International Journal of Computational Intelligence in Control
Vol. 12 No. 2 (July-December, 2020)

ROBUST *D*-STABILITY ANALYSIS VIA POSITIVE POLYNOMIALS AND LMIS

Tanagorn Jennawasin[*], MichihiroKawanishi[**] and Tatsuo Narikiyo[**] and Chun-Liang Lin[*]

[*]Department of Electrical Engineering, National Chung Hsing University, 250 KuaKuang Rd., Taichung 402, Taiwan, R.O.C., *E-mail: tanagorn,chunlin}@dragon.nchu.edu.tw*

[**]Control System Laboratory, Toyota Technological Institute, 2-12-1 Hisakata, Tempaku, Nagoya Japan 468-8511, *E-mail: kawa,n-tatsuo}@toyota-ti.ac.jp*

ABSTRACT: This paper is concerned with robust *D*-stability of linear systems depending polynomially on uncertain parameters which belong to semi-algebraic sets. The robust stability condition is converted into checking whether a polynomial is positive over a semi algebraic set. Based on sum-of-squares relaxations, a sufficient condition for the polynomial positivity can be formulated as solving a linear matrix inequality (LMI). Construction of a hierarchy of the LMI relaxations, which converge to the stability condition, is also possible via the degree increase of the polynomial. Moreover, a condition to verify instability amounts to solving polynomial equations and inequalities, whose LMI relaxations are available.

Keywords: Robust *D*-stability, Positive polynomials, Sum of squares, Instability Certificate, Linear Matrix Inequalities

1. INTRODUCTION

Robust stability analysis of linear systems affected by real parametric uncertainties is still an active and a challenge research field in control community. Precisely speaking, the main challenge of this research field is establishment of numerically tractable conditions for the robust stability analysis problems, most of which are known to be NP-hard [2].

Concerning robust Hurwitz stability, i.e., robust stability with respect to the left half of the complex plain, several appealing approaches based on Lyapunov functions have been extensively developed in the past two decades. In particular, a simple approach so called quadratic stability condition was firstly developed in the literature (see, for example, [3] and references therein). This approach is based on searching for a quadratic Lyapunov function independent of uncertain parameters, and hence a sufficient condition for the robust stability is formulated into solving a finite number of linear matrix inequalities (LMIs). In order to construct less conservative conditions, several approaches based on Lyapunov functions dependent of the uncertain parameters have been proposed [1, 5, 6, 17]. These approaches lead to less conservative LMI conditions for the robust stability. In particular, if the Lyapunov functions are assumed to depend polynomially on the uncertain parameters, construction of such Lyapunov functions can be relaxed into LMIs by some computational tools, for instance, Kalman-YakubovichPopov Lemma [1] or sum-of-squares technique [5, 15]. Extensions to robust *D*-stability, i.e., robust stability with respect to some subregion *D* of the complex plain, have been considered in [8, 12, 16]. In order to render LMI conditions for *D*-stability test, however, the region *D* must be described by an LMI [8, 12, 16]. Furthermore, it is still unproven whether existence of a polynomially parameter-dependent Lyapunov function is necessary for the robust *D*-stability.

Alternative approaches, which are not based on Lyapunov functions, have been proposed in [4, 9] for robust Hurwitz stability, in [7] for robust Schur stability, and in [19, 14, 18] for general robust D-stability. The approaches of [4, 9, 7] provide sequences of LMI relaxations that asymptotically become necessary for the robust Hurwitz stability or the robust Schur stability, i.e., robust stability with respect to the unit circle in the complex plain, as the relaxation orders increase. Extension of these approaches to general robust D-stability test, however, is still unclear. Siljak *et al.* [19] provided a different condition based on Bernstein polynomial expansions for the robust D-stability.

However, this approach is limited to the case of uncertain parameter regions described by multi-dimensional intervals. Moreover, large computational burden is required due to gridding on the parameter regions. For the case of scalar uncertain parameters, alternative methods based on computation of uncertainty intervals that guarantee D-stability have been developed by [14, 18].

