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ONE-DIMENSIONAL FLOW OF A COMPRESSIBLE VISCOUS AND HEAT-CONDUCTING MICROPOLAR FLUID WITH HOMOGENEOUS BOUNDARY CONDITIONS: A BRIEF SURVEY OF THE THEORY AND RECENT PROGRESS

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Abstract. We analyze the nonstationary 1-D flow of a compressible viscous and heat-conducting micropolar fluid which is in thermodynamical sense perfect and polytropic. The corresponding problem has homogeneous boundary conditions for velocity, microrotation and heat flux, as well as sufficiently smooth initial data. We also assume that the initial density and temperature are strictly positive.

In this work we give a brief survey and recent progress in mathematical analysis of the described problem, including the existence results, uniqueness, regularity and the large time behavior of the solution.

1. Introduction

The micropolar continuum allows modeling of physical processes in which phenomenas at the micro level can not be neglected, such as in modeling of smog, liquid crystals, or biological fluids. A model of micropolar continuum was developed in the 1960s by A. C. Eringen in a way that he added a microrotational vector field to the hydrodynamical state variables. Besides the microphenomena, Eringen's model describes the coupling of micro and macrophenomena. It is important to point out that Eringen's model neglects the microdeformations, assuming that the influence of the microphenomena can be described only with microrotations, while including microdeformations would only make the problem more complicated. Applicability of the Eringen's model was practically proved at the end of the last century, when Papautsky et al. showed that use of the concept of micropolarity improves predictions of experimental results concerning fluid flow through micropipes for 47% compared to the classical Navier-Stokes model (see [19]). This result, together with the proof of usability, opened a whole new range of applications of micropolar fluids in one of the most researched areas today - nanotechnology. Micropolar continuum is also applied in the modeling of solid bodies, but this area is out of the scope of this article. For general theory and applications of micropolar continuum we refer to [6].

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Regarding mathematical analysis of the micropolar fluid model, in majority of articles the model has been considered in incompressible case, while the compressible case (important in the analysis of gaseous fluids) was largely unexplored, particularly in cases that include temperature, i.e. thermodynamic component.

In this article we describe the compressible flow of an isotropic, viscous and heat conducting micropolar fluid, which is in the thermodynamical sense perfect and polytropic. The model for this kind of flow in the one-dimensional case was first described by Mujaković in [9]. In her later works she analyzed the one-dimensional model in relation to existence, regularity and stabilization for different kinds of problems with homogeneous and non-homogeneous boundary conditions. The first generalization to the three-dimensional case was done by Dražić and Mujaković in [5], where they analyzed the spherically symmetric model with homogeneous boundary conditions. Recently, the model with cylindrical symmetry has been considered (see [3],[17]).

In this work we analyze the one-dimensional case of the described fluid with homogeneous boundary conditions for velocity, microrotation and heat flux, which means that we consider the flow between solid and thermo insulated walls.

The paper is organized as follows. In the next section we describe the mathematical model of the described fluid and derive its form in the Lagrangian description. In the third section we give an overview of the progress in the mathematical analysis of this problem. We introduce the generalized solution to the problem together with the existence and uniqueness theorems. We also mention results concerning the regularity and the large time behavior of the solution. Finally, in the fourth section, we present another approach to global existence result, derived using the finite difference method, as well as the progress in numerical analysis of this problem.

2. The mathematical model

The mathematical model of the described fluid is stated for example in the book of G. Lukaszewicz [8] and reads:

$$\dot{\rho} = -\rho \nabla \cdot \mathbf{v}, \tag{2.1}$$

$$\rho \dot{\mathbf{v}} = \nabla \cdot \mathbf{T} + \rho \mathbf{f}, \tag{2.2}$$

$$\rho j_I \dot{\boldsymbol{\omega}} = \nabla \cdot \mathbf{C} + \mathbf{T}_x + \rho \mathbf{g}, \tag{2.3}$$

