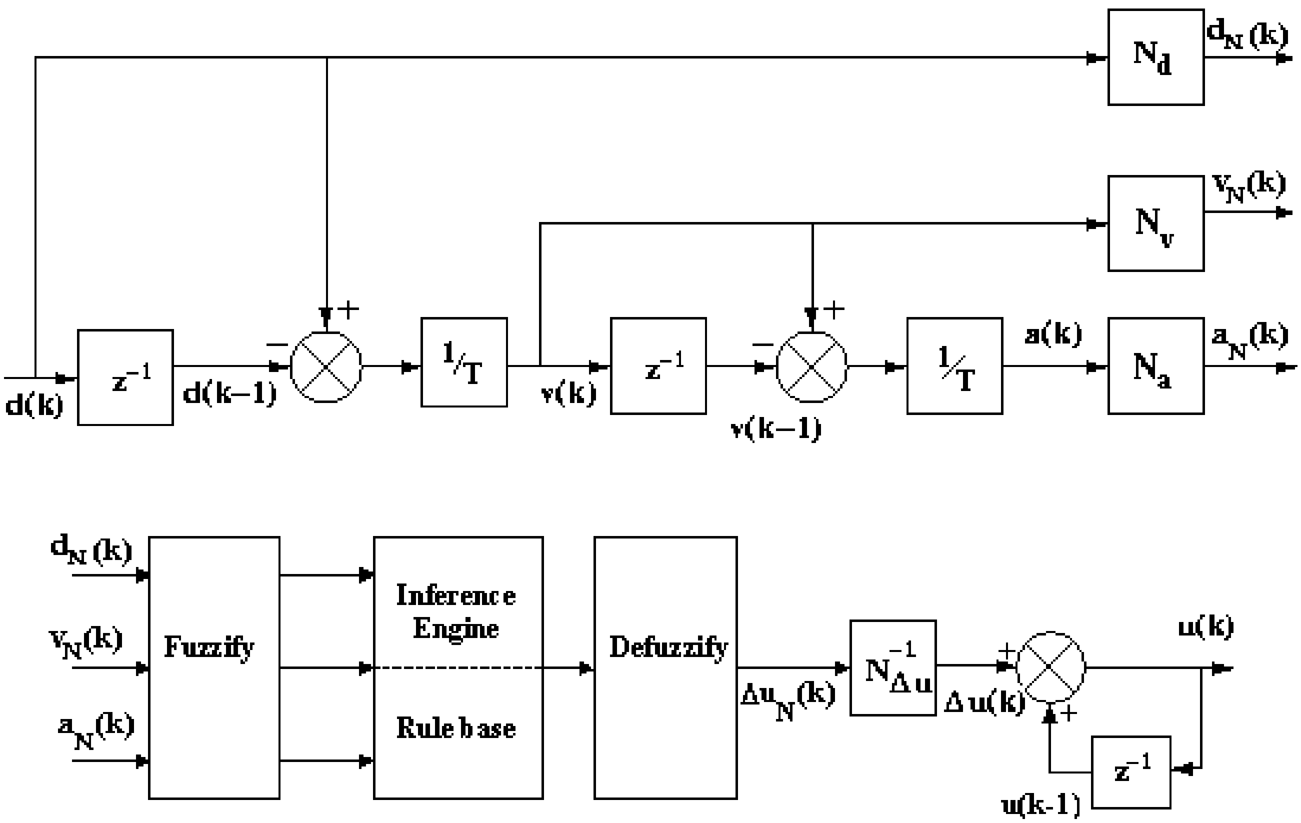

Figure 2: Block Diagram of a Typical Fuzzy PID Control System

$$\mu_\lambda(x_N)=\begin{cases} 0, & x_N \le x_{b_{\lambda-1}} \\ \dfrac{(x_N - x_{b_{\lambda-1}})}{(x_{a_\lambda} - x_{b_{\lambda-1}})}, & x_{b_{\lambda-1}} \le x_N \le x_{a_\lambda} \\ 1, & x_{a_\lambda} \le x_N \le x_{b_\lambda} \\ \dfrac{(x_{a_{\lambda+1}} - x_N)}{(x_{a_{\lambda+1}} - x_{b_\lambda})}, & x_{b_\lambda} \le x_N \le x_{a_{\lambda+1}} \\ 0, & x_{a_{\lambda+1}} \le x_N \end{cases} \tag{4}$$

For $\lambda = -J$

$$\mu_{-J}(x_N)=\begin{cases} 0, & x_N \le -L \\ 1, & -L \le x_N \le x_{b_{-J}} \\ \dfrac{(x_{a_{-(J-1)}} - x_N)}{(x_{a_{-(J-1)}} - x_{b_{-J}})}, & x_{b_{-J}} \le x_N \le x_{a_{-(J-1)}} \\ 0, & x_{a_{-(J-1)}} \le x_N \end{cases} \tag{5}$$

where x is d (for displacement), v (for velocity) and a (for acceleration), and L is L_1 (for displacement), L_2 (for velocity) and L_3 (for acceleration).

For $\lambda = J$

$$\mu_J(x_N)=\begin{cases} 0, & x_N \le x_{b_{(J-1)}} \\ \dfrac{(x_N - x_{b_{(J-1)}})}{(x_{a_J} - x_{b_{(J-1)}})}, & x_{b_{(J-1)}} \le x_N \le x_{a_J} \\ 1, & x_{a_J} \le x_N \le L \\ 0, & L \le x_N \end{cases} \tag{6}$$

Notice that

$$\mu_\lambda(x_N) + \mu_{\lambda+1}(x_N) = 1,\ x_N\ \varepsilon\,[-L, L] \tag{7}$$

and
$$i + j + n = m \tag{8}$$

Figure 3 shows membership functions $\mu_i(d_N)$, $\mu_j(v_N)$ and $\mu_n(a_N)$ corresponding to the input fuzzy sets D_i, V_j and A_n. Assume that there are $N_1 + N_2 + N_3 - 2$ (i.e. $2(J_1 + J_2 + J_3) + 1$)) number of fuzzy sets on the normalized output variable, the incremental control effort $\Delta u_N(kT)$. Among these, $J_1 + J_2 + J_3$ members are on negative output, $J_1 + J_2 + J_3$ members are on positive output and one member for zero output. The membership functions for normalized output is shown in Figure 4 and can be described by

$$\{O_{-J_0}, O_{-(J_0-1)}, \ldots, O_{-1}, O_0, O_1, \ldots, O_m, \ldots, O_{J_0-1}, O_{J_0}\} \tag{9}$$

