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INVOLUTIVE IDEALS DEFINING ALGEBRAIC GRADIENTS OF SOLUTIONS TO NONSTATIONARY HAMILTON-JACOBI EQUATIONS

Yu Kawano[1] and Toshiyuki Ohtsuka[1]

[1]Graduate School of Informatics, Kyoto University, Kyoto, Japan
E-mail: ykawano@i.kyoto-u.ac.jp; ohtsuka@i.kyoto-u.ac.jp

ABSTRACT: The Hamilton-Jacobi equation (HJE) plays an important role in the analysis and control of nonlinear systems and is very difficult to solve for general nonlinear systems. In this paper, a nonstationary HJE with coefficients belonging to meromorphic functions is considered, and its solutions with algebraic gradients are characterized in terms of commutative algebra. It is shown that there exists a solution with an algebraic gradient if and only if an involutive zero-dimensional radical ideal exists in a polynomial ring over the field of meromorphic functions of the time and state. If such an ideal is found, an algebraic gradient can be obtained simply by solving a set of algebraic equations.

Keywords: nonlinear systems, time-varying systems, Hamilton-Jacobi equation, optimal control, algebraic functions.

I. INTRODUCTION

The Hamilton-Jacobi equation (HJE) is one of the most important equations in systems and control theory. For example, a class of nonlinear optimal control problems [1], [2] leads to the HJE. However, it is difficult to solve the HJE analytically and numerically because it is a nonlinear partial differential equation. Recently, for a stationary HJE, a new representation of a solution has been proposed [3] in terms of commutative algebra. A solution is defined from a different viewpoint from the standard differential geometric approach [4], the viscosity solution [5] and the method of characteristics [6]. In [3], a set of algebraic equations satisfied by the gradient of a solution to the HJE was studied by restricting the Hamiltonian to a polynomial in the gradient of the solution with coefficients consisting of rational functions of the state.

Extending this approach, in this paper, we derive a necessary and sufficient condition for the existence of a set of algebraic equations that implicitly defines the gradient of a solution to a nonstationary HJE by restricting the Hamiltonian to a polynomial in the gradient of a solution with coefficients consisting of meromorphic functions of the time and state. Such a gradient is obtained simply by solving a set of algebraic equations instead of a nonlinear partial differential equation. Our condition is obtained in a similar form to one of the existence conditions for an algebraic gradient of a stationary HJE in [3]. In [7], the existence condition of an algebraic gradient has been derived for a nonstationary HJE, which has been obtained in a different form from the condition in this paper. In this paper, we also study the relation between our results and previous results.

The remainder of this paper is organized as follows. In Section II, the class of the HJE treated in this paper is stated. Then, a solution to the HJE with an algebraic gradient is defined. In Section III, the mathematical preliminaries based on commutative algebra are given. In Section IV, an existence condition for a solution with an algebraic gradient is presented for the HJE, which guarantees necessity and sufficiency. In Section V, a class of nonlinear time-varying optimal regulator problems is given such that explicit solutions are obtained as algebraic functions, and examples of explicit solutions are also presented. Conclusions are given in Section VI.

Notations: Throughout the paper, $x = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$ denotes the n-dimensional state vector of a dynamical system. For a scalar-valued function $V(t, x)$, we denote a row vector consisting of the partial derivatives of V with respect to x_i $(i = 1,2,\ldots,n)$ as $\partial V/\partial x$, and the column vector $(\partial V/\partial x)^{\mathrm{T}}$, which is the transpose of $\partial V/\partial x$, as ∇V. The sets of real and complex numbers are denoted by $\mathbf{R}$ and $\mathbf{C}$, respectively. Since $\mathbf{R}$ and $\mathbf{C}$ are closed under addition, subtraction, multiplication and division by nonzero elements, $\mathbf{R}$ and $\mathbf{C}$ are fields. The sets of meromorphic functions defined on a domain in $\mathbf{R}^n$ with variable x and a domain in $\mathbf{R} \times \mathbf{R}^n$ with variables (t, x) are denoted by K and K_t, respectively, where K and K_t are fields. Furthermore, the algebraic closures of K and K_t are denoted by $\overline{K}$ and $\overline{K}_t$, respectively.

II. SETTING OF THE PROBLEM

A. Hamilton-Jacobi Equation

For a scalar-valued function $H(t,x,p)$, we consider the following first-order partial differential equation:

$$\frac{\partial V}{\partial t} = -H(t, x, p), \quad p = \nabla V, \tag{1}$$

where $V(t,x)$ is a scalar-valued function. We call equation (1) the nonstationary Hamilton-Jacobi equation (HJE) or simply the HJE for the Hamiltonian H. The HJE (1) plays an important role in the analysis and control of time-varying nonlinear systems.