In this paper, we propose a new approach to the robust D-stability test. In the current approach, the uncertain parameter region and the region D are assumed to be semi-algebraic sets, i.e., they are described by polynomial inequalities. As opposed to [8, 12, 16], neither assumption on the LMI representation nor assumption on convexity of the region D is necessary in the proposed framework. As a result, the proposed approach is applicable to uncertain systems whose parameter regions and regions D are in more general form than those of [8, 12, 16, 14, 18, 19]. A sufficient condition for the robust stability amounts to checking whether a polynomial is positive over a semi-algebraic set. The positivity checking is relaxed into an LMI problem by the sum-of-squares (SOS) technique [15]. Moreover, a sequence of LMIs which asymptotically become necessary for the robust D-stability can be constructed based on the degree increase of the resulting polynomial. In other words, a sufficient LMI condition that is asymptotically necessary for the robust D-stability is derived in the current work. Several heuristic methods are provided to reduce computation complexity when applying the proposed stability test. We also propose a sufficient and necessary condition for the instability, which amounts to solving a system of polynomial equations and inequalities. Solving such polynomial equations and inequalities can be again relaxed into an LMI using the method of moment [11], which allows to construct a hierarchy of LMI relaxations asymptotically exact to the original problem. Note here that such a condition for the instability has never been proposed in the literature.

2. SUM-OF-SQUARES POLYNOMIALS

Let $\mathrm{R}[\theta]$ denote the set of polynomial in $\theta \in \mathrm{R}^p$. We define the notion of *sum-of-squares (SOS)* polynomials as follows.

Definition 1 *[15, 11] A polynomial* $S \in \mathrm{R}[\theta]$ *is said to be a sum of squares (SOS) if there exist polynomials* $q_i \in \mathrm{R}[\theta]$, $i = 1,\ldots,\nu$ *such that*

$$S(\theta) = \sum_{i=1}^{\nu} q_i^2(\theta)..$$

We use $\Sigma[\theta]$ to represent the set of SOS polynomials. It is clear that any polynomial $S \in \Sigma[\theta]$ is globally positive semidefinite, i.e., $S(\theta) \geq 0$, œ$\theta \in \mathrm{R}^p$, but the converse is not true in general.

A computational procedure for verifying whether $S(\theta)$ is an SOS proceeds as follows. Choose pairwise different monomials $u_1(\theta),\ldots,u_{nu}(\theta)$ and search for the coefficient matrix Y in the representation

$$\begin{bmatrix} q_1 & q_2 & \cdots & q_\nu \end{bmatrix}^T = Yu(\theta)$$

with $Y = (Y_1,\ldots,Y_{nu})$ and $u(\theta) = (u_1(\theta),\ldots,u_{nu}(\theta))^{\mathrm{T}}$. The polynomial $S(\theta)$ is said to be an SOS with respect to $u(\theta)$ if there exists some Y satisfying $S(\theta) = u(\theta)^{\mathrm{T}}(Y^{\mathrm{T}}Y)u(\theta)$. Substituting $Z = Y^{\mathrm{T}}Y$ yields the following result.

Proposition 1 *[15] A polynomial* $S \in \mathbb{S}[\theta]$ *is an SOS with respect to the monomial basis* $u(\theta)$ *if and only if there exists a symmetric matrix* $Z \succeq O$ *with*

$$S(\theta) = u(\theta)^{\mathrm{T}}Zu(\theta). \tag{1}$$

Expanding the right-hand side of (1) yields a polynomial of which coefficients depend affinely on elements of Z. As an identity in θ, we can match coefficients of the polynomials in both sides of (1). Hence the condition (1) can be interpreted as an affine constraint in Z. This implies that the problem to find $Z \succeq O$ with (1) can be formulated as an LMI. In other words, we can check whether $S \in \Sigma[\theta]$ with respect to some monomial basis by solving an LMI.

3. PROBLEM STATEMENT

Consider the uncertain linear time-invariant system described by

$$\delta[x(t)] = A(\theta)x(t) \tag{2}$$

where $\delta[x(t)] \triangleq dx(t)/dt$ for continuous-time systems, or $\delta[x(t)] \triangleq x(t+1)$ for discretetime systems, $x \in \mathbb{R}^n$ is the vector of state variables, and the matrix $A(\theta)$ depends polynomially on an uncertain parameter $\theta \in \mathbb{R}^m$ belonging to some compact semialgebraic set $\Theta = \{\theta \in \mathbb{R}^m \mid t_j(\theta) \geq 0,\ j = 1,2,\ldots,l\}$ with polynomials t_j's. We are interested in checking robust D-stability of (2), that is, whether all eigenvalues of matrix $A(\theta)$ lie in a given subregion D of the complex plane for all parameter $\theta \in \Theta$.

The parameter θ is assumed to be time-invariant through out this paper.

