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# DIRECT DESIGN OF A SUBOPTIMAL MODEL PREDICTIVE CONTROLLER WITH A POSITIVE POLYNOMIAL TECHNIQUE

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***ABSTRACT:*** Suboptimal design of a model predictive controller is considered for a discrete-time linear plant with the assumption that the controller is a piecewise polynomial defined on a given partition of the state space. The motivation is that the optimal controller is a piecewise affine function whose piecewise structure is irregular in general. Indeed, its computation requires a computational geometric technique and is difficult to carry out when the plant is of high order and/or when the piecewise structure is nearly degenerate. The proposed approach can avoid such geometric computation with a regular partition and a positive polynomial technique. It is more flexible than the existing suboptimal approaches with a regularly partitioned piecewise affine controller. It is also direct in that it does not need the computation of the optimal controller and is carried out with minimization of the distance to optimality. The asymptotic optimality of the proposed approach is discussed together with the stability of the resulting closed-loop system.

***Keywords:*** model predictive control, robust optimization, sum-of-squares technique, regiondividing technique, asymptotic optimality, stability.

## 1. INTRODUCTION

Model predictive control is one of the most widely used control methods in practice [21]. Formerly, its use was limited to plants of slow time response because it has to solve at each sampling instant an optimization problem on the behavior of the closed-loop system within some finite horizon. Bemporad– Morari–Dua–Pistikopoulos [2] broadened its scope by proposing the offline computation of the optimal solution as a function of the current state of the plant. At the application of the controller, only the evaluation of this function is required and hence the controller can be used for fast plants.

In the following, our attention is focused on the case of a discrete-time linear plant. In this case, the optimal solution is a piecewise affine function of the current state and the associated piecewise structure is irregular in general. Its computation requires a computational geometric technique, which is complicated and delicate especially in the case of a high-dimensional space, that is, in the case of a high-order plant in our context. Moreover, the computation can be numerically unstable when the partition is nearly degenerate. Note that this is the issue not only in the offline computation of this piecewise function but also in its online evaluation because the subregion containing a given state has to be identified there.

Motivated by these facts, several authors proposed approximation of the optimal controller, which gives a suboptimal but simpler controller. In particular, approximation with a piecewise affine function defined on a given regular partition has been actively investigated [1, 3, 10, 11, 12, 13]. There, the partition is chosen easy to handle even in the high-dimensional space. A problem of this approach is that such a piecewise affine controller does not give a good enough approximation of the optimal controller. Although the use of a fine partition improves the approximation, it also increases the computational cost. Moreover, the distance to optimality is not easy to measure in the existing works. In some works, *e.g.*, [3], this measurement needs the computation of the optimal controller, which requires problematic geometric computation. In other works, *e.g.*, [12, 13], the measurement becomes possible only after a suboptimal controller is fixed.

In this paper, a design of a piecewise polynomial model predictive controller is considered for a given regular partition. Since a polynomial is more flexible than an affine function, it can give a better approximation. The proposed approach is direct in the sense that it does not require the computation of the optimal controller and its design is performed by minimization of the distance to optimality. The first property means that geometric computation is not necessary any more; the second means that the distance to optimality can be taken into account in the design process. The proposed approach is asymptotically optimal in the sense that the resulting controller can be made arbitrarily close to the optimal one by the use of a high-degree polynomial or a fine partition. Finally, in the proposed approach, slight change of the formulation guarantees stability of the resulting closed-loop system. An early version of this paper has been presented at a conference [24].

A polynomial model predictive controller has been considered by Kvasnica–Löfberg–Fikar [16]. In their approach, the optimal controller is computed first and then approximated with a polynomial under the guarantee of closed-loop stability. A positive polynomial technique plays an important role there, which is the same in the present approach. On the other hand, it is different from the present approach that their approach requires the computation of the optimal controller and does not consider the distance to optimality. Domahidi–Zeilinger–Morari–Jones [8] proposed controller design with a general nonlinear function basis through minimization of the distance to optimality. They however considered the distance to optimality only at finitely many points in the state space while we consider the worst-case distance over the given domain. A positive polynomial technique has been used for other purposes in model predictive control. Namely, it was used for control of a linear parameter-varying plant [6] and for design of a separator [15].