Example 2.1 (Nonlinear optimal regulator): Consider $f : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$, $g : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^{n \times m}$ and $q : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$, and the following state equation and performance index:

(2)

(3)

where we assume that $f(t,0) = 0$ for all t, and q is positive definite with respect to x for each fixed t and uniformly bounded with respect to t for each fixed x. The value function $V(t_0, x_0) := \inf_u J$ satisfies the HJE for the Hamiltonian

(4)

The optimal regulator is given as $u_{opt} = -g^{\mathrm{T}} p$. If the system is affine in control and the performance index is quadratic in control, then the Hamiltonian is obtained as a function of (t, x, p).

Remark 2.1: To distinguish (1) from the HJE

$$H(x, p) = 0, \; p = \nabla V, \tag{5}$$

we call (5) the stationary HJE.

B. Solutions with Algebraic Gradient

In this paper, we consider how to obtain a set of algebraic equations that implicitly defines the gradient of a solution to the HJE. That is, our aim is to find the following solution.

Definition 2.1: An analytic function $\rho : D \to \mathbf{C}$ defined on a domain $D \subset \mathbf{R} \times \mathbf{R}^n$ is said to be an algebraic function over K_t if there exists a nonzero irreducible polynomial ψ with univariable X over K_t such that

$$\psi(t, x, \rho(t, x)) = 0 \tag{6}$$

holds for all $(t, x) \in D$.

Definition 2.2: For a solution $V(t, x)$ to the HJE (1), $\nabla V(t, x)$ is said to be an algebraic gradient if all of its components $\partial V(t, x)/\partial x_i$ $(i = 1,\ldots, n)$ are algebraic functions over K_t.

Remark 2.2: For elements p_{0i} $(i = 1,\ldots,n)$ of an algebraic gradient, there exist irreducible polynomials $\psi_i(t, x, p_i)$ with variables p_i $(i = 1,\ldots,n)$ over K_t such that $\psi_i(t, x, p_{0i}) = 0$. Thus, if there exists an algebraic gradient p_0, the gradient is obtained simply by solving a set of algebraic equations $[\psi_1,\ldots, \psi_n]^{\mathrm{T}}(t, x, p) = 0$ with respect to p instead of a nonlinear partial differential equation.

Example 2.2: Here, we give an example of an HJE having a solution with an algebraic gradient. Consider the nonlinear optimal regulator problem

$$\dot{x} = \frac{x}{2+2t} + xu, \tag{7}$$

(8)

The HJE in this case is given as

(9)

By using Proposition 5.1 in Section V, it is possible to find gradients of solutions to this HJE.

Actually, the gradients of the solutions are obtained as

(10)

which are algebraic functions defined by the irreducible polynomial $p^2 - x^2/(1 + t + x^2)$ over

III. MATHEMATICAL PRELIMINARIES

A. Ideals and Affine Varieties

In this paper, our objective is to characterize the class of HJEs having a solution with an algebraic gradient. From Definitions 2.1 and 2.2, algebraic gradients are defined by a set of algebraic equations, i.e., polynomials. The set of polynomials with coefficients in a field forms a commutative ring with the usual addition and multiplication. This commutative ring is called the polynomial ring and is studied in commutative algebra. In terms of commutative algebra, we study the existence of an algebraic gradient. Here, we summarize some tools of commutative algebra. For more details, see [8], [9].

Let $K[p] := K[p_1,...,p_n]$. We define an ideal, which is an important concept in commutative algebra. The set $\langle q_1,...,q_s \rangle$, defined as

(11)

is called the ideal generated by polynomials $q_1,...,q_s$. The definition of an ideal is similar to that of a vector subspace. However, an ideal is not a vector subspace because the coefficients of an ideal are elements in a polynomial ring, while those of a vector subspace are elements in a field.

Ideals are important for studying the set of common zeros of polynomials. Let $\mathbf{V}(I) \subset \bar{K}^n$ and $\mathbf{V}(q_1,...,q_s) \subset \bar{K}^n$ be the set of common zeros of all polynomials in $I := \langle q_1,...,q_s \rangle \subset K[p]$ and polynomials $q_1,...,q_s$, respectively. $\mathbf{V}(I) \subset \bar{K}^n$ and $\mathbf{V}(q_1,...,q_s) \subset \bar{K}^n$ are called affine varieties. From their definitions, we have $\mathbf{V}(I) = \mathbf{V}(q_1,...,q_s)$. This equality indicates that sets of polynomials defining the same ideal have the same common zeros but the converse is not always true. There is a case that two different ideals define the same affine variety.