where $J_0 = J_1 + J_2 + J_3$. Let $M = b_{J_0}$. The parameters L_1, L_2,

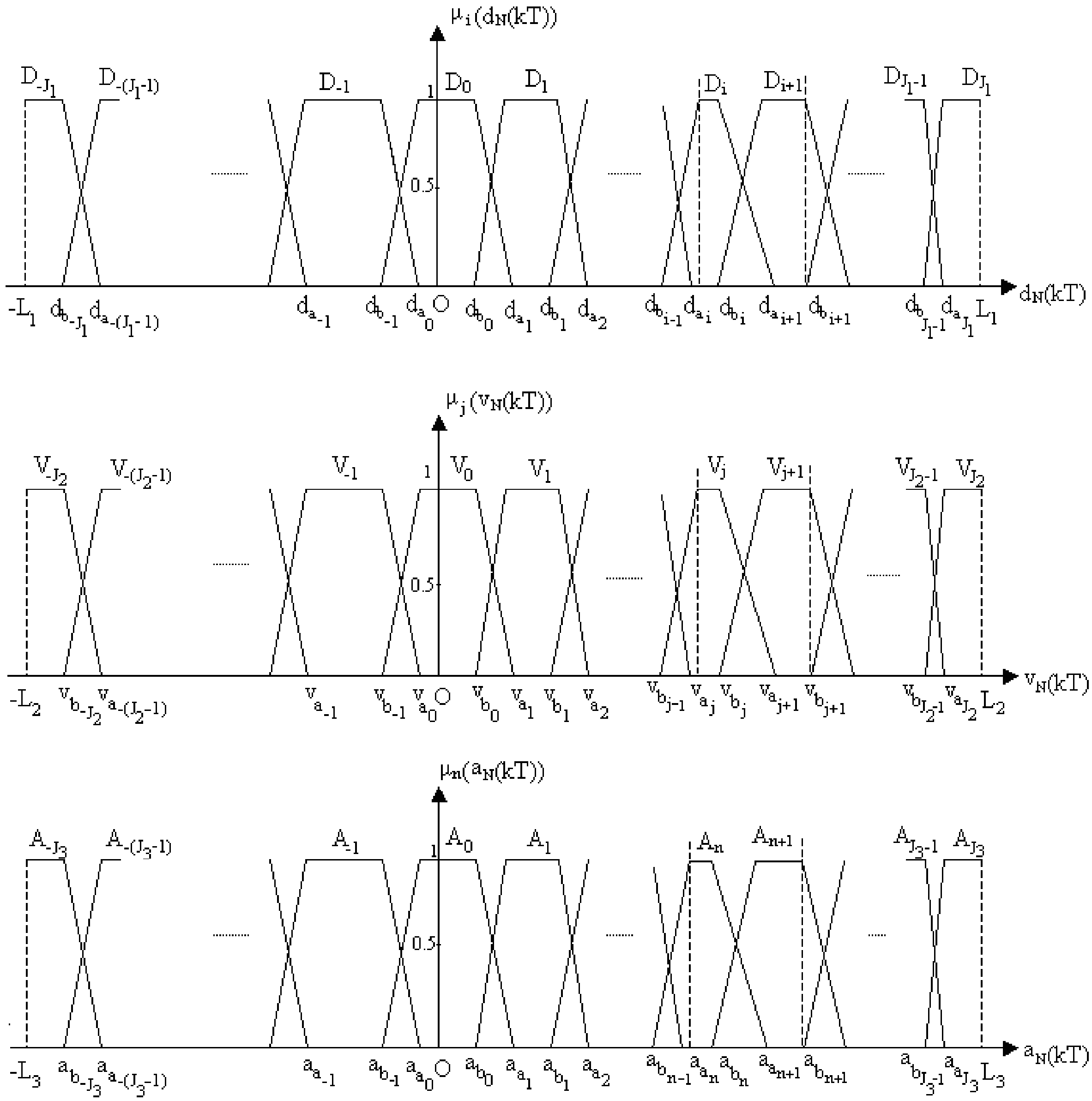

Figure 3: Membership Functions for Inputs $d_N(kT)$, $v_N(kT)$ and $a_N(kT)$

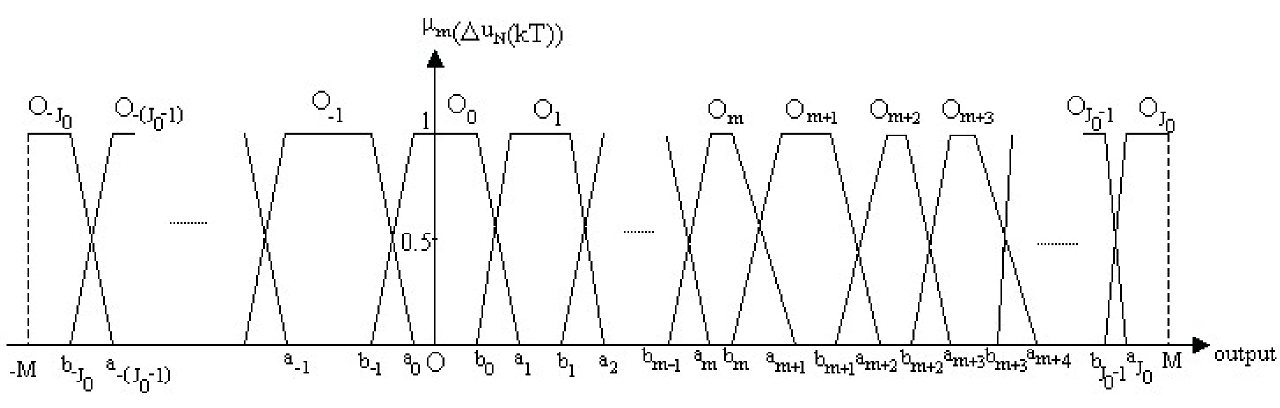

Figure 4: Output Membership Functions

L_3, M, d_{a_i} and d_{b_i} for $i = -J_1, -(J_1-1), \ldots -1, 0, 1, \ldots (J_1-1), J_1$; v_{a_j} and v_{b_j} for $j = -J_2, -(J_2-1), \ldots -1, 0, 1, \ldots (J_2-1), J_2$; a_{a_n} and a_{b_n} for $n = -J_3, -(J_3-1), \ldots -1, 0, 1, \ldots (J_3-1), J_3$; a_m and b_m for $m = -J_0, -(J_0-1), \ldots -1, 0, 1, \ldots (J_0-1), J_0$ are chosen by the designer.

2.3. Control Rule Base

The following linear control rules are considered in terms of the abovementioned input and output fuzzy sets.

(R_1) If d_N is D_i & v_N is V_j & a_N is A_n then Δu_N is O_m.
(R_2) If d_N is D_{i+1} & v_N is V_j & a_N is A_n then Δu_N is O_{m+1}.
(R_3) If d_N is D_{i+1} & v_N is V_j & a_N is A_{n+1} then Δu_N is O_{m+2}.
(R_4) If d_N is D_i & v_N is V_j & a_N is A_{n+1} then Δu_N is O_{m+1}.
(R_5) If d_N is D_i & v_N is V_{j+1} & a_N is A_{n+1} then Δu_N is O_{m+2}.
(R_6) If d_N is D_i & v_N is V_{j+1} & a_N is A_n then Δu_N is O_{m+1}.
(R_7) If d_N is D_{i+1} & v_N is V_{j+1} & a_N is A_n then Δu_N is O_{m+2}.
(R_8) If d_N is D_{i+1} & v_N is V_{j+1} & a_N is A_{n+1} then Δu_N is O_{m+3}.

The & symbol in the above rules represents the fuzzy 'AND' operation and the AND operation considered here is algebraic product triangular norm which is given by

$$\hat{\mu}(d_N, v_N, a_N) = \mu_p(d_N) \cdot \mu_q(v_N) \cdot \mu_r(a_N) \tag{10}$$

where $p \in \{D_i, D_{i+1}\}$, $q \in \{V_j, V_{j+1}\}$ and $r \in \{A_n, A_{n+1}\}$ are the p^{th}, q^{th} and r^{th} fuzzy sets on d_N, v_N and a_N respectively. It is to be noted here that the control rules are linear as the output fuzzy sets are linearly related to the input fuzzy sets.

2.4. Inference Engine

The degree of match is computed for each rule using algebraic product triangular norm given by Eq. (10). Then the degree of match is used to determine the inferred output fuzzy set via Mamdani minimum inference method, defined as $\min(\hat{\mu}, \mu(\Delta u))$. The reference output fuzzy set (trapezoid), and the inferred output fuzzy set (shown with hatching) are shown in Figure 5.

There are twelve possible input combinations, see Figure 6(a), of the normalized inputs $d_N(kT)$ and $v_N(kT)$ in the region

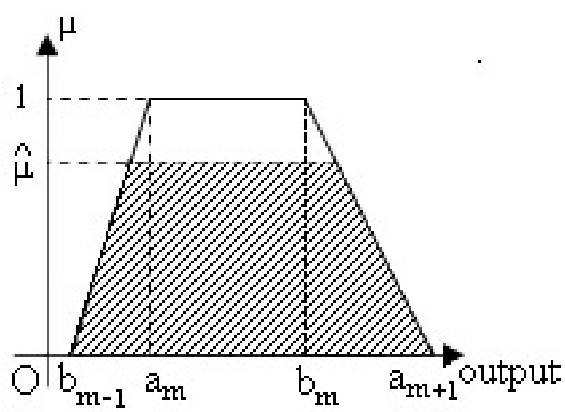

Figure 5: Mamdani Minimum Inference Method

defined by $d_{a_i} \le d_N(kT) \le d_{b_{i+1}}$ and $v_{a_j} \le v_N(kT) \le v_{b_{j+1}}$, twelve possible input combinations of the normalized inputs $d_N(kT)$ and $a_N(kT)$ in the region defined by $d_{a_i} \le d_N(kT) \le d_{b_{i+1}}$ and $a_{a_n} \le a_N(kT) \le a_{b_{n+1}}$, and twelve possible input combinations of the normalized inputs $v_N(kT)$ and $a_N(kT)$ in the region defined by $v_{a_j} \le v_N(kT) \le v_{b_{j+1}}$ and $a_{a_n} \le a_N(kT) \le a_{b_{n+1}}$. Similarly there are eight possible input combinations of the normalized inputs in each of the ($d_N v_N$ –, $d_N a_N$ –, and $v_N a_N$ –) plane shown in Figure 6(b). To represent a state point uniquely in the 3D input space, all possible input combinations in different planes are considered and thus 8000 different cells of the form (n_I, n_{II}, n_{III}) are obtained where n_I, n_{II}, n_{III} = 1, 2, …, 12 in Figure 6(a) and n_I, n_{II}, n_{III} = 13, 14, …, 20 in Figure 6(b). Not all 8000 cells are valid cells; only a few of them are valid. A cell (n_I, n_{II}, n_{III}) is said to be valid if and only if the relations between d_N and v_N, and d_N and a_N produce the relation between v_N and a_N. For example, the cell (4, 1, 4) is a valid cell because the relations $d_N \le v_N$ and $d_N \ge a_N$ produce the relation $v_N \ge d_N \ge a_N$ which is satisfied by the relation $v_N \ge a_N$. The outcomes of the control rules for all the valid cells with algebraic product triangular norm are listed in Table 1.