The characteristic polynomial of the system (2) is denoted by

$$H(s,\theta) = \det(sI - A(\theta)) = \sum_{i=0}^{n} h_i(\theta)s^i \tag{3}$$

It is clear that the system (2) is robustly D-stable if and only if all the roots of (3) lie inside the region D for all $\theta \in \Theta$.

4. ROBUST *D*-STABILITY TEST

The key idea in the current approach is to replace the complex variable s with two real variables, i.e., $s = x + jy$, with $x,y \in$ R. Based on this change of variable, the characteristic polynomial in (3) can be transformed as

$$H(s,\theta) = H_1(x,y,\theta) + jH_2(x,y,\theta) \tag{4}$$

where H_1 and H_2 are polynomials with real coefficients in $m + 2$ real variables. The degrees of the variables x,y in H_1 and H_2 are at most n.

We assume the region D', i.e., the complement of the region D in the complex plane is a semi-algebraic set

$$D' = \{(x, y) \mid d_j(x, y) \geq 0,\ j = 1, \ldots, k\}$$

where d_j's are polynomials. Some examples of D' are $D'_c = \{(x, y) \mid x \geq 0\}$ in the case of Hurwitz stability and $D'_d = \{(x, y) \mid x^2 + y^2 - 1 \geq 0\}$ in the case of Schur stability. Moreover, let

$$G = \{(x, y, \theta) \in \mathbb{R}^{m+2} \mid (x, y) \in D',\ \theta \in \Theta\}$$

Based on the assumptions on D' and Θ, it is obvious that the set G is also a semialgebraic set. Hence, G can be written as

$$G = \{(x, y, \theta) \in \mathbb{R}^{m+2} \mid g_j(x, y, \theta) \geq 0,\ j = 1, \ldots, r\} \tag{5}$$

where g_j's are polynomials.

It is not difficult to see that the system (2) is robustly D-stable if and only if there is no $(x, y, \theta) \in G$ such that $H_1(x, y, \theta) = H_2(x, y, \theta) = 0$. This condition can be transformed into positivity of a single polynomial as in the following lemma.

Lemma 1: *The system (2) is robustly D-stable if and only if there exist* $\tilde{M}, \tilde{N} \in \mathbb{R}[(x, y, \theta)]$ *such that*

$$\tilde{M}(x, y, \theta)H_1(x, y, \theta) + \tilde{N}(x, y, \theta)H_2(x, y, \theta) > 0, \quad \forall (x, y, \theta) \in G \tag{6}$$

Proof: It can be proved by contradiction that if the inequality (6) holds, then there is no $(x,y,\theta) \in G$ such that $H_1(x,y,\theta) = H_2(x,y,\theta) = 0$ and hence (2) is robustly$D$-stable.

Conversely, if (2) is robustly D-stable, then there holds $H_1(x,y,\theta)^2 + H_2(x,y,\theta)^2 > 0$, $\forall (x,y,\theta) \in G$ which satisfies (6) by taking $\tilde{M}(x,y,\theta) = H_1(x,y,\theta)$ and $\tilde{N}(x,y,\theta) = H_2(x,y,\theta)$.

Testing the positivity in (6) is known to be NP-hard. However, the notion of sum-of-squares (SOS) polynomials in Section 2 leads to construction of a computationally tractable condition for the positivity checking. Before proceeding, we state the following lemma which is useful for proving the main theorem. Note that arguments of polynomials will be omitted for notational simplicity.

Lemma 2 (Positivstellansatz [20]) *The following statements are equivalent: 1)*

2) There exist $t_i \in \mathbb{R}[\theta]$, $i = 1,\ldots,m$, *and* $s_0, s_1,\ldots, s_r, s_{12},\ldots, s_{k-1,k},\ldots,s_{12\ldots k}$ $\Sigma[\theta]$ *such that*

$$\sum_{i=1}^{m} q_i t_i = 1 + s_0 + \sum_{i=1}^{k} p_k s_k + p_1 p_2 s_{12} + \ldots + p_1 p_2 \ldots p_k s_{12\ldots k}$$

The main result, which provides a tractable condition for the robust stability, is stated as follows.