Example 3.1: Let $I = \langle p_1,...,p_n \rangle$ and be ideals in $K[p]$. Then, we have $\mathbf{V}(I) = \mathbf{V}(J) = \{0\} \subset \bar{K}^n$. However, $I \neq J$ holds.

In the above example, the two ideals I and J are distinct but the radicals of these two ideals are equivalent. The radical of an ideal $I \subset K[p]$, denoted by , is the set

$$\{f : f^m \in I \text{ for some integer } m \geq 1\}. \tag{12}$$

The radical is an ideal in $K[p]$ containing I, and thus is called a radical ideal. An ideal I is radical if and only if From the definition of the radical, for every ideal $I \subset K[p]$, we have $\mathbf{V}(I) = \mathbf{V}(\sqrt{I}) \subset \bar{K}^n$.

Example 3.2: Consider the ideals in Example 3.1. Then, I is the radical ideal, and the radical of J is I.

According to Hilbert's Nullstellensatz [8], [9], distinct radical ideals of a polynomial ring over an algebraically closed field [8], [9] define distinct affine varieties. An example of an algebraically closed field is the set of complex numbers $\mathbf{C}$. Let I and J be ideals in $\mathbf{C}[p]$. From Hilbert's Nullstellensatz, and only if $\mathbf{V}(I) = \mathbf{V}(J)$.

(B) Zero-Dimensional Ideals

In this paper, our aim is to find a set of polynomials defining an algebraic gradient. A set of polynomials having the same common zeros is characterized by an ideal in the polynomial ring. Thus, in this paper, we show a class of HJEs having a solution with an algebraic gradient in terms of ideals. From Definitions 2.1 and 2.2, an algebraic gradient is defined as an implicit function. Proposition 3.1 below shows that a set of algebraic equations determined by a zerodimensional radical ideal characterizes an implicit function. A zero-dimensional ideal is defined as follows [9].

Definition 3.1: A nontrivial ideal I in $K[p]$ is a zero-dimensional ideal if $\mathbf{V}(I) \subset \bar{K}^n$ is a finite set.

Example 3.3: Let I ($K[p]$ be an ideal such that there is no ideal J ($K[p]$ satisfying I (J ($K[p]$. This I is called a maximal ideal [9], and a maximal ideal is a zero-dimensional ideal [9].

Proposition 3.1: [7] A zero-dimensional radical ideal $I \subset K[p]$ can always be generated by n elements $F_1,\ldots, F_n \in K[p]$ such that $(\partial F_i/\partial x_j)$ is nonsingular at any element in the affine variety $\mathbf{V}(I) \subset \bar{K}^n$.

Example 3.4: Let $n = 2$, and let

We have

(13)

and thus I is a zero-dimensional ideal, where j is the unit imaginary number. Moreover, it can be shown that I is a radical ideal. This zero-dimensional radical ideal I is generated by two elements. Next, we compute the Jacobian matrix

$$\frac{\partial(F_1, F_2)}{\partial(p_1, p_2)} = \begin{bmatrix} 1 & 0 \\ 0 & 2p_2 + x_2 \end{bmatrix}, \tag{14}$$

which is nonsingular for any $p \in \mathbf{V}(I) \subset \bar{K}^2$.

IV. EXISTENCE OF SOLUTIONS

A. Algebraic Gradients of Stationary HJEs

Here, we summarize results for an algebraic gradient of a stationary HJE. In [3], the existence of algebraic gradients for a stationary HJE is characterized in terms of the involutiveness of an ideal. The involutiveness of an ideal is defined by using the Poisson bracket. The Poisson bracket for two functions Φ and Ψ is defined as

$$\{\Phi, \Psi\} := \sum_{i=1}^{n} \left(\frac{\partial \Phi}{\partial x_i}\frac{\partial \Psi}{\partial p_i} - \frac{\partial \Phi}{\partial p_i}\frac{\partial \Psi}{\partial x_i} \right). \tag{15}$$

If the class of functions Φ and Ψ is restricted to the polynomial ring $K[p]$, the Poisson bracket can be viewed as a mapping $\{\cdot,\cdot\} : K[p] \times K[p] \to K[p]$ because $K[p]$ is closed under partial differentiation.

We also define the involutiveness of an ideal as follows.

Definition 4.1: An ideal I in $K[p]$ is involutive if

$$\{\Phi, \Psi\} \in I \tag{16}$$

holds for all $\Phi \in I$ and for all $\Psi \in I$.