It may be seen from the control rules that each of the output fuzzy sets O_{m+1} and O_{m+2} are fired three times. In such a situation bounded sum triangular conorm is used to evaluate combined output fuzzy sets corresponding to the rule sets {(R_2), (R_4), (R_6)} and {(R_3), (R_5), (R_7)}. This triangular conorm is defined as $\min\{1, \mu_A(\Delta u_N) + \mu_B(\Delta u_N)\}$ where A and B are the fuzzy sets on the normalized output Δu_N.

Since the fuzzy controller is having three inputs, and algebraic product triangular norm is used, sum of all the outcomes corresponding to either rule set is less than unity. Therefore the combined membership using bounded sum triangular conorm is given by

$$\mu(R_2) + \mu(R_4) + \mu(R_6) < 1$$

or

$$\mu(R_3) + \mu(R_5) + \mu(R_7) < 1$$

2.5. Defuzzification

The most commonly used COS method is employed to defuzzify the incremental control output. This is expressed as

$$\Delta u_N(kT) = \frac{\begin{array}{c}\{A(\hat{\mu}_1)(h_1) + A(\hat{\mu}_2)(h_2) + A(\hat{\mu}_3)(h_3) + A(\hat{\mu}_4)(h_4) \\ + A(\hat{\mu}_5)(h_5) + A(\hat{\mu}_6)(h_6) + A(\hat{\mu}_7)(h_7) + A(\hat{\mu}_8)(h_8)\}\end{array}}{\sum_{i=1}^{8} A(\hat{\mu}_i)} \tag{11}$$

where $A(\hat{\mu}_i)$ is the area of the inferred output fuzzy set corresponding to the rule R_i and h_i, i = 1, 2, ... 8, is the centroid of inferred output fuzzy set (shown with hatching in Figure 5) corresponding to the rule R_i. As mentioned in

Table 1
The Outcomes of 'Algebraic Product' Operation of Premise Part of Fuzzy Control Rules (R_1) – (R_8) for Valid 3D Cells

Cells	(R_1) $\hat{\mu}_1$	(R_2) $\hat{\mu}_2$	(R_3) $\hat{\mu}_3$	(R_4) $\hat{\mu}_4$	(R_5) $\hat{\mu}_5$	R_6) $\hat{\mu}_6$	(R_7) $\hat{\mu}_7$	(R_8) $\hat{\mu}_8$
*(1, 1, 1) to (4, 4, 4)**	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8
(5, 9, 5)	0	μ_{V_j}	0	0	0	0	$\mu_{V_{j+1}}$	0
(5, 10, 7)	0	0	μ_{V_j}	0	0	0	0	$\mu_{V_{j+1}}$
(6, 6, 10)	0	0	$\mu_{D_{i+1}}$	μ_{D_i}	0	0	0	0
(6, 8, 9)	μ_{D_i}	$\mu_{D_{i+1}}$	0	0	0	0	0	0
(7, 11, 7)	0	0	0	μ_{V_j}	$\mu_{V_{j+1}}$	0	0	0
(7, 12, 5)	μ_{V_j}	0	0	0	0	$\mu_{V_{j+1}}$	0	0
(8, 6, 11)	0	0	0	0	μ_{D_i}	0	0	$\mu_{D_{i+1}}$
(8, 8, 12)	0	0	0	0	0	μ_{D_i}	$\mu_{D_{i+1}}$	0
(9, 5, 6)	0	μ_{A_n}	$\mu_{A_{n+1}}$	0	0	0	0	0
(9, 9, 9), (17, 17, 17)	0	1	0	0	0	0	0	0
(9, 10, 10), (17, 18, 18)	0	0	1	0	0	0	0	0
(10, 7, 6)	μ_{A_n}	0	0	$\mu_{A_{n+1}}$	0	0	0	0
(10, 11, 10), (18, 19, 18)	0	0	0	1	0	0	0	0
(10, 12, 9), (18, 20, 17)	1	0	0	0	0	0	0	0
(11, 7, 8)	0	0	0	0	$\mu_{A_{n+1}}$	μ_{A_n}	0	0
(11, 11, 11), (19, 19, 19)	0	0	0	0	1	0	0	0
(11, 12, 12), (19, 20, 20)	0	0	0	0	0	1	0	0
(12, 5, 8)	0	0	0	0	0	0	μ_{A_n}	$\mu_{A_{n+1}}$
(12, 9, 12), (20, 17, 20)	0	0	0	0	0	0	1	0
(12, 10, 11), (20, 18, 19)	0	0	0	0	0	0	0	1
(13, 17, 13)	0	$\mu_{V_{-J_2}}$	0	0	0	0	$\mu_{V_{J_2}}$	0
(13, 18, 15)	0	0	$\mu_{V_{-J_2}}$	0	0	0	0	$\mu_{V_{J_2}}$
(14, 14, 18)	0	0	$\mu_{D_{J_1}}$	$\mu_{D_{-J_1}}$	0	0	0	0
(14, 16, 17)	$\mu_{D_{-J_1}}$	$\mu_{D_{J_1}}$	0	0	0	0	0	0
(15, 19, 15)	0	0	0	$\mu_{V_{-J_2}}$	$\mu_{V_{J_2}}$	0	0	0
(15, 20, 13)	$\mu_{V_{-J_2}}$	0	0	0	0	$\mu_{V_{J_2}}$	0	0
(16, 14, 19)	0	0	0	0	$\mu_{D_{-J_1}}$	0	0	$\mu_{D_{J_1}}$
(16, 16, 20)	0	0	0	0	0	$\mu_{D_{-J_1}}$	$\mu_{D_{J_1}}$	0
(17, 13, 14)	0	$\mu_{A_{-J_3}}$	$\mu_{A_{J_3}}$	0	0	0	0	0
(18, 15, 14)	$\mu_{A_{-J_3}}$	0	0	$\mu_{A_{J_3}}$	0	0	0	0
(19, 15, 16)	0	0	0	0	$\mu_{A_{J_3}}$	$\mu_{A_{-J_3}}$	0	0
(20, 13, 16)	0	0	0	0	0	0	$\mu_{A_{-J_3}}$	$\mu_{A_{J_3}}$

where $\mu_1 = \mu_{D_i}\mu_{V_j}\mu_{A_n}$, $\mu_2 = \mu_{D_{i+1}}\mu_{V_j}\mu_{A_n}$, $\mu_3 = \mu_{D_{i+1}}\mu_{V_j}\mu_{A_{n+1}}$, $\mu_4 = \mu_{D_i}\mu_{V_j}\mu_{A_{n+1}}$, $\mu_5 = \mu_{D_i}\mu_{V_{j+1}}\mu_{A_{n+1}}$, $\mu_6 = \mu_{D_i}\mu_{V_{j+1}}\mu_{A_n}$, $\mu_7 = \mu_{D_{i+1}}\mu_{V_{j+1}}\mu_{A_n}$, $\mu_8 = \mu_{D_{i+1}}\mu_{V_{j+1}}\mu_{A_{n+1}}$
* only valid cells in (1, 1, 1) through (4, 4, 4)