$$\rho \dot{E} = -\nabla \cdot \mathbf{q} + \mathbf{T} : \nabla \mathbf{v} + \mathbf{C} : \nabla \boldsymbol{\omega} - \mathbf{T}_x \cdot \boldsymbol{\omega}, \tag{2.4}$$

$$\mathbf{T}_{ij} = (-p + \lambda \mathbf{v}_{k,k})\delta_{ij} + \mu \left(\mathbf{v}_{i,j} + \mathbf{v}_{j,i}\right) + \mu_r \left(\mathbf{v}_{j,i} - \mathbf{v}_{i,j}\right) - 2\mu_r \varepsilon_{mij} \boldsymbol{\omega}_m, \tag{2.5}$$

$$\mathbf{C}_{ij} = c_0 \boldsymbol{\omega}_{k,k} \delta_{ij} + c_d \left(\boldsymbol{\omega}_{i,j} + \boldsymbol{\omega}_{j,i}\right) + c_a \left(\boldsymbol{\omega}_{j,i} - \boldsymbol{\omega}_{i,j}\right), \tag{2.6}$$

$$\mathbf{q} = -k \nabla \theta, \tag{2.7}$$

$$p = R \rho \theta, \tag{2.8}$$

$$E = c_v \theta. \tag{2.9}$$

Here ρ, $\mathbf{v} = (v_1, v_2, v_3)$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, E, and θ are, respectively, mass density, velocity field, microrotation velocity field, internal energy density and absolute temperature. $\mathbf{T}$ is the stress tensor, $\mathbf{C}$ is the couple stress tensor, $\mathbf{q}$ is the heat flux density vector, $\mathbf{f}$ is the body force density and $\mathbf{g}$ is the body couple density. p denotes pressure and the positive constant j_I is microinertia density. λ and μ

are coefficients of viscosity and μ_r, c_0, c_d and c_a are coefficients of microviscosity. By the constant k ($k \geq 0$) we denote the heat conduction coefficient. The positive constant R is the specific gas constant and the positive constant c_v denotes the specific heat at a constant volume.

Equations (2.1)-(2.4) are, respectively, local forms of the conservation laws for the mass, momentum, momentum moment and energy. Equations (2.5)-(2.6) are constitutive equations for the micropolar continuum. Equation (2.7) is the Fourier law and equations (2.8)-(2.9) present the assumptions that the fluid is perfect and polytropic. The coefficients of viscosity and the coefficients of microviscosity are related through the Clausius-Duhamel inequalities, as follows:

$$\mu \geq 0, \qquad 3\lambda + 2\mu \geq 0, \qquad \mu_r \geq 0. \tag{2.10}$$

$$c_d \geq 0, \quad 3c_0 + 2c_d \geq 0, \quad |c_d - c_a| \leq c_d + c_a. \tag{2.11}$$

Vector $\mathbf{T}_x$ in the equations (2.3) and (2.4) is an axial vector with the Cartesian components

$$(\mathbf{T}_x)_i = \varepsilon_{ijk}\mathbf{T}_{jk}, \tag{2.12}$$

where ε_{ijk} is the Levi-Civita symbol, δ_{ij} is Kronecker delta symbol and we assume the Einstein notation for summation. The colon operator in equation (2.4) is the scalar product of tensors, for example

$$\mathbf{T} : \nabla\mathbf{v} = \mathbf{T}_{ji}\mathbf{v}_{i,j}. \tag{2.13}$$

The differential (dot) operator in equations (2.1)-(2.4) denotes material derivative defined by

$$\dot{\mathbf{a}} = \mathbf{a}_t + (\nabla\mathbf{a}) \cdot \mathbf{v}. \tag{2.14}$$