It can be checked whether or not a zero-dimensional radical ideal I = h$F_1,\ldots,F_n$iis involutive by computing the Poisson bracket for every pair of functions F_i and F_j ($i, j = 1,\ldots,n$) according to the following proposition [3].

Proposition 4.1: An ideal I := h$F_1,\ldots,F_n$i$\subset K[p]$ is involutive if and only if

$$\{F_i, F_j\} \in I \ (i, j = 1,\ldots, n) \tag{17}$$

holds.

An algebraic gradient is characterized in terms of an involutive ideal as follows [3].

Proposition 4.2: The stationary HJE (5) has a solution with an algebraic gradient on a contractible domain in $\mathbf{R}^n$ if and only if there exists an involutive maximal ideal containing the Hamiltonian $H \in K[p]$.

The maximality condition in Proposition 4.2 can be weakened as follows.

Theorem 4.1: The stationary HJE (5) has a solution with an algebraic gradient on a contractible domain in $\mathbf{R}^n$ if and only if there exists an involutive zero-dimensional radical ideal containing the Hamiltonian $H \in K[p]$.

Proof: Since a maximal ideal is a zero-dimensional radical ideal, the necessity is clear. We consider the sufficiency. A zero-dimensional radical ideal I can always be decomposed as a finite intersection of maximal ideals $M_1,\dots,M_s$[10]. Let I be an involutive ideal. Then all maximal ideals $M_i (i = 1,\dots,s)$ are also involutive [7]. If H is contained in I, then we have $H \in I \subset M_i (i = 1,\dots,s)$. Therefore, all maximal ideals $M_i (i = 1,\dots,s)$ are involutive ideals containing the Hamiltonian H. From Proposition 4.2, the stationary HJE (5) has a solution with an algebraic gradient.

A maximal ideal is a zero-dimensional radical ideal but the converse is not true in general. Therefore, it is easier finding an involutive zero-dimensional radical ideal containing $H \in K[p]$ than the maximal ideal in Proposition 4.2.

Remark 4.1: In analytical dynamics, functions F_i and F_j are said to be involutive if $\{F_i,F_j\} = 0$ holds identically over $\mathbf{R}^n \times \mathbf{R}^n$. It is readily shown that if every pair of functions F_i and $F_j (i,j = 1,\dots,n)$ are involutive, then from Proposition 4.1, the ideal $I := \langle F_1,\dots,F_n\rangle$ is also involutive. However, the converse is not true in general, and the involutiveness of an ideal is a weaker condition than the involutiveness of functions. Therefore, even if there exists a solution with an algebraic gradient, Liouville's theorem [11] is not necessarily applicable to guarantee the existence of a complete solution with arbitrary constants. Similarly, the existence of a solution with an algebraic gradient does not necessarily imply algebraic complete integrability [12], an algebraic expression of Liouville's theorem.

Let $I \subset K[p]$ be an involutive zero-dimensional radical ideal containing $H \in K[p]$. From Proposition 3.1, ideal I can be generated by n polynomials $F_1,\dots,F_n$, and the set of algebraic equations $F(x,p) := [F_1,\dots,F_n]^{\mathrm{T}}(x,p) = 0$ implicitly defines a function $p_0(x)$ on a suitable domain in $\mathbf{R}^n$. The involutiveness of I implies that $p_0(x)$ is the gradient of a scalar-valued function V, i.e., $p_0(x) = \nabla V$. Moreover, $H \in I$ implies $H(x,p_0(x)) = 0$. Thus, if there exists an involutive zero-dimensional radical ideal $I \subset K[p]$ containing $H \in K[p]$, then an algebraic gradient is obtained by solving $F(x,p) = 0$ with respect to p.

B. Algebraic Gradients of Nonstationary HJEs

Let $I \subset K_t[p_t] := K_t[p_0, p_1,\dots, p_n]$, where K_t is the field of meromorphic functions with variables (t, x). Note that $K_t[p_t]$ and $K[p] := K[p_1,\dots,p_n]$ are distinct, where K is the field of meromorphic functions with variable x. Here, we generalize Theorem 4.1 to the HJE (1).