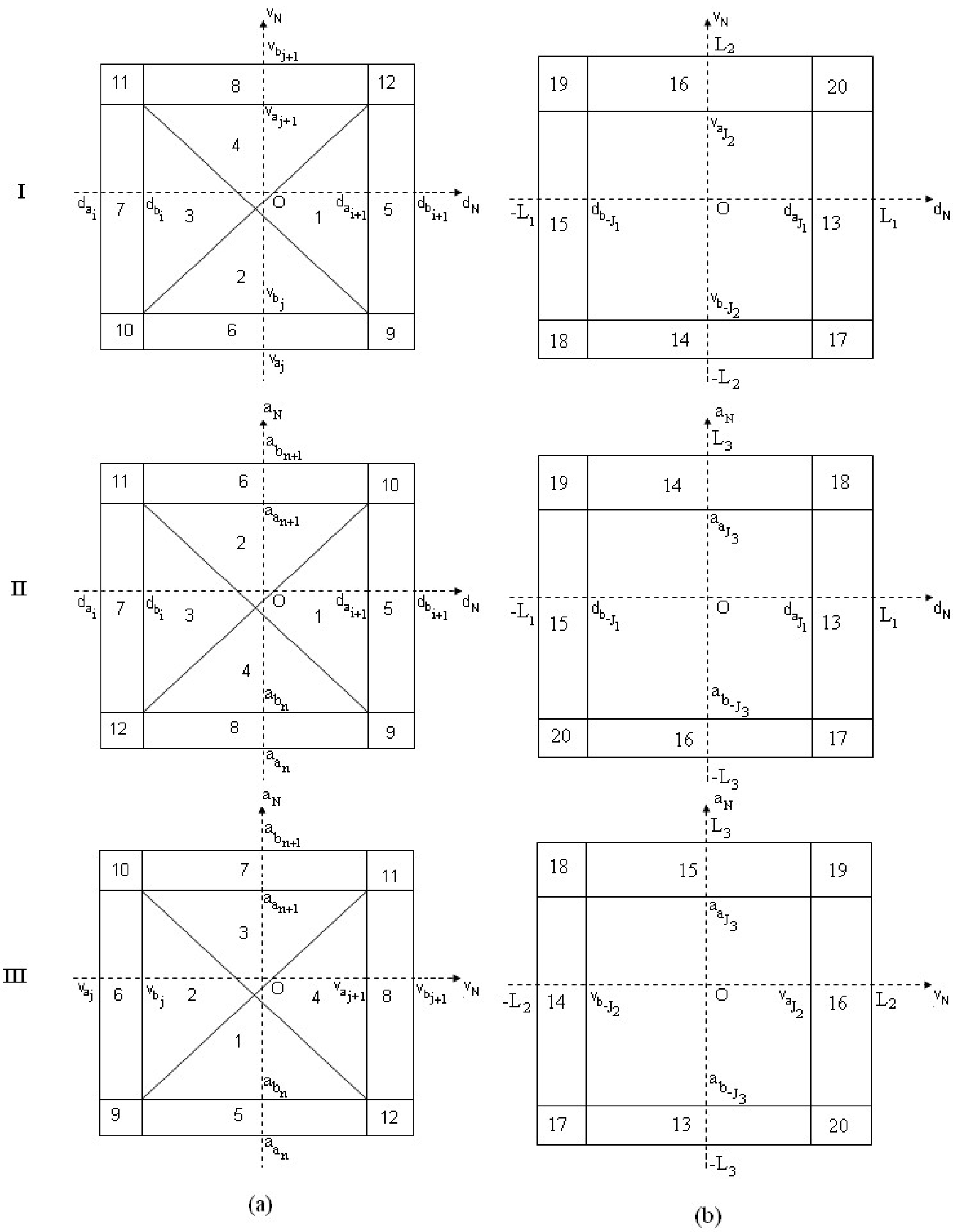

Figure 6: Possible Input Combinations of $d_N(kT)$, $v_N(kT)$ and $a_N(kT)$ (a) in the Region: $d_{a_i} \le d_N(kT) \le d_{b_{i+1}}$, $v_{a_j} \le v_N(kT) \le v_{b_{j+1}}$, $a_{a_n} \le a_N(kT) \le a_{b_{n+1}}$ (b) Outside the Region: $d_{b-j_1} \le d_N(kT) \le d_{aj_1}$, $v_{b-j_2} \le v_N(kT) \le v_{aj_2}$, $a_{b-j_3} \le a_N(kT) \le {}_{aaj_3}$

Section 2.4, the output fuzzy set O_{m+1} is fired three times for rule set $\{(R_2), (R_4), (R_6)\}$ and O_{m+2} is fired three times for the rule set $\{(R_3), (R_5), (R_7)\}$. In this situation, using the bounded sum triangular conorm, Eq. (11) can be written as

$$\Delta u_N(kT) = \frac{A(\hat{\mu}_1)(h_1) + A(\hat{\mu}_{2/4/6})(h_{2/4/6}) + A(\hat{\mu}_{3/5/7})(h_{3/5/7}) + A(\hat{\mu}_8)(h_8)}{A(\hat{\mu}_1) + A(\hat{\mu}_{2/4/6}) + A(\hat{\mu}_{3/5/7}) + A(\hat{\mu}_8)} \tag{12}$$

where $\hat{\mu}_{2/4/6}$ and $\hat{\mu}_{3/5/7}$ are the outcomes obtained using the triangular conorm. The area A and centroid h of the inferred output fuzzy set, obtained via Mamdani minimum inference and shown in Figure 5, are respectively given by $A = \hat{\mu}(a_{m+1} - b_{m-1})[2 - \hat{\mu}(1 - \theta_m)]/2$, and

$$h = \frac{\begin{array}{c}\{3(a_{m+1}^2 - b_{m-1}^2) - 3\hat{\mu}[b_{m-1}(a_m - b_{m-1}) \\ +a_{m+1}(a_{m+1} - b_m)] - \hat{\mu}^2[(a_m - b_{m-1})^2 - (a_{m+1} - b_m)^2]\}\end{array}}{3(a_{m+1} - b_{m-1})[2 - \hat{\mu}(1 - \theta_m)]} \tag{13}$$

where $$\theta_m = (b_m - a_m)/(a_{m+1} - b_{m-1}) \tag{14}$$

3. MATHEMATICAL MODELS OF THE FUZZY PID CONTROLLERS WITH SYMMETRIC TRIANGULAR FUZZY SETS

In this section mathematical model for fuzzy PID controllers with symmetric fuzzy sets is presented. For symmetric input membership functions, we consider $a_{a_{n+1}} - a_{b_n} = v_{a_{j+1}} - v_{b_j} = d_{a_{i+1}} - d_{b_i} = S$, and $a_{b_n} - a_{a_n} = v_{b_j} - v_{a_j} = d_{b_i} - d_{a_i} = 0$ for all i, j and n in Figure 3. Thus, with symmetric triangular input fuzzy sets, the cells (5, 5, 5) to (12, 12, 12) in Table 1 do not exist. Symmetric triangular output membership functions can be obtained by letting $a_m - b_{m-1} = a_{m+1} - b_m = W$ and $b_m - a_m = 0$ for all m in Figure 4. This results in $\theta_m = \theta_{m+1} = \theta_{m+2} = \theta_{m+3} = 0$, $a_m = b_m = m.W = (i + j + n)W$, $a_{m+1} = b_{m+1} = (m + 1)W = (i + j + n + 1)W$, $a_{m+2} = b_{m+2} = (m + 2)W = (i + j + n + 2)W$, and $a_{m+3} = b_{m+3} = (m + 3)W = (i + j + n + 3)W$ in view of Eq. (8), and the expression in Eq. (12) for incremental control output modifies to

$$\Delta u = \left(\frac{1}{N_{\Delta u}}\right)\frac{[A(\hat{\mu}_1)(i+j+n)W + A(\hat{\mu}_{2/4/6})(i+j+n+1)W + A(\hat{\mu}_{3/5/7})(i+j+n+2)W + A(\hat{\mu}_8)(i+j+n+3)W]}{A(\hat{\mu}_1)+A(\hat{\mu}_{2/4/6})+A(\hat{\mu}_{3/5/7})+A(\hat{\mu}_8)}$$

$$= \frac{(i+j+n+1.5)W}{N_{\Delta u}} + \left(\frac{W}{N_{\Delta u}}\right)\frac{1.5[A(\mu_8)-A(\mu_1)]-0.5[A(\mu_{2/4/6})-A(\mu_{3/5/7})]}{A(\hat{\mu}_1)+A(\hat{\mu}_{2/4/6})+A(\hat{\mu}_{3/5/7})+A(\hat{\mu}_8)}$$

$$= \Delta u_g + \Delta u_l \qquad (15)$$

where
$$\Delta u_g = \frac{(i+j+n+1.5)W}{N_{\Delta u}} \qquad (16)$$

and

$$\Delta u_l = \left(\frac{W}{N_{\Delta u}}\right)\frac{1.5[A(\mu_8)-A(\mu_1)]-0.5[A(\mu_{2/4/6})-A(\mu_{3/5/7})]}{A(\hat{\mu}_1)+A(\hat{\mu}_{2/4/6})+A(\hat{\mu}_{3/5/7})+A(\hat{\mu}_8)} \qquad (17)$$

In view of Eq. (15) we have

Theorem: The structure of the fuzzy PID controller with linear control rules is the sum of a global three-dimensional multi-level relay and a local nonlinear PID controller.

In the following expressions, for simplicity we define $z_1(k) = d_N(k) - (i + 0.5)S$, $z_2(k) = v_N(k) - (j + 0.5)S$ and $z_3(k) = a_N(k) - (n + 0.5)S$ and consider kT as k. The mathematical model of fuzzy PID controller in different cells follows now.