For simplicity reasons, we also assume that

$$\mathbf{f} = \mathbf{g} = 0. \tag{2.15}$$

By substituting (2.5)-(2.11) into the system (2.1)-(2.4), together with (2.15), we obtain:

$$\dot{\rho} = -\rho\nabla \cdot \mathbf{v}, \tag{2.16}$$

$$\rho\dot{\mathbf{v}} = -R\nabla(\rho\theta) + (\lambda + \mu - \mu_r)\nabla(\nabla \cdot \mathbf{v}) + (\mu + \mu_r)\Delta\mathbf{v} + 2\mu_r\nabla \times \boldsymbol{\omega}, \tag{2.17}$$

$$j_I\rho\dot{\boldsymbol{\omega}} = 2\mu_r(\nabla \times \mathbf{v} - 2\boldsymbol{\omega}) + (c_0 + c_d - c_a)\nabla(\nabla \cdot \boldsymbol{\omega}) + (c_d + c_a)\Delta\boldsymbol{\omega}, \tag{2.18}$$

$$\begin{aligned} c_v\rho\dot{\theta} = k_\theta\Delta\theta - R\rho\theta(\nabla \cdot \mathbf{v}) + \lambda(\nabla \cdot \mathbf{v})^2 + \\ \frac{\mu}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T) : (\nabla\mathbf{v} + (\nabla\mathbf{v})^T) + 4\mu_r\left(\frac{1}{2}\nabla \times \mathbf{v} - \boldsymbol{\omega}\right)^2 \\ + c_0(\nabla \cdot \boldsymbol{\omega})^2 + (c_d + c_a)\nabla\boldsymbol{\omega} : \nabla\boldsymbol{\omega} + (c_d - c_a)\nabla\boldsymbol{\omega} : (\nabla\boldsymbol{\omega})^T. \end{aligned} \tag{2.19}$$

Now, we consider the model (2.16)-(2.19) for one-dimensional flow. It is assumed (in the Cartesian coordinate frame) that $v_2 = v_3 = \omega_2 = \omega_3 = 0$ and that the functions ρ, $v = v_1$, $\omega = \omega_1$ and θ depend on $x = x_1$ and t only. We obtain the model in the Eulerian description:

$$\dot{\rho} + \rho v_x = 0, \tag{2.20}$$

$$\rho\dot{v} = -(R\rho\theta)_x + \sigma_1 v_{xx}, \tag{2.21}$$

$$j_I\rho\dot{\omega} = \sigma_2\omega_{xx} - 4\mu_r\omega, \tag{2.22}$$

$$c_v\rho\dot{\theta} = -R\rho\theta v_x + \sigma_1(v_x)^2 + \sigma_2(\omega_x)^2 + 4\mu_r\omega^2 + k\omega_{xx}, \tag{2.23}$$

where

$$\sigma_1 = \lambda + 2\mu, \quad \sigma_2 = c_0 + 2c_d, \tag{2.24}$$

$$\sigma_1, \sigma_2, k, \mu_r > 0. \tag{2.25}$$

We will consider the system (2.20)-(2.23) in the domain $]0, L[\times\mathbf{R}^+$, under the homogeneous boundary conditions:

$$v(0,t) = v(L,t) = 0, \quad \omega(0,t) = \omega(L,t) = 0, \quad \theta_x(0,t) = \theta_x(L,t) = 0, \tag{2.26}$$

for $t > 0$, and non-homogeneous initial conditions:

$$\rho(x,0) = \rho_0(x), \quad v(x,0) = v_0(x), \quad \omega(x,0) = \omega_0(x), \quad \theta(x,0) = \theta_0(x), \tag{2.27}$$

for $x \in]0, L[$, where ρ_0, v_0, ω_0 and θ_0 are known real functions defined on $]0, L[$.