The rest of this paper is organized as follows. Section 2 sets up the problem to be considered. Section 3 gives the proposed approach. Section 4 is for the discussion on the approach and Section 5 is for an example. Section 6 concludes the paper.

The following notation is used. The symbols $O$ and $I$ stand for the zero matrix and the identity matrix, respectively, of appropriate size. The symbol $^T$ expresses the transpose of a matrix or a vector. For a vector $u$, the inequalities $u > 0$ and $u \geq 0$ express the elementwise positivity and nonnegativity of $u$, respectively. The inequalities $u > v$ and $u \geq v$ are equivalent to $u - v > 0$ and $u - v \geq 0$, respectively. For a symmetric matrix $Q$, the inequalities $Q \succ O$ and $Q \succeq O$ stand for the positive definiteness and the positive semidefiniteness of $Q$, respectively. For a real number $a$, the symbol $\lceil a \rceil$ designates the smallest integer larger than or equal to $a$. Finally, for a minimization problem $P$, its optimal value (if exists) is denoted by $\min P$.

## 2. PROBLEM

Suppose that a plant to be controlled has the dynamics

$$x(t+1) = Ax(t) + Bu(t)$$

for $t = 0, 1,\ldots$ and the constraints

$$u(t) \in U_0, \quad x(t) \in X_0$$

for $t = 0,1,\ldots$, where $U_0$ and $X_0$ are given convex polytopes. The dimensions of the state $x(t)$ and the input $u(t)$ are denoted by $p$ and $p_u$, respectively. In the model predictive control, the input $u(t)$ at the time $t$ is computed by the following procedure. We first measure $x(t)$ and solve the following optimization problem parametrized by $x$ with substituting $x := x(t)$:

$$\begin{aligned} O_x : \text{minimize} \quad & J_x(u) := \sum_{k=0}^{N-1} \left( \frac{1}{2} x_k^T Q x_k + \frac{1}{2} u_k^T R u_k \right) + \frac{1}{2} x_N^T Q_{fxN} \\ \text{subject to} \quad & x_0 = x, \\ & x_{k+1} = Ax_k + Bu_k \ (k = 0,1,\ldots, N-1), \\ & u_k \in U_0 \ \ (k = 0, 1,\ldots, N-1) \\ & x_k \in X_0 \ \ (k = 0, 1, \ldots, N-1) \end{aligned}$$

$$x_N \in X_f,$$

where the design variable is $u := \left(u_0^T \ u_1^T \cdots u_{N-1}^T\right)^T$ for a positive integer $N$. In this problem, $Q \succeq O$, $R \succ O$, and $Q_f \succeq O$ are given symmetric matrices and $X_f$ is a given convex polytope. Then, the first $p_u$ elements of the optimal solution $u$, *i.e.*, $u_0$, is the input $u(t)$ to be applied. In the following, we make a natural assumption that $U_0$, $X_0$, and $X_f$ contain the origin in their interiors.

As in Bemporad *et al.* [2], the function $J_x(u)$ can be expressed without $x_k (k = 0,1,...,N)$ by the repeated substitution of $x_{k+1} = Ax_k + Bu_k$. Let us write the resulting expression as

$$J_x(u) = \frac{1}{2}u^T Hu + x^T Fu + \frac{1}{2}x^T Yx$$

with appropriate matrices $H$, $F$, and $Y$, where $H$ and $Y$ are symmetric. By completing the square, we can rewrite $J_x(u)$ as $(1/2)z^THz + (1/2)x^T (Y - FH^{-1}F^T)x$ with $z := u + H^{-1}F^Tx$. Hence, the minimization of $J_x(u)$ is equivalent to that of $V_x(z) := (1/2)z^THz$. We rewrite the constraints of $O_x$ in terms of $z$ to have $Gz \leq f(x)$, where the right-hand side $f(x)$ turns out to be affine in $x$. Consequently, the problem to be solved is

$$P_x: \qquad \text{minimize} \qquad V_x(z) = \frac{1}{2}z^T Hz$$
$$\text{subject to} \qquad Gz \leq f(x),$$

whose design variable is $z$. The dimension of $z$ is written as $n(= Np_u)$ and that of $f(x)$ as $m(= N(m_u+m_x)+m_f)$, where $m_u$, $m_x$, and $m_f$ are the numbers of inequalities necessary for the description of $U_0$, $X_0$, and $X_f$, respectively. Due to the assumptions on $O_x$, we have $H \succ O$ and $f(0) > 0$. We also make an additional assumption that the problem $P_x$ as well as the original problem $O_x$ is considered only for $x \in X$ with $X$ being a closed convex polytope in $\mathbb{R}^p$ having the origin in its interior. Moreover, we assume for any $x \in X$ the existence of $z$ (which is dependent on $x$) such that $Gz < f(x)$. The latter assumption is satisfied if the domain $X$ is chosen small enough, because of the assumption $f(0) > 0$.