Case (a): $iS \le d_N(k) \le (i + 1)S$, $jS \le v_N(k) \le (j + 1)S$, $nS \le a_N(k) \le (n + 1)S$

$$\Delta u(k) = \frac{(i+j+n+1.5)W}{N_{\Delta u}} + \left(\frac{2W}{N_{\Delta u}}\right)\frac{N_{m1}z_1(k)+N_{m2}z_2(k)+N_{m3}z_3(k)}{D} \qquad (18)$$

where

$$N_{m1} = 7S^5 - 4(z_2^2(k)+z_3^2(k))S^3 - 16z_2^2(k)z_3^2(k)S \qquad (19)$$

$$N_{m2} = 7S^5 - 4(z_3^2(k)+z_1^2(k))S^3 - 16z_3^2(k)z_1^2(k)S \qquad (20)$$

$$N_{m3} = 7S^5 - 4(z_1^2(k)+z_2^2(k))S^3 - 16z_1^2(k)z_2^2(k)S \qquad (21)$$

and
$$D = 15S^6 - 4(z_1^2(k)+z_2^2(k)+z_3^2(k))S^4 - 16(z_1^2(k)z_2^2(k) + z_2^2(k)z_3^2(k)+z_3^2(k)z_1^2(k))S^2 - 64z_1^2(k)z_2^2(k)z_3^2(k) \qquad (22)$$

Case (b): One input lies in the cube defined by $iS \le d_N(k) \le (i + 1)S$, $jS \le v_N(k) \le j + 1S$, $nS \le a_N(k) \le (n + 1)S$ and the remaining two inputs do not lie within this cube. The $\Delta u(k)$ in different cells is as follows:

$$\Delta u(k) = \frac{(y+0.5)W}{N_{\Delta u}} + \left(\frac{2WS}{N_{\Delta u}}\right)\frac{x}{3S^2-4x^2} \qquad (23)$$

with x and y as defined in Table 2.

$$\Delta u(k) = \frac{(y+2J-0.5)W}{N_{\Delta u}} + \left(\frac{W}{N_{\Delta u}}\right)\frac{3S^2+2Sx-4x^2}{3S^2-4x^2} \qquad (24)$$

with x and y as defined in Table 2.

$$\Delta u(k) = \frac{(y-2J+1.5)W}{N_{\Delta u}} - \left(\frac{W}{N_{\Delta u}}\right)\frac{3S^2-2Sx-4x^2}{3S^2-4x^2} \qquad (25)$$

with x and y as defined in Table 2.

Table 2
Attributes of x and y

x	y	*Equation (23) with cells*	*Equation (24) with cells*	*Equation (25) with cells*
$z_1(k)$	i	(14, 14, 18), (16, 16, 20)	(16, 14, 19)	(14, 16, 17)
$z_2(k)$	j	(13, 17, 13), (15, 19, 15)	(13, 18, 15)	(15, 20, 13)
$z_3(k)$	n	(17, 13, 14), (19, 15, 16)	(20, 13, 16)	(18, 15, 14)

Case (c): All the three inputs do not lie within the inner cube defined by $iS \le d_N(k) \le (i + 1)S$, $jS \le v_N(k) \le (j + 1)S$, $nS \le a_N(k) \le (n + 1)S$.

$$\Delta u(k) = \frac{-M}{3N_{\Delta u}} \text{ for cells (17,17,17), (18,19,18), (19,20,20)} \qquad (26)$$

$$\Delta u(k) = \frac{M}{3N_{\Delta u}} \text{ for cells (17,18,18), (19,19,19), (20,17,20)} \qquad (27)$$

$$\Delta u(k) = \frac{-M}{N_{\Delta u}} \text{ for cell (18,20,17)} \qquad (28)$$

$$\Delta u(k) = \frac{M}{N_{\Delta u}} \text{ for cell (20,18,19)} \qquad (29)$$

4. PROPERTIES OF FUZZY PID CONTROLLERS WITH SYMMETRIC TRIANGULAR FUZZY SETS

The mathematical model of fuzzy PID controller presented in the previous section is in the form of Eq. (15), in which Δu_g represents the global three-dimensional multilevel relay with respect to *i*, *j*, and *n*, and Δu_l represents the local nonlinear PID controller. Since $W = M/(3J)$ and $S = l/J$, Eq. (16) can be expressed as

$$\Delta u_g = \Delta u_g(i, j, n) = \frac{M \times [(i+0.5)S + (j+0.5)S + (n+0.5)S]}{N_{\Delta u} \times 3l}$$

where $l = d_{a_{J_1}} = v_{a_{J_2}} = a_{a_{J_3}} = J$ when $J_1 = J_2 = J_3 = J$ in Figure 3. Considering symmetric fuzzy sets, the point $((i + 0.5)S, (j + 0.5)S)$ is the center of the square in the $d_N v_N$–plane, $((i + 0.5)S, (n + 0.5)S)$ is the center of the square in the $d_N a_N$–plane and $((j + 0.5)S, (n + 0.5)S)$ is the center of the square in the $v_N a_N$–plane, all shown in Figure 6(a). The control action produced by the multilevel relay depends on the current position of squares in the respective scaled input planes. As the multilevel relay sweeps the entire input planes when all values of *i*, *j*, *n* are used, the first part of the incremental control output Δu_g is known as the 'global' multilevel relay. By studying Eqs. (18) to (25), it is found that the second part of the controller output $\Delta u_l(k)$ is calculated based on the relative position of the normalized inputs with respect to center of each square. Thus, $\Delta u_l(k)$ locally adjusts the control action generated by the global multilevel relay. Therefore the second part is known as the 'local' nonlinear PID controller.

For the fuzzy controller considered in Section 3, the maximum absolute value of $\Delta u_g(i, j, n)$ occurs at $i, j, n = J - 1$ or at $i, j, n = -J$ and its value is

$$|\Delta u_g|_{max} = \frac{M \times (N-2)}{N_{\Delta u} \times (N-1)} \tag{30}$$

and the maximum absolute value of $\Delta u_l(k)$, occurring at $(d_N, v_N, a_N) = ((i+1)S, (j+1)S, (n+1)S)$ or (*iS*, *jS*, *nS*), is given by

$$|\Delta u_l|_{max} = \frac{M}{N_{\Delta u} \times (N-1)} \tag{31}$$

To study the roles of global three-dimensional multilevel relay and local nonlinear PID controller in total control action, and the degree of nonlinearity of fuzzy controllers as *N* changes, we define a constant η as

$$\eta = \frac{|\Delta u_l|_{max}}{|\Delta u_g|_{max} + |\Delta u_l|_{max}} \times 100 = \frac{1}{N-1} \times 100 \tag{32}$$

It can be seen from Eq. (32) that (i) for larger values of *N*, η approaches a smaller value which leads to improvement in the resolution of the global multilevel relay output and the fuzzy controller becomes less nonlinear, and (ii) as $N \geq 3$, η attains a maximum value of 50% for $N = 3$ which implies that the global multilevel relay and the local nonlinear PID controller play equal roles in total control action.

5. MATHEMATICAL MODELS OF FUZZY PID CONTROLLERS WITH ASYMMETRIC FUZZY SETS

In the following mathematical models of fuzzy PID controllers with asymmetric fuzzy sets are presented.

5.1. Output Fuzzy Sets–Trapezoidal

Case (a): All the three inputs lie within the 3D input space defined by $d_{b_i} \leq d_N(k) \leq d_{a_{i+1}}$, $v_{b_j} \leq v_N(k) \leq v_{a_{j+1}}$, $a_{b_n} \leq a_N(k) \leq a_{a_{n+1}}$ i.e. cells (1, 1, 1) to (4, 4, 4)

$$\Delta u(k) = \frac{1}{3N_{\Delta u}}\left(\frac{Num}{Den}\right) \tag{33}$$

where

$$\begin{aligned} Num = 3\{&\hat{\mu}_1(a_{m+1}^2 - b_{m-1}^2) + (\hat{\mu}_2 + \hat{\mu}_4 + \hat{\mu}_6)(a_{m+2}^2 - b_m^2) + (\hat{\mu}_3 + \hat{\mu}_5 + \hat{\mu}_7)(a_{m+3}^2 - b_{m+1}^2) \\ &+ \hat{\mu}_8(a_{m+4}^2 - b_{m+2}^2) - \hat{\mu}_1^2[b_{m-1}(a_m - b_{m-1}) + a_{m+1}(a_{m+1} - b_m)] - (\hat{\mu}_2^2 + \hat{\mu}_4^2 + \hat{\mu}_6^2) \\ &[b_m(a_{m+1} - b_m) + a_{m+2}(a_{m+2} - b_{m+1})] - (\hat{\mu}_3^2 + \hat{\mu}_5^2 + \hat{\mu}_7^2)[b_{m+1}(a_{m+2} - b_{m+1}) + a_{m+3} \\ &(a_{m+3} - b_{m+2})] - \hat{\mu}_8^2[b_{m+2}(a_{m+3} - b_{m+2}) + a_{m+4}(a_{m+4} - b_{m+3})]\} - \hat{\mu}_1^3[(a_m - b_{m-1})^2 \\ &- (a_{m+1} - b_m)^2] - (\hat{\mu}_2^3 + \hat{\mu}_4^3 + \hat{\mu}_6^3)[(a_{m+1} - b_m)^2 - (a_{m+2} - b_{m+1})^2] - (\hat{\mu}_3^3 + \hat{\mu}_5^3 \\ &+ \hat{\mu}_7^3)[(a_{m+2} - b_{m+1})^2 - (a_{m+3} - b_{m+2})^2] - \hat{\mu}_8^3[(a_{m+3} - b_{m+2})^2 - (a_{m+4} - b_{m+3})^2] \end{aligned}$$

and

$$\begin{aligned} Den = &\hat{\mu}_1(a_{m+1} - b_{m-1})[2 - \hat{\mu}_1(1 - \theta_m)] + (a_{m+2} - b_m)[2(\hat{\mu}_2 + \hat{\mu}_4 + \hat{\mu}_6) \\ &- (\hat{\mu}_2^2 + \hat{\mu}_4^2 + \hat{\mu}_6^2)(1 - \theta_{m+1})] + (a_{m+3} - b_{m+1})[2(\hat{\mu}_3 + \hat{\mu}_5 + \hat{\mu}_7) \\ &- (\hat{\mu}_3^2 + \hat{\mu}_5^2 + \hat{\mu}_7^2)(1 - \theta_{m+2})] + \hat{\mu}_8(a_{m+4} - b_{m+2})[2 - \hat{\mu}_8(1 - \theta_{m+3})] \end{aligned}$$

with $\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3, \hat{\mu}_4, \hat{\mu}_5, \hat{\mu}_6, \hat{\mu}_7$, and $\hat{\mu}_8$, as defined in Table 1, θ_m as defined in Eq. (14), and

$$\theta_{m+1} = \frac{b_{m+1} - a_{m+1}}{a_{m+2} - b_m},\ \theta_{m+2} = \frac{b_{m+2} - a_{m+2}}{a_{m+3} - b_{m+1}},\ \theta_{m+3} = \frac{b_{m+3} - a_{m+3}}{a_{m+4} - b_{m+2}} \tag{34}$$