In the mathematical analysis of compressible fluids, it is convenient to use Lagrangian description. The Eulerian coordinates (x, t) are connected to the Lagrangian coordinates (ξ, t) by the relation

$$x(\xi, t) = \varphi(\xi, t), \tag{2.28}$$

where, for $\xi \in]0, L[$, $t \mapsto \varphi(\xi, t)$ is a solution of the Cauchy problem

$$\frac{d\varphi(\xi,t)}{dt} = v(\varphi(\xi,t),t), \quad \varphi(\xi,0) = \xi. \tag{2.29}$$

We introduce the new coordinates

$$x' = \eta^{-1}\psi(\xi), \quad t' = \zeta^{-1}t, \tag{2.30}$$

where

$$\psi(\xi) = \int_0^\xi \rho_0(s)ds, \quad \eta = \psi(L), \quad \zeta = \eta\sigma_1^{-1}(2\chi)^{-\frac{1}{2}}\sigma_2^{\frac{1}{2}}. \tag{2.31}$$

Using the coordinates (2.30), the spatial domain becomes $]0, 1[$, and we get the following initial-boundary problem (that we write omitting the primes for simplicity reasons):

$$\rho_t + \rho^2 v_x = 0, \tag{2.32}$$

$$v_t = (\rho v_x)_x - K(\rho\theta)_x, \tag{2.33}$$

$$\rho\omega_t = A\left(\rho(\rho\omega_x)_x - \omega\right), \tag{2.34}$$

$$\rho\theta_t = -K\rho^2\theta v_x + \rho^2(v_x)^2 + \rho^2(\omega_x)^2 + \omega^2 + D\rho(\rho\theta_x)_x, \tag{2.35}$$

$$v(0,t) = v(1,t) = 0, \quad \omega(0,t) = \omega(1,t) = 0, \quad \theta_x(0,t) = \theta_x(1,t) = 0, \tag{2.36}$$

$$\rho(x,0) = \rho_0(x), \quad v(x,0) = v_0(x), \quad \omega(x,0) = \omega_0(x), \quad \theta(x,0) = \theta_0(x), \tag{2.37}$$

considered in the domain $]0, 1[\times\mathbf{R}^+$, where

$$K = Rc_v^{-1}, \quad A = j_I^{-1}\sigma_1^{-1}\sigma_2, \quad D = kc_v^{-1}\sigma_1^{-1}. \tag{2.38}$$

These positive constants were introduced in order to reduce the number of constants in the system.

We assume that the functions ρ_0 and θ_0 are strictly positive and bounded,

$$m \le \rho_0(x), \theta_0(x) \le M, \quad x \in]0, 1[, \tag{2.39}$$

where $m, M \in \mathbf{R}^+$.

Hereafter we will consider the problem (2.32)-(2.37), which is equivalent to the problem (2.20)-(2.23), (2.26), (2.27).

3. Properties of the solution

In this section we consider the properties of the so-called generalized solution to the problem (2.32)-(2.37).

Definition 3.1. Given any $T \in \mathbf{R}^+$, a generalized solution to the problem (2.32)-(2.37) in the domain $Q_T =]0,1[\times]0,T[$ is a function

$$(x,t) \mapsto (\rho, v, \omega, \theta)(x,t), \quad (x,t) \in Q_T, \tag{3.1}$$

where

$$\rho \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(]0,1[)) \cap \mathrm{H}^1(Q_T)\ ,\ \inf_{Q_T} \rho > 0\ , \tag{3.2}$$

$$v, \omega, \theta \in \mathrm{L}^\infty(0,T;\mathrm{H}^1(]0,1[)) \cap \mathrm{H}^1(Q_T) \cap \mathrm{L}^2(0,T;\mathrm{H}^2(]0,1[)), \tag{3.3}$$

that satisfies the equations (2.32)-(2.35) a.e. in Q_T and conditions (2.36)-(2.37) in the sense of traces.

Let us mention that by using the embedding and interpolation theorems one can conclude that our generalized solution could be treated as a strong solution. In fact, we have

$$\rho \in \mathrm{L}^\infty(0,T;\mathrm{C}([0,1])) \cap \mathrm{C}([0,T],\mathrm{L}^2(]0,1[))\ , \tag{3.4}$$

$$v, \omega, \theta \in \mathrm{L}^2(0,T;\mathrm{C}^1([0,1])) \cap \mathrm{C}([0,T],\mathrm{H}^1(]0,1[)), \tag{3.5}$$

$$v, \omega, \theta \in \mathrm{C}(\overline{Q}_T). \tag{3.6}$$

In her work on one-dimensional flow of a compressible viscous micropolar fluid, Mujaković first analyzed the existence of the generalized solution to the problem (2.32)-(2.37). Using the Faedo-Gelerikin method, she proved in [9] the existence locally in time and the uniqueness of the solution. Then, based on extension principle and series of a priori estimates, she proved in [10] the global existence theorem for the problem (2.32)-(2.37). These results are summarized in the following theorem.