Bemporad *et al.* [2] showed that the optimal solution of $P_x$ is a piecewise affine function of $x \in X$ and considered its explicit computation. Once such a function is obtained, its evaluation for the measured state $x(t)$ immediately gives the control input $u(t)$. Unfortunately, the computation of this piecewise affine function has the problem discussed in the introduction. In order to circumvent the problem, we consider to obtain a suboptimal solution of $P_x$ as a piecewise polynomial in $x$ defined on a given partition of $X$.

## 3. PROPOSED APPROACH

In the proposed approach, we arbitrarily partition the domain $X$ into several convex polytopes and design a polynomial controller for each subpolytope. For simplicity, we first focus on the special case that the partition consists of only one subpolytope, *i.e.*, the domain $X$ itself. The general case will be considered in Section 3.3; in fact, the same procedure is just repeated for each subpolytope.

### 3.1. Reduction to a Robust Optimization Problem

We consider to obtain a polynomial $z(x)$ that nearly optimizes $P_x$ for each $x \in X$. This problem is in fact reduced to a robust optimization problem. Let us fix $x \in X$ for the moment.

The problem $P_x$ has a unique optimal solution due to the strict convexity of the objective function $V_x(z)$ and the compactness of its sublevel sets. A necessary and sufficient condition for $z \in \mathbb{R}^n$ to be the optimal solution is the existence of $\lambda \in \mathbb{R}^m$ such that

$$Gz \leq f(x), \ \ \lambda \geq 0, \ \ Hz + G^T\lambda = 0, \ \ \lambda^T[f(x) - Gz] = 0$$

(*e.g.*, [9, p. 340][5, p. 366]). This is the Karush–Kuhn–Tucker condition for optimality and the vector $\lambda$ is a Lagrange multiplier.

By solving the equality above as $z = -H^{-1}G^T\lambda$ and substituting it to the remaining inequalities, we reach the problem

$$\begin{aligned} &\text{minimize} && c \\ &\text{subject to} && -GH^{-1}G^T \lambda \leq f(x), \\ &&& \lambda \leq 0, \\ &&& \lambda^T[f(x) + GH^{-1}G^T\lambda] \leq c, \end{aligned}$$

whose design variable is $(\lambda, c) \in \mathbb{R}^m \times \mathbb{R}$. If some $(\lambda, c)$ attains the optimal value $c = 0$ ($c$ cannot be negative), the mentioned optimality condition guarantees that the vector $z = -H^{-1} G^T\lambda$ gives the optimal solution of the problem $P_x$. On the other hand, if some $(\lambda, c)$ attains a suboptimal value $c > 0$, the vector $z = -H^{-1}G^T\lambda$ gives a suboptimal solution of $P_x$ and its distance to optimality is less than or equal to $c$. To see the latter statement to hold, observe that, for this $\lambda$ and any $z$ satisfying $Gz \leq f(x)$,

Minimize both sides subject to $Gz \leq f(x)$ to have

$$\min\ P_x \ \geq \min_{Gz \leq f(x)} \left\{ \frac{1}{2} z^T Hz - \lambda^T [f(x) - Gz] \right\}$$

$$\geq$$

$$= -\frac{1}{2}\lambda^T GH^{-1}G^T\lambda - \lambda^T f(x).$$

Multiply $-1$ to both sides and add $(1/2)z^T Hz$ for $z = -H^{-1} G^T\lambda$ to have

$$\begin{aligned} \frac{1}{2} z^T Hz - \min\ P_x &\leq \frac{1}{2} z^T Hz + \frac{1}{2}\lambda^T GH^{-1}G^T\lambda + \lambda^T f(x) \\ &= \lambda^T [f(x) + GH^{-1}G^T\lambda] \\ &\leq c. \end{aligned} \tag{1}$$