Theorem 1: *The (2) is robustly D-stable if and only if there exist polynomials* $M, N \in \mathbb{R}[(x,y,\theta)]$, *and SOS polynomials* $S_0, S_1,\ldots, S_r, S_{12},\ldots, S_{r-1,r},\ldots,S_{12\ldots r} \in \Sigma[(x,y,\theta)]$ *such that*

$$MH_1 + NH_2 = 1 + S_0 + \Sigma_{i=1}^{r} g_i S_i + g_1 g_2 S_{12} \qquad (7)$$

Proof: We firstly prove the sufficiency. By the definition of SOS polynomials, the constraint (7) implies that $MH_1 + NH_2 \geq 1 > 0$ for all $(x,y,\theta) \in G$, and thus the system (2) is robustly D-stable by Lemma 1.

We then prove the necessity. Suppose that the system (2) is robustly D-stable, then there exists no $(x,y,\theta) \in G$ such that $H_1(x,y,\theta) = H_2(x,y,\theta) = 0$. Thus

$$\left\{\theta \middle| \begin{array}{ll} g_i(\theta) \geq 0 & i = 1,\ldots,r \\ H_j(\theta) = 0 & j = 1,2 \end{array}\right\} = \emptyset$$

By Positivstellensatz, it implies that there exist polynomials $M,N \in R[(x,y,\theta)]$ and SOS polynomials $S_0,S_1,\ldots,S_r,S_{12},\ldots,S_{r-1,r},\ldots,S_{12\ldots r} \in \Sigma[(x,y,\theta)]$ such that (7) holds. The constraint (7) is affine in decision variables M,N and the SOS polynomials S_i's. If the degrees of M, N and the related SOS polynomials are fixed a priori, solving for the decision variables can be cast as a standard LMI problem using the methodology in Section 2. Theorem 1 implies that the gap between the LMI condition and the robust stability condition can be arbitrarily reduced by the degree increase of M, N and the SOS polynomials. Note here that solving an SOS problem can be easily implemented with the help of available softwares, such as YALMIP [13].

The representation in (7) contains many terms involving SOS polynomials. The number of such terms increases exponentially with respect to r, which is the number of polynomials in the characterization of the set G. Therefore, the problem might be difficult to solve due to high computational complexity when the number r is large. In order to reduce the computational complexity, we can consider a representation with small number of SOS terms. A specific example is the representation which contains only terms involving a single polynomial g_i, i.e.,

$$M\,H_1 + NH_2 = 1 + S_0 + \sum_{i=1}^{r} g_i S_i$$

Moreover, the degrees of the polynomials M,N and S_i's in (7) are not known a priori and can be theoretically very high. In order to make the problem solvable at moderate cost, these degrees should be bounded by some small values. A possible strategy is to take M,N and S_i's in the way that the total degree p of each term appearing in (7) is the same. Note here that increase of the degree p yields a tighter condition for the stability test, but at the expense of computational complexity.

Remark: The test proposed in [19] is based on the strict positivity of $|H(x,y,\theta)|^2 = H_1(x,y,\theta)^2 + H_2(x,y,\theta)^2$. The positivity test is based on Bernstein polynomials, and is only on the unit circle $x^2 + y^2 = 1$. It is clear that this is a special case of our test, corresponding to $M = H_1$, $N = H_2$, and $p = 2\deg H$. It is shown by numerical experiments that our test is tight even for smaller values of the degree p.

5. A CONDITION FOR INSTABILITY

We have mentioned in the previous section that the degrees of the M,N and S_i's in (7) to achieve robust D-stability are not known a priori. Hence, infeasibility of (7) at some finite degrees of M,N and S_i's does not imply that the system is not robustly D-stable. In this section, we provide a tractable condition to certify the instability of the system.

Based on the decomposition in (4), the system (2) is *not* robustly D-stable if and only if there is a point $(x,y,\theta) \in G$ such that $H_1(x,y,\theta) = H_2(x,y,\theta) = 0$. In order to verify the instability, therefore, it is required to solve the following polynomial systems

$$\begin{aligned} &(x, y) \in D', \\ &H_1(x, y, \theta) = 0, \\ &H_2(x, y, \theta) = 0 \end{aligned} \tag{8}$$

Any (x, y) and θ satisfying (8) are referred to as an unstable pole and a destabilizing parameter, respectively. Since we want to find only one unstable pole, as well as an associated destabilizing parameter, to prove the instability of the system, it is possible to add a criterion to characterize the unstable pole. In the case of Hurwitz stability, for example, it is possible to find the unstable pole closest to the imaginary axis by minimizing x subject to the constraints in (8). For general regions D, we consider the following optimization problem:

where $p(x,y)$ is a polynomial that characterizes the unstable pole. Note that the problem P_D is minimizing a multivariate polynomial over a semi-algebraic set, and hence is nonconvex in general. However, the methodology in [11] can be applied to construct a hierarchy of convex LMI relaxations to P_D. In particular, we can build a sequence of