Theorem 3.2. *Let the functions $\rho_0, \theta_0 \in \mathrm{H}^1(]0,1[)$ satisfy the conditions (2.39) and let $v_0, \omega_0 \in \mathrm{H}^1_0(]0,1[)$. Then for any $T \in \mathbf{R}^+$ there exists unique generalized solution to the problem (2.32)-(2.37) in the domain Q_T having the property*

$$\theta > 0 \ \ \text{in}\ \overline{Q}_T. \tag{3.7}$$

In the second stage of her research, Mujaković proved a regularity theorem for the problem (2.32)-(2.37). More specifically, assuming that the initial data are Hölder continuous on $]0,1[$, she proved that, for any $T > 0$, the mass density, velocity, microrotation velocity and temperature are Hölder continuous on Q_T. In the proof she followed the method of A. V. Kazhykhov, applied to the case of a classical fluid (see [1]), basing the proof on some inequalities for Hölder norms of the solution. The regularity result is given in the following theorem:

Theorem 3.3. *Let the functions*

$$\rho_0 \in \mathrm{C}^{1+\alpha}([0,1])\,, \quad v_0, \omega_0, \theta_0 \in \mathrm{C}^{2+\alpha}([0,1])\,, \quad 0 < \alpha < 1 \tag{3.8}$$

satisfy the compatibility conditions

$$v_0(0) = v_0(1) = \omega_0(0) = \omega_0(1) = \theta_0'(0) = \theta_0'(1) = 0, \tag{3.9}$$

$$(\rho_0(v_0)')' - K(\rho_0\theta_0)' = 0, \text{ for } x = 0, 1, \tag{3.10}$$

$$(\rho_0(\omega_0)')' - \frac{\omega_0}{\rho_0} = 0, \text{ for } x = 0, 1. \tag{3.11}$$

Then the generalized solution to the problem (2.32)-(2.37) has the properties:

$$\rho \in \mathrm{C}^{1+\alpha}(Q_T), \quad v, \omega, \theta \in \mathrm{C}^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T), \quad 0 < \alpha < 1. \tag{3.12}$$

In the next stage of her research, Mujaković considered stabilization problem for (2.32)-(2.37). Theorem 3.2 ensures the existence of the solution on the arbitrary but finite time interval $]0, T[$, so the main difficulty was to prove a priori estimates for the solution independent of T. She obtained these estimates in [12], using the results from [10]. In her proof she followed some ideas of S. N. Antontsev, A. V. Kazhykhov and V. N. Monakhov, applied to 1-D initial-boundary value problem for a classical fluid (see [1]). The result, which gives the existence of the solution on the time interval $]0, \infty[$, is summerized in the next theorem.

Theorem 3.4. *Let the initial functions ρ_0, v_0, ω_0 and θ_0 satisfy the same conditions as in Theorem 3.2. Then the problem (2.32)-(2.37) has a solution*

$$(x, t) \mapsto (\rho, v, \omega, \theta)(x, t) \tag{3.13}$$

in the domain $Q =]0, 1[\times]0, \infty[$ with the properties:

$$\rho \in \mathrm{L}^\infty(0, \infty; \mathrm{H}^1(]0, 1[)), \tag{3.14}$$

$$\rho_t \in \mathrm{L}^\infty(0, \infty; \mathrm{L}^2(]0, 1[)) \cap \mathrm{L}^2(Q), \tag{3.15}$$