Now, we allow $x$ to move in $X$. The preceding discussion motivates us to assume $\lambda(x)$ to be a polynomial of some fixed degree $d$ and solve the optimization problem:

$$\begin{aligned} T:\ & \textit{minimize } c \\ & \text{subject to } -GH^{-1}G^T\lambda(x) \leq f(x) \quad (x \in X), \\ & \qquad \lambda(x) \geq 0 \quad (x \in X), \\ & \qquad \lambda(x)^T [f(x) + GH^{-1}G^T\lambda(x)] \leq c \quad (x \in X) \end{aligned}$$

with the coefficients of $\lambda(x)$ and the scalar $c$ being the design variables. The problem $T$ is a robust optimization problem because the constraints have to be satisfied for any $x \in X$. Its solution has the following property.

**Theorem 1.** *Let a polynomial $\lambda(x)$ and a scalar $c$ be a feasible solution of the problem $T$. Then, $z(x) := -H^{-1}G^T\lambda(x)$ is a polynomial in $x$ and gives a feasible solution of the problem $P_x$ for each $x \in X$. Moreover, the corresponding value of the objective function $(1/2)z(x)^T Hz(x)$ differs from the optimal value* $\min P_x$ *at most $c$ for each $x \in X$.*

*Proof.* The proof is almost clear from the discussion so far.

The polynomiality of $z(x)$ is obvious from its definition. Since $Gz(x) = -GH^{-1}G^T\lambda(x) \leq f(x)$ for each $x \in X$, this $z(x)$ gives a feasible solution of $P_x$ for each $x \in X$. Moreover, it satisfies the inequality (1) for each $x \in X$, which implies the last statement of the theorem.

This theorem implies that a feasible solution $\lambda(x)$ of $T$ gives a feasible solution $z(x)$ of $P_x$ and further a suboptimal controller $u_0(x) = (I_{pu}\ \ O)u(x) := (I_{pu}\ \ O)\ [z(x) - H^{-1}F^T x]$. The associated $c$ means the worst-case distance to optimality over $X$. Therefore, $c$ has to be made as small as possible. Computation of such $\lambda(x)$ and $c$ will be discussed in the next section.

This approach has the following advantages. The resulting controller $u_0(x)$ is a polynomial in $x$ and can be closer to the optimal controller than an affine interpolation considered in the literature. Computation of the optimal controller is not necessary and thus geometric computation is not necessary either. A controller design is performed through minimization of $c$, which stands for the distance to optimality.

### 3.2. Solution of the Robust Optimization Problem

In order to solve the robust optimization problem $T$, we use the sum-of-squares technique, which has made a remarkable progress in this decade [7, 17, 19, 26, 27]. A general treatment of this type of problems has been presented in [18]. With the sum-of-squares technique, a sufficient condition is obtained for each constraint of $T$. Since these sufficient conditions are independent of $x$, their use in $T$ instead of the original constraints gives a solvable problem and its solution is a feasible solution of $T$. Moreover, the sufficient conditions can be made arbitrarily tight at the cost of increasing complexity of the conditions.

Let us briefly see how a sufficient condition is obtained for the second constraint $\lambda(x) \geq 0$ ($x \in$ X). Suppose that the domain $X$ is expressed as $\{x \in \mathbb{R}^n \mid r_1(x) \geq 0, r_2(x) \geq 0, \ldots, r_\ell(x) \geq 0\}$ with scalar polynomials $r_1(x)$, $r_2(x)$, …,$r_\ell(x)$ whose degrees are $d_1$, $d_2$, ..., $d_\ell$, respectively. Then, $\lambda(x) \geq 0$ ($x \in X$) follows if the vector polynomial

$$\lambda(x) - s_1(x)r_1(x) - s_2(x)r_2(x) - \cdots - s_\ell(x)r_\ell(x) \quad (2)$$