Case (b): One normalized input is in the 3D input space defined by $d_{b_i} \leq d_N(k) \leq d_{a_{i+1}}$, $v_{b_j} \leq v_N(k) \leq v_{a_{j+1}}$, $a_{b_n} \leq a_N(k) \leq a_{a_{n+1}}$ and the remaining two normalized inputs are outside this 3D input space, see Figure 6. The terms *Num* and *Den* for different cells are as follows:

In Eq. (33)

$$Num = 3\{k_2(a_{m+2}^2 - b_m^2) + k_1(a_{m+3}^2 - b_{m+1}^2) - k_2^2[b_m(a_{m+1} - b_m) + a_{m+2}(a_{m+2} - b_{m+1})] - k_1^2[b_{m+1}(a_{m+2} - b_{m+1}) + a_{m+3}(a_{m+3} - b_{m+2})]\} - k_2^3[(a_{m+1} - b_m)^2 - (a_{m+2} - b_{m+1})^2] - k_1^3[(a_{m+2} - b_{m+1})^2 - (a_{m+3} - b_{m+2})^2] \quad (35)$$

and

$$Den = k_2[2 - k_2(1 - \theta_{m+1})](a_{m+2} - b_m) + k_1[2 - k_1(1 - \theta_{m+2})](a_{m+3} - b_{m+1}) \quad (36)$$

where k_1 and k_2 are defined in Table 3.

In Eq. (33)

$$Num = 3\{k_1(a_{m+3}^2 - b_{m+1}^2) + k_2(a_{m+4}^2 - b_{m+2}^2) - k_1^2[b_{m+1}(a_{m+2} - b_{m+1}) + a_{m+3}(a_{m+3} - b_{m+2})] - k_2^2[b_{m+2}(a_{m+3} - b_{m+2}) + a_{m+4}(a_{m+4} - b_{m+2})]\} - k_1^3[(a_{m+2} - b_{m+1})^2 - (a_{m+3} - b_{m+2})^2] - k_2^3[(a_{m+3} - b_{m+2})^2 - (a_{m+4} - b_{m+3})^2] \quad (37)$$

and

$$Den = k_1[2 - k_1(1 - \theta_{m+2})](a_{m+3} - b_{m+1}) + k_2[2 - k_2(1 - \theta_{m+3})](a_{m+4} - b_{m+2}) \quad (38)$$

where k_1 and k_2 are defined in Table 3.

In Eq. (33)

$$Num = 3\{k_1(a_{m+1}^2 - b_{m-1}^2) + k_2(a_{m+2}^2 - b_m^2) - k_1^2[b_{m-1}(a_m - b_{m-1}) + a_{m+1}(a_{m+1} - b_m)] - k_2^2[b_m(a_{m+1} - b_m) + a_{m+2}(a_{m+2} - b_{m+1})]\} - k_1^3[(a_m - b_{m-1})^2 - (a_{m+1} - b_m)^2] - k_2^3[(a_{m+1} - b_m)^2 - (a_{m+2} - b_{m+1})^2] \quad (39)$$

and

$$Den = k_1[2 - k_1(1 - \theta_m)](a_{m+1} - b_{m-1}) + k_2[2 - k_2(1 - \theta_{m+1})](a_{m+2} - b_m) \quad (40)$$

where k_1 and k_2 are defined in Table 3

Case (c): Normalized inputs $d_N(k), v_N(k), a_N(k)$ are not in the 3D input space defined by $d_{b_i} \le d_N(k) \le d_{a_{i+1}}$, $v_{b_j} \le v_N(k) \le v_{a_{j+1}}$, $a_{b_n} \le a_N(k) \le a_{a_{n+1}}$.

In the cells (9, 9, 9), (10, 11, 10), (11, 12, 12), (17, 17, 17), (18, 19, 18) and (19, 20, 20):

$$\Delta u(k) = \frac{a_{m+2}^2 + b_{m+1}^2 - a_{m+1}^2 - b_m^2 + b_{m+1}a_{m+2} - b_m a_{m+1}}{3N_{\Delta u}(a_{m+2} - b_m)(1 + \theta_{m+1})} \quad (41)$$

In the cells (9, 10, 10), (11, 11, 11), (12, 9, 12), (17, 18, 18), (19, 19, 19) and (20, 17, 20):

$$\Delta u(k) = \frac{a_{m+3}^2 + b_{m+2}^2 - a_{m+2}^2 - b_{m+1}^2 + b_{m+2}a_{m+3} - b_{m+1}a_{m+2}}{3N_{\Delta u}(a_{m+3} - b_{m+1})(1 + \theta_{m+2})} \quad (42)$$

In the cells (10, 12, 9) and (18, 20, 17):

Table 3
Attributes of k_1 and k_2

Equations	*Cells*	k_1	k_1
Eq. (35) and (36) for *Num* and *Den*	(5, 9, 5), (13, 17, 13)	$\hat{\mu}_7$	$\hat{\mu}_2$
or	(6, 6, 10), (14, 14, 18)	$\hat{\mu}_3$	$\hat{\mu}_4$
Eq. (47) and (48) for Num_1 and Den_1	(7, 11, 7), (15, 19, 15)	$\hat{\mu}_5$	$\hat{\mu}_4$
	(8, 8, 12), (16, 16, 20)	$\hat{\mu}_7$	$\hat{\mu}_6$
	(9, 5, 6), (17, 13, 14)	$\hat{\mu}_3$	$\hat{\mu}_2$
	(11, 7, 8), (19, 15, 16)	$\hat{\mu}_5$	$\hat{\mu}_6$
Eq. (37) and (38) for *Num* and *Den*	(5, 10, 7), (13, 18, 15)	$\hat{\mu}_3$	$\hat{\mu}_8$
or	(8, 6, 11), (16, 14, 19)	$\hat{\mu}_5$	$\hat{\mu}_8$
Eq. (49) and (50) for Num_1 and Den_1	(12, 5, 8), (20, 13, 16)	$\hat{\mu}_7$	$\hat{\mu}_8$
Eq. (39) and (40) for *Num* and *Den*	(6, 8, 9), (14, 16, 17)	$\hat{\mu}_1$	$\hat{\mu}_2$
or	(7, 12, 5), (15, 20, 13)	$\hat{\mu}_1$	$\hat{\mu}_6$
Eq. (51) and (52) for Num_1 and Den_1	(10, 7, 6), (18, 15, 14)	$\hat{\mu}_1$	$\hat{\mu}_4$

$$\Delta u(k) = \frac{a_{m+1}^2 + b_m^2 - a_m^2 - b_{m-1}^2 + b_m a_{m+1} - b_{m-1}a_m}{3N_{\Delta u}(a_{m+1} - b_{m-1})(1 + \theta_m)} \quad (43)$$

In the cells (12, 10, 11) and (20, 18, 19):

$$\Delta u(k) = \frac{a_{m+4}^2 + b_{m+3}^2 - a_{m+3}^2 - b_{m+2}^2 + b_{m+3}a_{m+4} - b_{m+2}a_{m+3}}{3N_{\Delta u}(a_{m+4} - b_{m+2})(1 + \theta_{m+3})} \quad (44)$$

The above mathematical model obtained with asymmetric trapezoidal output fuzzy sets is complex in nature and thus cannot be separated into global and local parts. By considering $a_{m+4} = b_{m+4}$, $a_{m+3} = b_{m+3}$, $a_{m+2} = b_{m+2}$, $a_{m+1} = b_{m+1}$, $a_m = b_m$, $a_{m-1} = b_{m-1}$ (such that $\theta_m = \theta_{m+1} = \theta_{m+2} = \theta_{m+3}$ $\theta = 0$) in the above model, model with asymmetric triangular output fuzzy sets can be obtained. This model, shown below, is less complicated than that with trapezoidal output fuzzy sets and its expressions can be decomposed into global and local parts.