$$\rho_x \in \mathrm{L}^2(0, \infty; \mathrm{L}^2(]0, 1[)), \tag{3.16}$$

$$v, \omega \in \mathrm{L}^\infty(0, \infty; \mathrm{H}^1(]0, 1[)) \cap \mathrm{H}^1(Q) \cap \mathrm{L}^2(0, \infty; \mathrm{H}^2(]0, 1[)), \tag{3.17}$$

$$\theta \in \mathrm{L}^\infty(0, \infty; \mathrm{H}^1(]0, 1[)), \tag{3.18}$$

$$\theta_x \in \mathrm{L}^2(0, \infty; \mathrm{H}^1(]0, 1[)), \tag{3.19}$$

$$\theta_t \in \mathrm{L}^2(Q). \tag{3.20}$$

In the following theorem, which is proved in [13], Mujaković showed the stabilization of the solution when $t \to \infty$.

Theorem 3.5. *Let $(\rho, v, \omega, \theta)$ be a generalized solution to the problem (2.32)-(2.37) in the domain Q. Then we have the convergence*

$$(\rho, v, \omega, \theta) \to ((\rho^*)^{-1}, 0, 0, \theta^*) \tag{3.21}$$

in the space $\left(\mathrm{H}^1(]0, 1[)\right)^4$ when $t \to \infty$, where

$$\rho^* = \int_0^1 \frac{1}{\rho_0(x)} dx, \tag{3.22}$$

$$\theta^* = \frac{1}{c_v}\left(\frac{1}{2}\|v_0\|^2_{L^2(Q)} + \frac{1}{2A}\|\omega_0\|^2_{L^2(Q)} + c_v\|\theta_0\|_{L^1(Q)}\right). \tag{3.23}$$

The proof of Theorem 3.5 is based on the results of Theorem 3.4 and application of Friedrichs and Poincare inequalities.

Huang and Nie continued the work of Mujaković in [7] and studied the exponential stability of the solutions in H^i $(i = 1, 2)$. They use modified idea from [18] to prove the exponential stability of solutions in $(H^1(]0, 1[))^4$ for the 1-D micropolar fluid system (2.32)-(2.37) and then obtain the global existence and decay rate of solutions in $(H^2(]0, 1[))^4$. The results are stated in the following theorems.

Theorem 3.6. *Let the initial functions ρ_0, v_0, ω_0 and θ_0 satisfy the same conditions as in Theorem 3.2 and the compatibility conditions (3.9). Then there exist constants $C_1 > 0$ and $\gamma_1 = \gamma_1(C_1) > 0$ such that for any fixed $\gamma \in]0, \gamma_1]$, problem (2.32)-(2.37) admits a unique global solution*

$$(\rho(t), v(t), \omega(t), \theta(t)) \in H^1(]0, 1[) \times H_0^1(]0, 1[) \times H_0^1(]0, 1[) \times H^1(]0, 1[)$$

verifying the following estimate for any $t > 0$:

$$0 < C_1^{-1} \leq \rho(x, t) \leq C_1, \quad 0 < C_1^{-1} \leq \theta(x, t) \leq C_1 \quad \text{on } Q, \tag{3.24}$$

$$\begin{aligned} &e^{\gamma t} \left(\|\rho - \rho^*\|_{H^1}^2 + \|v\|_{H^1}^2 + \|\omega\|_{H^1}^2 + \|\theta - \theta^*\|_{H^1}^2 \right) + \int_0^t e^{\gamma s} \left(\|\rho - \rho^*\|_{H^1}^2 + \right. \\ &\left. \|v\|_{H^2}^2 + \|\theta - \theta^*\|_{H^2}^2 + \|\omega\|_{H^2}^2 + \|v_t\|_{L^2}^2 + \|\omega_t\|_{L^2}^2 + \|\theta_t\|_{L^2}^2 \right)(s) ds \leq C_1, \end{aligned} \tag{3.25}$$

where ρ^* and θ^* are defined by (3.22) and (3.23).