is nonnegative all over for some vector polynomials $s_1(x)$, $s_2(x)$, …, $s_\ell(x)$ that are again nonnegative all over $\mathbb{R}^p$. We choose the degrees of $s_1(x)$, $s_2(x)$, ...,$s_\ell(x)$ as $2(D–[d_1/2])$, $2(D– [d_2/2])$, ..., $2(D – [d_\ell/2])$, respectively, with some positive integer $D$ larger than or equal to $d/2$, $d_1/2$, $d_2/2$, …, $d_\ell/2$, where $d$ is the degree of $\lambda(x)$. Then the polynomial (2) has the degree $2D$ at most. This polynomial is nonnegative all over if each of its elements is expressed as the sum of squares of polynomials and this sufficient condition is expressed as positive semidefiniteness of some matrix having the size $\binom{D+p}{p} \times \binom{D+p}{p}$. Similarly, for each $i$ = 1, 2,...,‘, nonnegativity of each element of $s_i(x)$ is expressed as positive semidefiniteness of a matrix having the size $\binom{D-\lceil d_i/2 \rceil + p}{p} \times \binom{D-\lceil d_i/2 \rceil + p}{p}$. Thus, we obtain a sufficient condition for $\lambda(x) \geq 0$ ($x \in X$) in terms of positive semidefiniteness of appropriate matrices. These matrices are independent of $x$ and are affine in the coefficients of $\lambda(x)$. Hence, the sufficient condition is suitable to use in $T$ in place of $\lambda(x) \geq 0$ ($x \in X$). Moreover, the sufficient condition can be made arbitrarily tight by the increase of the positive integer $D$.

The same technique is applied to the first and the third constraints of the problem $T$. In the case of the third constraint, this is performed after the constraint is equivalently rewritten as

$$\begin{pmatrix} c - \lambda(x)^T F \lambda(x) & \lambda(x)^T G \\ G^T \lambda(x) & H \end{pmatrix} \succeq O.$$

Replacing the constraints of $T$ by the obtained sufficient conditions, we have a semidefinite programming problem, which is solvable with the standard interior-point method. The required manipulations can be easily performed with a software such as YALMIP [20].

The matrices in the sufficient conditions have the sizes exponential in $D$ and $p$. Hence the positive integer $D$ cannot be chosen large. This also means that the degree $d$ of $\lambda(x)$ cannot be large.

### 3.3. Partition of the Domain

In the case that the domain $X$ is partitioned into several convex polytopes, the procedure discussed so far is applied to each subpolytope. It is notable that the problem $T$ can be solved independently for each subpolytope. Hence, the computational cost depends only linearly on the number of subpolytopes.

Blind refinement of the partition of $X$ increases the number of subpolytopes rapidly especially when the dimension $p$ is large. In order to avoid it, the technique of adaptive partition is often useful. Namely, we start with a coarse partition, solve the problem $T$ for each subpolytope, find the subpolytope having the largest (*i.e.*, worst) optimal value, and then partition that subpolytope. Repeating this procedure, we can improve the controller with suppressing the rapid increase of the number of subpolytopes. This technique has been used in a more general context of robust optimization [22, 23].

## 4. DISCUSSION

### 4.1. Asymptotic Optimality

In general, a Lagrange multiplier may not be unique; it may not be continuous to the perturbation of the problem. Hence, it is not obvious in the problem $T$ whether the objective function $c$ can be made small enough with a piecewise polynomial $\lambda(x)$.

This question can be answered affirmatively. Arbitrarily good approximation is possible by increasing the degree of the polynomial or the resolution of the partition. This is because, for any positive $c$, there exists a continuous function $\hat{\lambda}(x)$ that satisfies the constraints of $T$ with the strict inequalities. This continuous λb($x$) can be approximated arbitrarily well with a piecewise polynomial, which implies the preceding claim.

**Theorem 2.** *For any positive number c, there exists a function* $\hat{\lambda}(x)$ *that is continuous in X and* satisfies

$$-GH^{-1}G^T\hat{\lambda}(x) < f(x) \quad (x \in X),$$

$$\hat{\lambda}(x) > 0 \quad (x \in X),$$

$$(x \in X).$$

*Proof*. For the proof of the theorem, we introduce the following unconstrained optimization problem parametrized by $x \in X$ and $t > 0$:

where $[f(x) - Gz]_i$ stands for the $i$th element of the vector $f(x) - Gz$. This problem often appears in the context of the interior-point method (*e.g.*, [4, Section 6.4]). The objective function takes a finite value for some $z$ due to the assumed strict feasibility of $P_x$. Moreover, it is strictly convex and has a compact sublevel set, which implies that the problem $P_{x,t}$ has the unique optimal solution denoted by $\hat{z}(x,t)$. At the objective function has to have the zero gradient, that is,