That is, for the HJE (1), we present a necessary and sufficient condition for the existence of an algebraic gradient. First, we define the involutive ideal in $K_t[p_t]$. The Poisson bracket in $K_t[p_t]$ for two functions $\Phi \in K_t[p_t]$ and $\Psi \in K_t[p_t]$ is defined as

$$\{\Phi,\Psi\}_t := \sum_{i=0}^{n}\left(\frac{\partial\Phi}{\partial x_i}\frac{\partial\Psi}{\partial p_i} - \frac{\partial\Phi}{\partial p_i}\frac{\partial\Psi}{\partial x_i}\right), \tag{18}$$

where $x_0 := t$. The Poisson bracket in $K_t[p_t]$ can also be viewed as a mapping $\{\cdot,\cdot\} : K_t[p_t] \times K_t[p_t] \to K_t[p_t]$ because $K_t[p_t]$ is also closed under partial differentiation. We also define the involutiveness in $K_t[p_t]$ of an ideal with respect to the Poisson bracket in $K_t[p_t]$ as follows.

Definition 4.2: An ideal I in $K_t[p_t]$ is involutive in $K_t[p_t]$ if

$$\{\Phi,\Psi\}_t \in I \tag{19}$$

holds for all $\Phi \in I$ and for all $\Psi \in I$.

We next give the main result of this paper.

Theorem 4.2: The HJE (1) has a solution with an algebraic gradient on a contractible domain in $\mathbf{R} \times \mathbf{R}^n$ if and only if there exists an involutive zero-dimensional radical ideal in $K_t[p_t]$ containing $p_0 + H \in K_t[p_t]$.

Proof: First, we show that the HJE (1) has a solution V with an algebraic gradient on a contractible domain in $\mathbf{R} \times \mathbf{R}^n$ if and only if $p_i := \partial V/\partial x_i (i = 0,1\dots,n)$ satisfying the HJE (1) are algebraic functions. The sufficiency is obvious, and thus we consider the necessity. Since p is an algebraic gradient, $p_i : = \partial V/\partial x_i$ ($i = 1...,n$) satisfy the HJE (1) and are algebraic functions. From (1) and $H \in K_t[p_t]$, $p_0 = \partial V/\partial t$ is also an algebraic function. Next, in a similar manner to the proofs of Proposition 4.2 and Theorem 4.1, it is possible to show that p_i ($i = 0,1...,n$) satisfying the HJE (1) are algebraic functions if and only if there exists an involutive zero-dimensional radical ideal in $K_t[p_t]$ containing $p_0 + H \in K_t[p_t]$.

An involutive zero-dimensional radical ideal I in $K_t[p_t]$ containing $p_0 + H \in K_t[p_t]$ can be generated by $n + 1$ polynomials $F_0, F_1,...,F_n$. By solving $[F_0,...,F_n]^{\mathrm{T}}(t, x, p_t) = 0$ with respect to p_t, we have $p_i := \partial V/\partial x_i$ ($i = 0,1,...,n$), i.e., $p_0 = \partial V/\partial t$ and an algebraic gradient $p := \partial V/\partial x$. However, the value of $p_0 = \partial V/\partial t$ is not required to determine the optimal control input. In fact, we can construct a set of algebraic equations only defining an algebraic gradient, as shown below.

According to Theorem 4.2, to obtain an algebraic gradient, we need to find an involutivezerodimensional radical ideal $I = \langle F_0,...,F_n\rangle \subset K_t[p_t]$ containing $p_0 + H \in K_t[p_t]$. If we choose F_0 as $p_0 + H$, the condition $p_0 + H \in I$ holds. Let us consider the intersection $I \cap K_t[p]$, which is an ideal called an elimination ideal. Note that since an algebraic gradient of the HJE depends on time in addition to the state variable, the field considered here is the field of meromorphic functions with variables t and x. When $F_0 = p_0 + H$, we have the following for the elimination ideal $I \cap K_t[p]$.

Lemma 4.1: [9] A nontrivial ideal $I \subset K_t[p_t]$ is a zero-dimensional ideal if and only if $I \cap K_t[p_i]$ ($i= 0,1,...,n$) is a nontrivial ideal.

Theorem 4.3: Let $I = \langle p_0 + H, F_1,...,F_n\rangle \subset K_t[p_t]$ be a zero-dimensional radical ideal, where the Hamiltonian H does not depend on p_0. Then, there exist $G_i \in K_t[p]$ ($i = 1,...,n$) such that $I = \langle p_0 + H(p), G_1,...,G_n\rangle$ holds. Moreover, the elimination ideal $I \cap K_t[p]$ is the zero-dimensional radical ideal generated by $\{G_1,...,G_n\}$. Conversely, if $J = \langle G_1,...,G_n\rangle \subset K_t[p]$ is a zero-dimensional radical ideal, $J_t := \langle p_0 + H, G_1,...,G_n\rangle \subset K_t[p_t]$ is also a zerodimensional radical ideal for any $H \in K_t[p]$.