5.2. Output Fuzzy Sets–Triangular

Case(a): All the three inputs lie within the 3D input space defined by $d_{b_i} \le d_N(k) \le d_{a_{i+1}}$, $v_{b_j} \le v_N(k) \le v_{a_{j+1}}$, $a_{b_n} \le a_N(k) \le a_{a_{n+1}}$ i.e. cells (1, 1, 1) to (4, 4, 4)

$$\Delta u(k) = \Delta u_g + \Delta u_l(k) \quad (45)$$

$$= \frac{b_{m+1} + b_{m+2}}{3N_{\Delta u}} + \left(\frac{1}{3N_{\Delta u}}\right)\frac{Num_1}{Den_1} \quad (46)$$

where

$$Num_1 = \hat{\mu}_1(b_{m+1}-b_{m-1})(3b_{m-1}+b_{m+1}-2b_{m+2})+(\hat{\mu}_2+\hat{\mu}_4+\hat{\mu}_6)(b_{m+2}-b_m)$$
$$(3b_m-2b_{m+1}+b_{m+2})+(\hat{\mu}_3+\hat{\mu}_5+\hat{\mu}_7)(b_{m+3}-b_{m+1})$$
$$(b_{m+1}-2b_{m+2}+3b_{m+3})+\hat{\mu}_8(b_{m+4}-b_{m+2})(-2b_{m+1}+b_{m+2}+3b_{m+4})$$
$$-\hat{\mu}_1^2(b_{m+1}-b_{m-1})(3b_{m-1}-3b_m+2b_{m+1}-b_{m+2})-(\hat{\mu}_2^2+\hat{\mu}_4^2+\hat{\mu}_6^2)$$
$$(b_{m+2}-b_m)(3b_m-4b_{m+1}+2b_{m+2})-(\hat{\mu}_3^2+\hat{\mu}_5^2+\hat{\mu}_7^2)(b_{m+3}-b_{m+1})$$
$$(2b_{m+1}-4b_{m+2}+3b_{m+3})-\hat{\mu}_8^2(b_{m+4}-b_{m+2})(-b_{m+1}+2b_{m+2}$$
$$-3b_{m+3}+3b_{m+4})-\hat{\mu}_1^3[(b_m-b_{m-1})^2-(b_{m+1}-b_m)^2]-(\hat{\mu}_2^3+\hat{\mu}_4^3+\hat{\mu}_6^3)$$
$$[(b_{m+1}-b_m)^2-(b_{m+2}-b_{m+1})^2]-(\hat{\mu}_3^3+\hat{\mu}_5^3+\hat{\mu}_7^3)[(b_{m+2}-b_{m+1})^2$$
$$-(b_{m+3}-b_{m+2})^2]-\hat{\mu}_8^3[(b_{m+3}-b_{m+2})^2-(b_{m+4}-b_{m+3})^2]$$

and
$$Den_1 = \hat{\mu}_1[2-\hat{\mu}_1](b_{m+1}-b_{m-1})+[2(\hat{\mu}_2+\hat{\mu}_4+\hat{\mu}_6)$$
$$-(\hat{\mu}_2^2+\hat{\mu}_4^2+\hat{\mu}_6^2)](b_{m+2}-b_m)+[2(\hat{\mu}_3+\hat{\mu}_5+\hat{\mu}_7)$$
$$-(\hat{\mu}_3^2+\hat{\mu}_5^2+\hat{\mu}_7^2)](b_{m+3}-b_{m+1})+\hat{\mu}_8[2-\hat{\mu}_8](b_{m+4}-b_{m+2})$$

Case (b): One normalized input is in the 3D input space defined by $d_{b_i} \le d_N(k) \le d_{a_{i+1}}$, $v_{b_j} \le v_N(k) \le v_{a_{j+1}}$, $a_{b_n} \le a_N(k) \le a_{a_{n+1}}$ and the remaining two normalized inputs are outside this 3D input space, see Figure 6. The terms Num_1 and Den_1 for different cells are as follows:

In Eq. (46)

$$Num_1 = k_2(b_{m+2}-b_m)(3b_m-2b_{m+1}+b_{m+2})+k_1(b_{m+3}-b_{m+1})(b_{m+1}-2b_{m+2}$$
$$+3b_{m+3})-k_2^2(b_{m+2}-b_m)(3b_m-4b_{m+1}+2b_{m+2})-k_1^2(b_{m+3}-b_{m+1})$$
$$(2b_{m+1}-4b_{m+2}+3b_{m+3})-k_2^3[(b_{m+1}-b_m)^2-(b_{m+2}-b_{m+1})^2]$$
$$-k_1^3[(b_{m+2}-b_{m+1})^2-(b_{m+3}-b_{m+2})^2] \quad (47)$$

$$Den_1 = k_2[2-k_2](b_{m+2}-b_m)+k_1[2-k_1](b_{m+3}-b_{m+1}) \quad (48)$$

where k_1 and k_2 are defined in Table 3

In Eq. (46)

$$Num_1 = k_1(b_{m+3}-b_{m+1})(b_{m+1}-2b_{m+2}+3b_{m+3})+k_2(b_{m+4}-b_{m+2})(-2b_{m+1}$$
$$+b_{m+2}+3b_{m+4})-k_1^2(b_{m+3}-b_{m+1})(2b_{m+1}-4b_{m+2}+3b_{m+3})$$
$$-k_2^2(b_{m+4}-b_{m+2})(-b_{m+1}+2b_{m+2}-3b_{m+3}+3b_{m+4})$$
$$-k_1^3[(b_{m+2}-b_{m+1})^2-(b_{m+3}-b_{m+2})^2]$$
$$-k_2^3[(b_{m+3}-b_{m+2})^2-(b_{m+4}-b_{m+3})^2] \quad (49)$$

$$Den_1 = k_1[2-k_1](b_{m+3}-b_{m+1})+k_2[2-k_2](b_{m+4}-b_{m+2}) \quad (50)$$

where k_1 and k_2 are defined in Table 3

In Eq. (46)

$$Num_1 = k_1(b_{m+1}-b_{m-1})(3b_{m-1}+b_{m+1}-2b_{m+2})+k_2(b_{m+2}-b_m)$$
$$(3b_m-2b_{m+1}+b_{m+2})-k_1^2(b_{m+1}-b_{m-1})(3b_{m-1}-3b_m$$
$$+2b_{m+1}-b_{m+2})-k_2^2(b_{m+2}-b_m)(3b_m-4b_{m+1}+2b_{m+2})$$
$$-k_2^3[(b_{m+1}-b_m)^2-(b_{m+2}-b_{m+1})^2]$$
$$-k_2^3[(b_{m+1}-b_m)^2-(b_{m+2}-b_{m+1})^2] \quad (51)$$

$$Den_1 = k_1[2-k_1](b_{m+1}-b_{m-1})+k_2[2-k_2](b_{m+2}-b_m) \quad (52)$$

where k_1 and k_2 are defined in Table 3

Case (c): Normalized inputs $d_N(k), v_N(k), a_N(k)$ are not in the 3D input space defined by $d_{b_i} \le d_N(k) \le d_{a_{i+1}}$, $v_{b_j} \le v_N(k) \le v_{a_{j+1}}$, $a_{b_n} \le a_N(k) \le a_{a_{n+1}}$

In the cells (9, 9, 9), (10, 11, 10), (11, 12, 12), (17, 17, 17), (18, 19, 18) and (19, 20, 20):

$$\Delta u(k) = \frac{b_{m+2}+b_{m+1}+b_m}{3N_{\Delta u}}$$

In the cells (9, 10, 10), (11, 11, 11), (12, 9, 12), (17, 18, 18), (19, 19, 19) and (20, 17, 20):

$$\Delta u(k) = \frac{b_{m+3}+b_{m+2}+b_{m+1}}{3N_{\Delta u}}$$

In the cells (10, 12, 9) and (18, 20, 17):

$$\Delta u(k) = \frac{b_{m+1}+b_m+b_{m-1}}{3N_{\Delta u}}$$

In the cells (12, 10, 11) and (20, 18, 19):

$$\Delta u(k) = \frac{b_{m+4}+b_{m+3}+b_{m+2}}{3N_{\Delta u}}$$

By substituting $a_{a_{n+1}} - a_{b_n} = v_{a_{j+1}} - v_{b_j} = d_{a_{i+1}} - d_{b_i} = S$ and $a_{b_n} - a_{a_n} = v_{b_j} - v_{a_j} = d_{b_i} - d_{a_i} = 0$ for all *i*, *j* and *n* in Figure 3, and $a_{m+3} = b_{m+3} = (i+j+n+3)W$, $a_{m+2} = b_{m+2} = (i+j+n+2)W$, $a_{m+1} = b_{m+1} = (i+j+n+1)W$, $a_m = b_m = (i+j+n)W$, and $\theta_m = \theta_{m+1} = \theta_{m+2} = \theta_{m+3} = 0$, in the above mathematical model, the model of fuzzy PID controller in Section 3 can be obtained.