with $G^T = (g_1 \quad g_2 \quad \cdots \quad g_m)$. Hence, defining $\hat{\lambda}(x,t) = (\hat{\lambda}_1(x,t)\ \hat{\lambda}_2(x,t) \cdots \hat{\lambda}_m(x,t))^T$ with

$$\hat{\lambda}_i(x,t) := \frac{t}{[f(x) - G\hat{z}(x,t)]_i}$$

for $i = 1, 2, \ldots, m$, we are to have

$$G\hat{z}(x,t) < f(x), \ \ \hat{\lambda}(x,t) > 0, \ \ H\hat{z}(x,t) + G^T\hat{\lambda}(x,t) = 0,$$

We will show that $\hat{\lambda}(x) := \hat{\lambda}(x, c/2m)$ is the desired function. Indeed, it satisfies all the requiredinequalities by definition. The only thing necessary to show is its continuity in $X$. Let us consider the equalities to be satisfied by $\hat{z}(x,t)$ and  *i.e.*, $H\hat{z} + G^T\hat{\lambda} = 0$ and  ($i = 1, 2, \ldots, m$) which are $n+m$ equalities with respect to the n-dimensional vector  and the $m$-dimensional vector . The Jacobian of the left-hand sides is

with $\hat{\Lambda}$ being the $m \times m$ diagonal matrix whose diagonal elements are the $m$ elements of the vector . This Jacobian is nonsingular for $t > 0$ and  $> 0$. Hence, the implicit function theorem implies thedesired continuity of  and  for $x \in X$ and $t > 0$. The proof is complete.

## 4.2. Stability

In the model predictive control, the closed-loop stability is not obvious and has been a big issue. In the case of the optimal model predictive controller, if the matrix $Q_f$ and the polytope $X_f$ are appropriately chosen, the optimal value of $O_x$, *i.e.*, min$O_x$, becomes a Lyapunov function and assures the stability of the closed-loop system [21]. In the case of a suboptimal controller, several results are available [1, 3, 10, 11, 12, 13].

We here discuss stability of our suboptimal controller with the condition used by Jones–Morari [13]. In particular, if a suboptimal solution $u(x) = (u_0(x)\ u_1(x)\ \cdots\ u_{N-1}(x))^T$ of the problem $O_x$ satisfies

$$J_x(u(x)) - \min\ O_x \le \frac{1}{2}x^TQx + \frac{1}{2}u_0(x)^TRu_0(x)$$

for any $x \in X$, the controller $u_0(x)$ asymptotically stabilizes the plant. It is possible to rephrase this condition in terms of $P_x$ by noting the relation $z(x) = u(x) + H^{-1}F^{T}x$ and $V_x(z(x)) = J_x(u(x)) -(1/2)x^{T}(Y - FH^{-1}F^{T})x$. Indeed, the stabilityis guaranteed if a suboptimal solution $z(x)$ of $P_x$ satisfies

$$V_x(z(x)) - \min\ P_x \le \frac{1}{2}x^T Qx + \frac{1}{2}u_0(x)^T Ru_0(x) \quad (3)$$

for any $x \in X$, where $u_0(x) = (I_{p_u}\ \ O)[z(x) - H^{-1}F^T x]$.

This motivates us to consider the following variation of the problem $T$:

$T$: *minimize* $c$

$$\begin{aligned} \text{subject}\quad & -GH^{-1}G^T\lambda(x) \le f(x)\ (x \in X), \\ & \lambda(x) \ge 0\ \ (x \in X), \\ & \lambda(x)^T[f(x) + GH^{-1}G^T\lambda(x)] \le \\ & \frac{c}{2}x^T Qx + \frac{c}{2}[\lambda(x)^T G + x^T F]H^{-1}\begin{pmatrix} I_{p_u} \\ O \end{pmatrix} R \times \\ & \begin{pmatrix} I_{p_u} & O \end{pmatrix} H^{-1}[G^T\lambda(x) + F^T x]\ (x \in X). \end{aligned}$$

Again, $\lambda(x)$ is a polynomial of some fixed degree $d$. This problem has the following property.