Proof: By dividing $F_i(p_0,p)$ ($i= 1,...,n$) by $p_0 + H(p)$ we have

$$F_i(p_0, p) = a_i(p_0, p)(p_0 + H(p)) + G_i(p), \quad G_i(p) := F_i(-H(p), p) \tag{20}$$

for suitable $a_i \in K_t[p_t]$ ($i = 1,...,n$). Thus, we have $I = \langle p_0 + H, G_1,...,G_n\rangle$. We show that the elimination ideal $I \cap K_t[p]$ is the zero-dimensional radical ideal generated by $\{G_1,...,G_n\}$. For $I \cap K_t[p]$, we have $(I \cap K_t[p]) \cap K_t[p_i] = I \cap K_t[p_i]$ ($i = 1,...,n$). Since I is a zero-dimensional ideal, from Lemma 4.1, $I \cap K_t[p_i]$ ($i = 0,1,...,n$) is a nontrivial ideal. Therefore, $I \cap K_t[p]$ is also a zero-dimensional ideal. For a zero-dimensional ideal

holds [8]. Since I is a radical ideal, we have

and consequently

Therefore,

holds.

That is, $I \cap K_t[p]$ is a radical ideal. It remains to show that $I \cap K_t[p]$ is generated by $\{G_1,...,G_n\}$. Let $\{g_1,...,g_s\}$ be a Grobner basis of I with respect to the lexicographic order $p_0 > p_1 > ... > p_n$ such that $g_r,...,g_s$ ($r \leq s$) do not depend on p_0. Since $p_0 + H$ is a generator of I, it is possible to show that $\{g_r,...,g_s\}$ and $\{G_1,...,G_n\}$ generate the same ideal in $K_t[p]$. Moreover, $\{g_r,...,g_s\}$ is a set of generators of $I \cap K_t[p]$ [9].

Finally, we show that J_t is a zero-dimensional radical ideal if J is a zero-dimensional radical ideal. Since J is a zero-dimensional ideal, $\mathbf{V}(J) \subset \bar{K}_t^n$ is a finite set. From the definition of J_t, we have $\mathbf{V}(J_t) = \{(p_0,p) \in \bar{K}_t \times \bar{K}_t^n : p_0 = -H(p), p \in \mathbf{V}(J)\}$, which is also a finite set. Thus, J_t is a zero-dimensional ideal. Consider $f \in K_t[p_t]$ such that $f^m \in J_t$. By dividing $f(p_0, p)$ by

$p_0 + H(p)$ we have

$$f(p_0,p) = a(p_0,p)(p_0 + H) + g(p), \quad g(p) := f(-H(p),p) \tag{21}$$

for a suitable $a \in K_t[p_t]$. From $p_0 + H \in J_t$ and $f^m = (a(p_0 + H) + g(p))^m \in J_t$, we obtain $g^m \in J_t$, where g does not depend on p_0. Thus, we have $g^m \in J \subset K_t[p]$. Since J is a radical ideal, we have $g \in J$. Therefore, $f = a(p_0 + H) + g$ belongs to J_t, i.e., J_t is a radical ideal.

Let $I =$ h$p_0 + H(p), F_1, \ldots, F_n$i $\subset K_t[p_t]$ be a zero-dimensional radical ideal in $K_t[p_t]$. Suppose that F_i $(i = 1,\ldots,n)$ does not depend on p_0 without loss of generality. From Theorem 4.3, $I \cap K_t[p]$ is a zero-dimensional radical ideal in $K_t[p]$ generated by $\{F_1,\ldots,F_n\}$. Finally,

we consider the involutiveness condition in $K_t[p_t]$, which yields, from Proposition 4.1,

$$\{F_i, F_j\}_t \in I \ (i, j = 1,\ldots,n), \tag{22}$$

$$\{F_k, p_0 + H\}_t \in I \ (k = 1,\ldots,n). \tag{23}$$

Condition (22) leads to

$$\{F_i, F_j\} = \{F_i, F_j\} \in I \ (i, j = 1, \ldots, n) \tag{24}$$

which is simply the involutiveness condition (17). Condition (23) leads to

$$= \frac{\partial F_k}{\partial x_0} + \{F_k, H\} \in I \ (k = 1, \ldots, n) \tag{25}$$

This condition is simply the following H-invariance condition.

Definition 4.3: For a given $H \in K_t[p]$, an ideal I in $K_t[p]$ is H-invariant if

$$\frac{\partial \Phi}{\partial t} + \{\Phi, H\} \in I \tag{26}$$

holds for all $\Phi \in I$.