LMI problems

$$\begin{aligned} P_D^k: &\quad \text{minimize} \quad f(w) \\ \text{subject to} &\quad F_k(w) \succeq 0 \end{aligned}$$

for $k = 1,2,...$, where $f(w)$ is a linear function of the vector w of decision variables, and $F_k(w)$ is an LMI constraint constructed from moment matrices and localization matrices of appropriate orders (see [11] for details). The problem is referred to as the LMI relaxation of order k to the original problem P_D. Let d denotes the maximum degree of the polynomials p, H_1, H_2 and g_j in P_D. Valid relaxation orders are $k = k_0, k_0 + 1, k_0 + 2, ...$, where k_0 is the minimal relaxation order such that $k_0 = d/2$ if d is even, and $k_0 = (d + 1)/2$ if d is odd [11, 10]. It was proved in [11] that solving a sequence of LMI relaxations , $k = k_0, k_0 + 1, k_0 + 2, ...$ provides a sequence of lower bounds which converges to the optimal value of P_D, i.e.,

and

where P_D^{k*} and denote the optimal values of the problems and P_D, respectively. Moreover, if the optimal solution of satisfies some algebraic conditions at some k, then the LMI relaxation is exact, that is,

In this case, we can extract the optimal solution of P_D from that of P_D^k and verify that the system is not robustly D-stable. Note here that construction of the LMI relaxations can be easily performed using the software Gloptipoly [10].

The rest of this section is devoted for discussion of some computational issues in solving the polynomial optimization problem P_D. In fact, it was shown by numerical experiments [11, 10] that the optimal value P_D^* is often attained by solving an LMI problem with a low order k. In some cases, however, the sequence of lower bounds converges slowly to and thus we need to solve of high order to obtain the optimal value and the optimal solution of P_D. Since the size of the matrices and the number of variables in grow rapidly as k increases, the LMI problem might be difficult to solve due to high computational complexity. This issue can be partially addressed by following the heuristic method suggested in [9]. The idea is illustrated here by considering a robust instability certificate problem with respect to the Hurwitz stability, which amounts to solving the following polynomial optimization problem

$$
\begin{aligned}
&\text{minimize}_{x,y,\theta} && x \\
&\text{subject to} && x \geq 0 \\
& && t_j(\theta) \geq 0,\ j = 1,\dots, l \\
& && H_1(x, y, \theta) = 0, \\
& && H_2(x, y, \theta) = 0,
\end{aligned}
$$

where $t_j(\theta)$'s are polynomials characterizing the uncertainty region. It is clear that the system is not robustly Hurwitz stable if and only if the optimal value of the optimization problem above is nonnegative, that is, $x^* \geq 0$. By considering an LMI relaxation P^k for the problem above, if $P^{k*} \geq 0$ at some order k, then $x^* \geq P^{k*} \geq 0$. Therefore, we can conclude that the system is not robustly Hurwitz stable even though the optimal value x^{*}has not been found, and further increase of the relaxation order is not necessary in this case. Generalization of this idea to other robust instability certificate problems with respect to general regions D can be done in similar manners.

6. NUMERICAL EXAMPLES

The tests described in Sections 4 and 5 are applied with the following example. All the computation is performed on YALMIP [13] and Gloptipoly [10]. withSeDuMi [21] as an SDP solver. The computer is equipped with Pentium-III 1200 MHz and 248 MBytememory.

Example1 (Hurwitz stability): Consider the uncertain system

(10)

studied in [9] with the system matrix

$$
A(\theta) = \begin{bmatrix} -2+\theta_1 & 0 & -1+\theta_1 \\ 0 & -3+\theta_2 & 0 \\ -1+\theta_1 & -1+\theta_2 & -4+\theta_1 \end{bmatrix},
$$

where $\theta = (\theta_1, \theta_2) \in [-\gamma, \gamma]^2$ is the uncertain parameter.

In this example, we want to compute the maximum value of $\gamma \in R$ such that the system (10) is robustly Hurwitz stable over $[-\gamma, \gamma]^2$.