**Theorem 3.** *Suppose that a polynomial* $\lambda(x)$ *and a scalar* $c$ *are feasible in the problem* $T_0$ *and satisfy* $c \le 1$. *Then, the controller* $u_0(x)$ *defined by* $u_0(x) = (I_{pu}O)[z(x) - H^{-1}F^{T}x]$ *and* $z(x) = -H^{-1}G^{T}\lambda(x)$ *asymptotically stabilizes the plant.*

*Proof.* As was shown in Subsection 3.1, the left-hand side of the third constraint of $T0$ is larger than or equal to $V_x(z(x))$–min $P_x$. On the other hand, the right-hand side is equal to $c[(1/2)x^{T} Qx + (1/2)u_0(x)^{T}Ru_0(x)]$ for $u_0(x)$ defined in the theorem. Since $c \le 1$, the condition (3) is satisfied for any $x \in X$, which implies the desired stability.

The problem $T0$ is a robust optimization problem. Although its third constraint looks more complicated than the counterpart in $T$, it can be equivalently restated by a matrix inequality affine in $\lambda(x)$. Hence, an approach similar to $T$ is possible and a feasible solution $(\lambda(x),c)$ can be found with the sum-of-squares technique.

## 5. EXAMPLE

The approach in Section 3 was applied to a plant

$$x(t+1) = \begin{pmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{pmatrix} x(t) + \begin{pmatrix} 0.0609 \\ 0.0064 \end{pmatrix} u(t)$$

under the input constraint $-2 \le u(t) \le 2$. This example was taken from Bemporad *et al.* [2]. For the model predictive control of this plant, the weight matrices were chosen as $Q = I$ and $R = 0.01$ and the terminal weight $Q_f$ was set to the matrix such that $Q_f - A^{T}Q_f A = Q$. The length of the horizon was chosen as $N = 2$ and the domain $X$ was set to $[-1, 1]^2$. The semidefinite programming problems were solved with the solver SeDuMi [28] and the modeling language YALMIP [20]. The used computer was equipped with Intel Core 2 Duo P8800 (2.66 GHz and 2.67 GHz) and memory of 4 GB.

First, the domain $X$ was not partitioned and our approach was applied with $\lambda(x)$ of various degree. The result is summarized in Table 1. It is seen that the attained optimal value of $T$, which means the distance to optimality, decreases as the degree of $\lambda(x)$ increases. This is consistent with the discussion in Subsection 4.1. On the other hand, the computational time increases rapidly.

Next, the degree of $\lambda(x)$ was fixed to six and the domain $X$ was partitioned into subpolytopes. The result is found in Table 2. Here, in the second row, the partition consists of two subpolytopes $[-1, 0] \times [-1, 1]$ and $[0, 1] \times [-1, 1]$; in the third row, the partition consists of four subpolytopes $[-1, 0]^2$, $[-1, 0]\times[0, 1]$, $[0, 1]\times[-1, 0]$, and $[0, 1]^2$. Although the increase of the computational time is not as rapid as in Table 1, the distance to optimality reaches a small value in the third row of the table. The controller $u_0(x)$ in this case is presented in Figure 1 (a). It is close to the optimal controller in Figure 1 (b), which is computed with the Multi-Parametric Toolbox [14] and is a piecewise affine function defined on nine irregular subpolytopes.

When the two controllers in Figure 1 were applied to the plant, they produced the state trajectories in Figure 2 for three initial states $(0.9\ \ 0.7)^{T}$, $(0.9\ \ 0.2)^{T}$, and $(0.9\ -0.3)^{T}$. The

**Table 1**
**Design of Suboptimal Controllers in the Case that the Polynomial Degree is Changed and the Partition is Fixed**

| *poly. deg.* | *# ofsubpoly.* | *attained opt. val.* | *comp. time* |
|---|---|---|---|
| 6 | 1 | 8.890 | 16.3s |
| 8 | 1 | 1.604 | 103.0s |
| 10 | 1 | 0.402 | 459.4s |

**Table 2**
**Design of Suboptimal Controllers in the Case that the Polynomial Degree is Fixed and the Partition is Changed**

| *poly. deg.* | *# ofsubpoly.* | *attained opt. val.* | *comp. time* |
|---|---|---|---|
| 6 | 1 | 8.890 | 16.3s |
| 6 | 2 | 0.225 | 34.9s |
| 6 | 4 | 0.029 | 66.7s |