Proposition 4.3: For a given $H \in K_t[p]$, an ideal $I :=$ h$F_1,\ldots,F_n$i $\subset K_t[p]$ is H-invariant if and only if

$$\tag{27}$$

holds.

Remark 4.2: In analytical dynamics, a function F_i is said to be a first integral if $\partial F_i/\partial t + \{F_i, H\} = 0$ holds identically over $\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n$ [11]. From Proposition 4.3, it is readily shown that if F_i $(i = 1,\ldots,s)$ are first integrals, then the ideal $I =$ h$F_1,\ldots,F_s$i is H-invariant. However, the converse is not true in general. Thus, the H-invariance of an ideal is a weaker condition than the first integrability of functions.

In summary, $I =$ h$p_0 + H, F_1 \ldots, F_n$i $\subset K_t[p_t]$ is an involutive zero-dimensional radical ideal in $K_t[p_t]$ containing $p_0 + H$ if and only if $I \cap K_t[p]$ is an H-invariant and involutive zero-dimensional radical ideal. In fact, the following theorem is presented in [7].

Theorem 4.4: The HJE (1) has a solution with an algebraic gradient on a contractible domain in $\mathbf{R} \times \mathbf{R}^n$ if and only if there exists an H-invariant and involutive zero-dimensional radical ideal in $K_t[p]$.

Theorem 4.4 indicates that generators of an involutive zero-dimensional radical ideal in $K_t[p_t]$ containing $p_0 + H$ can always be chosen as $\{p_0 + H, F_1,\ldots, F_n\}$ without loss of generality. Moreover, if $F(t, x, p_t) := [F_1,\ldots, F_n]^T$ does not depend on p_0, $I =$ h$F_1,\ldots,F_n$i $\subset K_t[p]$ is an H-invariant and involutive zero-dimensional radical ideal, and a solution to $F(t, x, p) = 0$ with respect to p is an algebraic gradient.

In general, it is not always easy to solve a set of algebraic equations over K_t in contrast to over $\mathbf{R}$. At each time and state $(\hat{t}, \hat{x}) \in \mathbf{R} \times \mathbf{R}^n$, $F(\hat{t}, \hat{x}, \hat{p}) = 0$ is a set of algebraic equations over $\mathbf{R}$. There are various techniques for solving algebraic equations over $\mathbf{R}$ such as the Newton method, and various numerical solution techniques have been reported that are based on a Gröbner basis [9]. By exploiting these techniques, the value of the algebraic gradient can be determined pointwise.

C. Stabilizing Solution

When an algebraic gradient exists, it is not necessarily unique. In systems and control theory, a stabilizing solution to the HJE is important. Here, we characterize the gradient of the stabilizing solution.

Definition 4.4: A solution $V(t, x)$ to the HJE is called a stabilizing solution if it is defined on an open set containing the origin, if $V(t, 0) = 0$ and $\nabla V(t,0) = 0$ hold for all $t \geq 0$, and if the origin is a uniformly asymptotically stable equilibrium of $dx/dt = (\partial H/\partial p)^{\mathrm{T}}(t,x, \nabla V)$.

A stabilizing solution to the HJE is characterized by the stabilizing solution to the Riccati differential equation (RDE):

(28)

where

(29)

$$C(t) := \frac{\partial^2 H}{\partial x^2}(t,0,0), \quad X(t) := \frac{\partial^2 V}{\partial x^2}(t,0) \tag{30}$$

A solution $X(t)$ is called the stabilizing solution if the origin is the uniformly asymptotically stable equilibrium of $dx/dt = (A(t) + B(t)X(t))x$.

Proposition 4.4: [13] Let V be a solution to the HJE with $V(t, 0) = 0$ and $\nabla V(t,0) = 0$ for all $t \geq 0$. If $X(t) := \partial^2 V(t,0)/\partial x^2$ is a stabilizing solution to the RDE, V is a stabilizing solution.

The RDE can be viewed as a particular form of the Lyapunov equation. From the Lyapunov stability theorem [14], a stabilizing solution to the RDE $X(t)$ is positive semidefinite, and consequently $X(t_0) \geq 0$. Thus, by checking whether or not $\partial p(t_0, 0)/\partial x \geq 0$, we can narrow down the range of candidate stabilizing solutions.