The characteristic polynomial of the system (10) is

$$
\begin{aligned}
H(s, \theta) &= s^3 + (9 - 2\theta_1 - \theta_2)s^2 \\
&\quad + (25 - 10\theta_1 - 6\theta_2 + 2\theta_1\theta_2)s \\
&\quad + (21 - 12\theta_1 - 7\theta_2 + 4\theta_1\theta_2)
\end{aligned}
$$

After substitution of $s = x + iy$ in the above equation, we obtain the polynomials $H_1(x,y,\theta_1,\theta_2)$ and $H_2(x,y,\theta_1,\theta_2)$ of degree 3 in (4). The region G in (5) is given by

We perform the stability test by searching for polynomials $M,N \in \mathbb{R}[(x,y,\theta_1,\theta_2)]$ and SOS polynomials $S_0,S_1,S_2,S_3 \in \Sigma[(x,y,\theta_1,\theta_2)]$ satisfying

$$
MH_1 + NH_2 = 1 + S_0 + S_1 x + S_2(\gamma^2 - \theta_1^2) + S_3(\gamma^2 - \theta_2^2). \quad (11)
$$

Here, we fix the degree of polynomials M, N and SOS polynomials S_0,S_1,S_2,S_3 in (11) such that the degree of the terms appearing there is 4, e.g., deg M= deg$N$= 1, degS_0 = 4, degS_1 = degS_2 = degS_3 = 2.

Let γ^* denote the maximum value of γ such that (11) is feasible. By performing bisection on γ, we obtain $\gamma^* = 1.7416$. For each fixed γ, the computational time for solving the LMI from (11) is about 0.9 seconds by using YALMIP. When $\gamma = 1.76$, we verify instability of (10) by solving the optimization problem

$$
\begin{aligned}
&P_D : \text{minimize}_{x,y,\theta 1,\ \theta 2} && x \\
&\text{subject to} && x \geq 0, \\
& && H_j(x, y, \theta_1, \theta_2) = 0,\ \ j = 1,2.
\end{aligned}
$$

By using Gloptipoly, we obtain the unstable pole $x = 0.0806$ with the destabilizing parameter $\theta = (1.76,1.76)$ at the LMI relaxation of order $k = k_0 = 2$. The computational time for solving the LMI is about 0.4 seconds. Tightness of the testing conditions is revealed by the numerical results. In fact, the system (10) is marginally stable at $\theta = (1.75,1.75)$.

Note that we can not verify the robust stability of (10) when $\gamma \in (1.7416,1.75)$ by solving (11). However, the conservatism in this case can be reduced by increasing the degree of the terms in (11), or solving the SOS constraint (7) instead.

Example 2 (Schur stability): Consider the following characteristic polynomial

$$H(s,\theta) = -(11+3\theta)s^5 + (5+3\theta)s^4 + \alpha(1+\theta)s + \alpha(1-\theta) \quad (12)$$

with an uncertain parameter $\theta \in [-1,1]$, and with a constant α. We want to check the robust Schur stability of the polynomial (12) for various values of α.

The total degree of $H(s,\theta)$ as a polynomial in (s,θ) is 6, and hence the total degrees of the polynomials H_1 and H_2 in (4) are 6. We fix the degree of polynomials M, N and SOS polynomials in (7) such that the degree of the terms appearing there is 8. Solving (7) for various values of α, it turns out that the polynomial (12) is robustly Schur for $-3.3261 \le \alpha \le 2.9999$. For a fixed α, the time required for solving the resulting LMI from (7) is about 2.0 seconds.

We also verify the instability of (12) for α outside the interval $[-3.3261, 2.9999]$. By applying the methodology in Section 5, the instability of (12) can be proved for the most values of $\alpha \notin [-3.3261, 2.9999]$, by solving LMI relaxations of low orders.

7. CONCLUSIONS

A necessary and sufficient condition for the robust *D*-stability of linear systems was derived as checking positivity of a polynomial, whose hierarchy of LMI relaxations was constructed by the notions of sum-of-squares polynomials. On the other hand, instability of the systems can be ensured by solving a system of polynomial equations and inequalities. A hierarchy of LMI relaxations for the problem can be constructed using the method of moments. Numerical experiments shown that the proposed conditions, for both the stability and instability test, yield accurate results in many cases even the relaxations of low orders were used for the tests.

ACKNOWLEDGMENT

This research was supported in part by National Science Council, Taiwan, R.O.C under the grant NSC-98-2221-E-005-08-MY3.

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