6. ILLUSTRATIVE EXAMPLES

Comparison of the performances of linear PID controller and the fuzzy PID controller with multiple symmetric triangular input and output fuzzy sets is presented here by considering the following examples:

(i) a linear third-order aircraft attitude-control system [1]

$$G_{p1}(s) = \frac{2.718\times 10^9}{s(s+400.26)(s+3008)} \quad (53)$$

with unit step reference signal,

Table 4
Attributes of the Linear, Nonminimum Phase, and Nonlinear Processes

Process	T	K_P^d	K_I^d	K_D^d	$\lvert d \rvert_{max}$	$\lvert v \rvert_{max}$	$\lvert a \rvert_{max}$
$G_{p1}(s)$	0.002	0.309T	4.5T	0.006T	1	18.3066	5016.8934
$G_{p2}(s)$	0.001	10.5T	20000T	0.0005T	1	131.343	19960.067
Nonlinear	0.1	1.8T	1.8T	0.008T	4	4.10676	994.592

Table 5
Attributes and Time-domain Performance Data of Plants with Linear and Fuzzy PID Controllers

Process	*PID controller*	N_d	N_v	N_a	$N_{\Delta u}$	$l = M$	N	M_p (%)	t_r (sec)	t_s (sec)
$G_{p1}(s)$	linear	-	-	-	-	-	-	0	0.146	0.2
	fuzzy	38	0.25	0.48×10^{-3}	240	38	3	5.58*	0.006	0.0095
	fuzzy	38	0.25	0.48×10^{-3}	240	38	5	3.405*	0.0057	0.0093
	fuzzy	38	0.25	0.48×10^{-3}	240	38	7	3.48*	0.0056	0.0091
$G_{p2}(s)$	linear	-	-	-	-	-	-	85.2357	0.009	0.4020
	fuzzy	1800	4.0	0.41×10^{-4}	1.6	3800	3	0.0709	0.0024	0.005
	fuzzy	1800	4.0	0.41×10^{-4}	1.6	3800	5	0.0257	0.0021	0.0041
	fuzzy	1800	4.0	0.41×10^{-4}	1.6	3800	7	0.0935	0.0023	0.0042
Nonlinear	linear	-	-	-	-	-	-	37.4442	1.0	5.6
	fuzzy	3	3	0.045	2.1	20	3	0	1.148	1.75
	fuzzy	3	3	0.045	2.1	20	5	0	1.176	1.62
	fuzzy	3	3	0.045	2.1	20	7	0	1.144	1.58

* Desired $M_p \leq 5\%$

(ii) a linear third-order nonminimum phase system [4]

$$G_{p2}(s) = \frac{s^2 - s - 2}{s^3 + 3s^2 - 10s - 24} \tag{54}$$

with unit step reference, and

(iii) a nonlinear first-order system [4]

$$\dot{y}(t) = y(t) + sin^2(\sqrt{\lvert y(t) \rvert}) + u(t) \tag{55}$$

with step input of magnitude 4 as the reference signal. In Eqs. (53) and (54), $G_{p1}(s)$ and $G_{p2}(s)$ represent the transfer functions of the plants to be controlled.

Fuzzy PID controllers are designed for all the above three plants. The values of sampling period T, proportional gain K_P^d, integral gain K_I^d, derivative gain K_D^d, maximum absolute displacement (error) $\lvert d \rvert_{max}$, maximum absolute velocity $\lvert v \rvert_{max}$, and maximum absolute acceleration $\lvert a \rvert_{max}$ are given in Table 4 for all the three processes.

The parameters N_d, N_v, N_a, $N_{\Delta u}$, N, l, and M of the fuzzy PID controllers, which gave rise to the responses in Figures 7-9, are listed in Table 5 in which M_p, t_r and t_s denote peak overshoot, rise time and settling time, respectively. Figures 7-9 also show the responses with conventional PID controllers. Upon comparison, it is evident from the plots that the fuzzy PID controllers perform better, demonstrating their superiority over their counterparts- conventional PID controllers. The parameter N takes only odd integer values with the minimum value being 3. Since the basic objective of this simulation study is to demonstrate the influence of the parameter N on the performance of the fuzzy PID controller, the values of functional parameters are fixed and only N is varied as shown in Table 5. From the time-domain performance data in Table 5, it is observed that as the value of N increases from 3 to 7, there is some improvement in performance, in general. The best performance is obtained with N = 5 for the plants $G_{p1}(s)$ and $G_{p2}(s)$, and with N = 7 for the nonlinear plant.

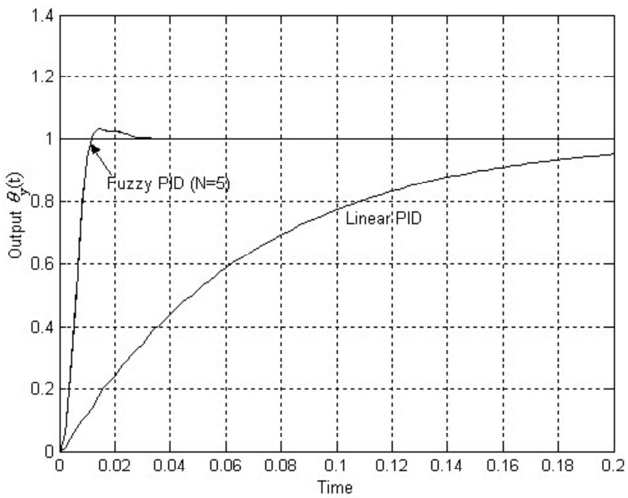

Figure 7: Unit Step Response of Closed Loop System with $G_{p1}(s)$

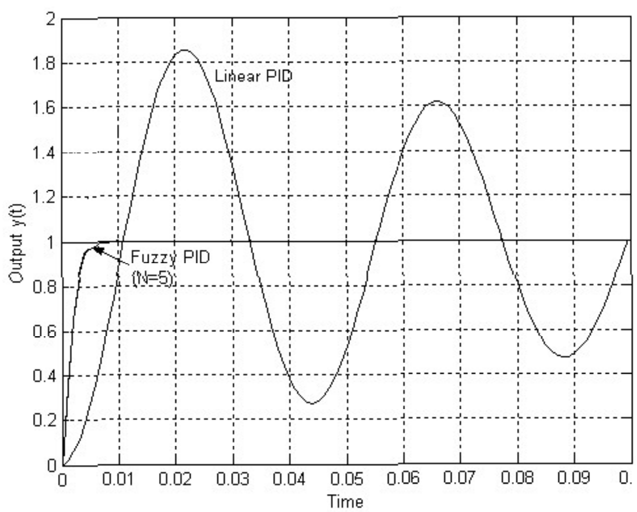

Figure 8: Unit Step Response of Closed Loop System with $G_{p2}(s)$

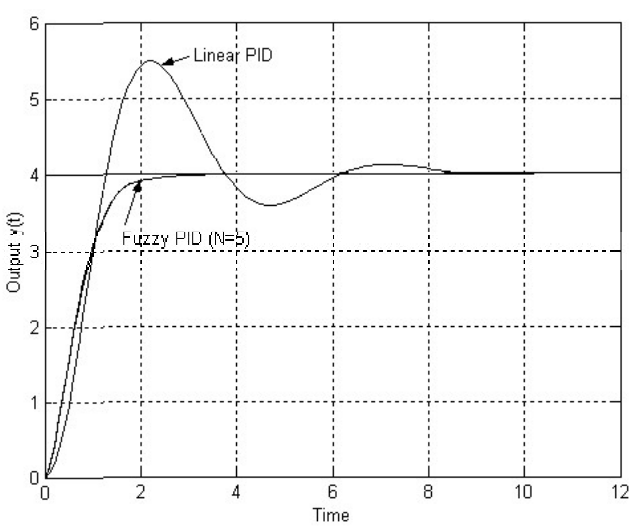

Figure 9: Step (Magnitude 4) Response of Closed Loop System with Nonlinear Process

7. CONCLUSION

In this paper, mathematical modeling of fuzzy PID controllers with multiple fuzzy sets has been considered. First, a model is derived using symmetric triangular membership functions for fuzzification of input variables and output variable, linear control rules, algebraic product triangular norm, bounded sum triangular conorm, Mamdani minimum inference method and COS defuzzification. It has been shown that the resulting controller is equivalent to the sum of a global three-dimensional multi-level relay and a local nonlinear PID controller. Properties of the fuzzy controller have been investigated. Next, mathematical model of generalized fuzzy PID controller has been derived analytically using asymmetric trapezoidal membership functions for inputs and output. The mathematical model obtained with asymmetric trapezoidal output fuzzy sets is complex in nature and thus cannot be separated into global and local parts. But, when model with asymmetric triangular output fuzzy sets is considered, it is found to be less complicated than that with trapezoidal output fuzzy sets, and its expressions are decomposed into global and local parts. The superiority of fuzzy PID controller over linear PID controller has been demonstrated through a simulation study on three different processes. Influence of N on system performance has been studied for all the processes.

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