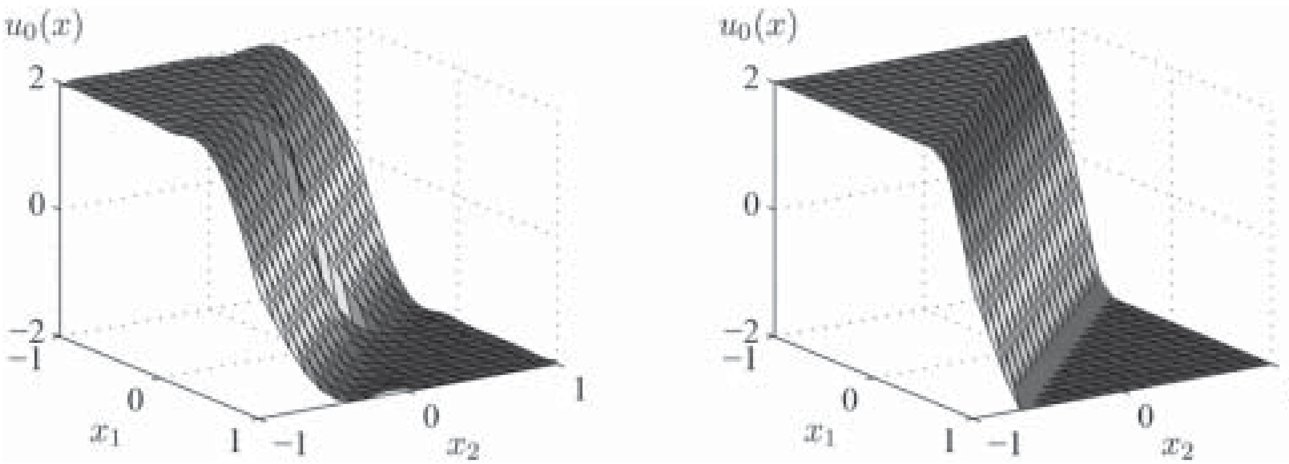


**Figure 1:** Model predictive controllers as functions of the current state.(a) A suboptimal controller, which is a piecewise sixth-degree polynomial on a regular partition consisting of four subpolytopes; (b) The optimal controller, which is a piecewise affine function on an irregular partition consisting of nine subpolytopes.

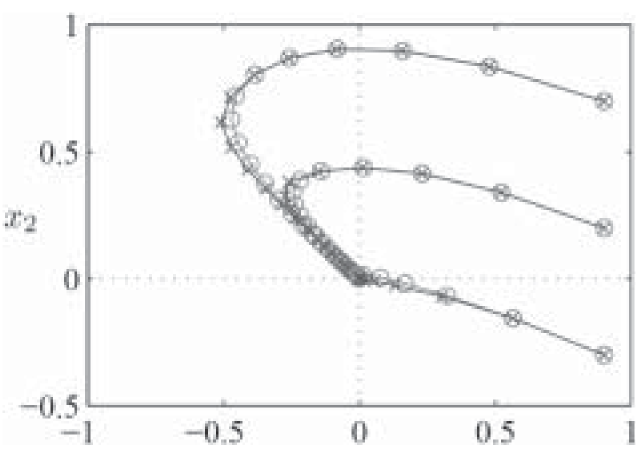


**Figure 2:** The state trajectories produced by the suboptimal controller in Figure 1 (a) (marked by "o") and the optimal controller in Figure 1 (b) (marked by "x"). Three initial states are tried, *i.e.*, $(0.9\ 0.7)^T$, $(0.9\ 0.2)^T$, and $(0.9\ -0.3)^T$.

trajectories of the two controllers are reasonably close to each other.

## 6. CONCLUSION

In this paper, a design of a suboptimal model predictive controller is considered with a regularly partitioned piecewise polynomial. This approach can make the distance to optimality smaller than the existing approaches with a regularly partitioned piecewise affine function. It does not need the computation of the optimal controller and thus does not need geometric computation. Moreover, it is performed by direct minimization of the distance to optimality. The approach is asymptotically optimal in the sense that the distance to optimality can be made arbitrarily small by the use of a high-degree polynomial or a fine partition. It is also possible to guarantee the stability of the resulting closed-loop system.

It has to be admitted that the computational time in Section 5 is rather large for the simple control problem considered there. In fact, it is possible to reduce the computational time by a slight change of the problem formulation. Details can be found in [25].

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