V. EXAMPLE

It is generally difficult to solve the HJE for a given Hamiltonian because the HJE is a nonlinear partial differential equation. This difficulty also applies to the involutiveness in $K_t[p_t]$ even when the class of solutions to the HJE is restricted to solutions with algebraic gradients. Therefore, it is still difficult to find an involutive ideal in $K_t[p_t]$ for a given Hamiltonian. Considering the difficulty in solving the HJE, it is worth finding solvable examples of the HJE in a particular form. Solvable examples can be constructed by assuming the structure of elements in a basis of an ideal. From Proposition 3.1, the zero-dimensionality specifies the number of elements. The involutiveness condition in $K_t[p_t]$ reduces to conditions determining unknown coefficients of elements. These conditions are helpful for finding solvable examples as shown in this section. Then, it is possible to check if a given HJE belongs to these solvable examples.

Let us consider $H \in K_t[p_t]$ given by Example 2.1. We consider an ideal generated by the following polynomials:

$$F0 = p0 + H(t,x,p)$$

$$F1 = p21 + b11(x1)p1 + b12(x1),$$

$$F_i = p_i + b_i(x_i,t) \ (i = 2,3,\dots,n). \tag{31}$$

On the basis of the results in [7], we have the following proposition.

Proposition 5.1: Assume that
If for some b_{11}, b_{12} and b_i ($i = 2,\dots,n$),

$$f_1 + \sum_{j=1}^{m} g_{1j}\left(\sum_{k=2}^{n} b_k g_{kj}\right) + \frac{b_{11}}{2}\sum_{j=1}^{m} g_{1j}^2 = 0, \tag{32}$$

$$2\sum_{k=2}^{n}\left(b_k f_k + \int_0^{x_k}\frac{\partial b_k}{\partial t}dx_k\right) - \sum_{j=1}^{m} b_{12} g_{1j}^2 + \sum_{j=1}^{m}\left(\sum_{k=2}^{n} b_k g_{kj}\right)^2 - 2q = 0 \tag{33}$$

are satisfied, then $I = \mathrm{h}p_0 + H, F_1, \dots, F_n$iis an involutive zero-dimensional radical ideal in $K_t[p_t]$.

Since F_i ($i = 1,\dots,n$) does not depend on p_0, an algebraic gradient is obtained by solving the set of algebraic equations $[F_1,\dots,F_n]^{\mathrm{T}}(t,x,p) = 0$ with respect to p. For a suitable class of optimal control problems for n-dimensional systems, an algebraic gradient can be obtained by using Proposition 5.1. On the basis of Proposition 5.1, examples of an explicit solution are given.

Example 5.1: Let $n = 1$ and $m = 1$. Consider f and q in Example 2.2. By choosing $b_{11} = 0$ and $b_{12} = x^2/(1 + t + x^2)$, the conditions in Proposition 5.1 hold. Thus, the algebraic gradients in Example 2.2 are obtained by solving $p^2 - x^2/(1 + t + x^2) = 0$ with respect to p.

Example 5.2: Let $n = 2$ and $m = 1$. Suppose a state equation and cost function are given as follows:

$$\dot{x} = \begin{bmatrix} -\dfrac{x_1^3}{2} + \dfrac{x_2}{1+e^{-t}} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \tag{34}$$

$$J = \frac{1}{2}\int_{t_0}^{\infty} \left(x_1^2 + x_2^2 + u^2 \right) dt. \tag{35}$$

According to Proposition 5.1, an algebraic gradient p is obtained as a zero of the following polynomials:

(36)

Algebraic gradients are obtained by solving the above algebraic equations. On the basis of the discussion of the stabilizing solution in Section IV-C, an algebraic gradient p is chosen such that $\partial p / \partial x(0,0)$ is positive

In fact, it is possible to show that the closed-loop system using the feedback law

(38)

is uniformly asymptotically stable at the origin. Finally, the value function is expressed as

(39)

The line integral yields an explicit value function:

(40)

Note that the value function is not an algebraic function even though the gradients are algebraic functions.

VI. CONCLUSION

In this paper, a polynomial-type HJE for a Hamiltonian with coefficients consisting of meromorphic functions of the time and state was considered. A necessary and sufficient condition for the existence of a solution with an algebraic gradient was characterized in terms of an involutive zero-dimensional radical ideal. This existence condition is obtained in a similar form to that for the stationary HJE. In previous research, the existence of an algebraic gradient of the HJE has been studied. In this paper, we studied the relation between our results and the results of previous research. If an involutive zero-dimensional radical ideal is found, an algebraic gradient is obtained simply by solving a set of algebraic equations. Finally, a class of nonlinear optimal regulator problems has been given such that gradients of explicit solutions are obtained as algebraic functions, and examples of explicit solutions were also presented.

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