Theorem 3.7. *Let the initial functions satisfy (2.39), (3.9) and the condition*

$$\rho_0, \theta_0 \in H^2(]0, 1[), \quad \theta_0' \in H_0^1(]0, 1[), \quad v_0, \omega_0 \in H^2(]0, 1[) \cap H_0^1(]0, 1[). \tag{3.26}$$

Then there exist constants $C_2 > 0$ and $\gamma_2 = \gamma_2(C_2) > 0$ such that for any fixed $\gamma \in]0, \gamma_2]$, problem (2.32)-(2.37) admits a unique global solution $(\rho(t), v(t), \omega(t), \theta(t))$ from

$$H^2(]0, 1[) \times (H^2(]0, 1[) \cap H_0^1(]0, 1[) \times (H^2(]0, 1[) \cap H_0^1(]0, 1[)) \times H^2(]0, 1[)$$

verifying the following estimate for any $t > 0$:

$$\begin{aligned} &e^{\gamma t} \left(\|\rho - \rho^*\|_{H^2}^2 + \|v\|_{H^2}^2 + \|\omega\|_{H^2}^2 + \|\theta - \theta^*\|_{H^2}^2 \right) + \int_0^t e^{\gamma s} \left(\|\rho - \rho^*\|_{H^2}^2 + \right. \\ &\left. \|v\|_{H^3}^2 + |\theta - \theta^*\|_{H^3}^2 + \|\omega\|_{H^3}^2 + \|v_t\|_{H^1}^2 + \|\omega_t\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 \right)(s) ds \leq C_2, \end{aligned} \tag{3.27}$$

where ρ^* and θ^* are the same as in Theorem 3.6.

4. Finite difference approximation and numerical solution of considered problem

The global existence result for the solution to the described problem is also derived in [15] using the finite difference method. This proof is technically more demanding, but it has some advantages:

(1) The local existence theorem for the proof of the global existence is not needed.

(2) The Faedo-Galerkin method is limited to the problems with the smooth enough initial functions and to the problems with homogeneous boundary data, while the finite difference method could be extended to other classes of initial functions and to the problems with non-homogeneous boundary conditions.
(3) The convergent and easy applicable numerical scheme is obtained too.

Let us briefly describe the applied approach. In order to obtain an approximate system for the problem (2.32)-(2.37), a space discrete difference scheme is introduced. More precisely, semi-discrete finite difference approximate solutions on a uniform staggered grid are constructed.

The scheme is obtained in the following way. Let h be an increment in x such that $Nh = 1$ for $N \in \mathbf{Z}^+$. The staggered grid points are denoted with $x_k = kh$, $k \in \{0, 1, \ldots, N\}$ and $x_j = jh$, $j \in \left\{\frac{1}{2}, \ldots, N - \frac{1}{2}\right\}$. For each integer N, the following time dependent functions are constructed:

$$\rho_j(t), \theta_j(t), \quad j = \tfrac{1}{2}, \ldots, N - \tfrac{1}{2}, \tag{4.1}$$

$$v_k(t), \omega_k(t), \quad k = 0, 1, \ldots, N, \tag{4.2}$$

that form a discrete approximation to the solution at defined grid points

$$\rho(x_j, t), \theta(x_j, t), \quad j = \tfrac{1}{2}, \ldots, N - \tfrac{1}{2},$$
$$v(x_k, t), \omega(x_k, t), \quad k = 0, 1, \ldots, N.$$

First, the functions $\rho_j(t), v_k(t), \omega_k(t), \theta_j(t)$, $j = \frac{1}{2}, \ldots, N - \frac{1}{2}$, $k = 1, \ldots, N-1$, are determined by using appropriate spatial discretization of equation system (2.32)-(2.35):

$$\dot{\rho}_j = -\rho_j^2 \delta v_j, \tag{4.3}$$

$$\dot{v}_k = \delta(\rho\delta v)_k - K\delta(\rho\theta)_k, \tag{4.4}$$

$$\rho_k \dot{\omega}_k = A\left[\rho_k \delta(\rho\delta\omega)_k - \omega_k\right], \tag{4.5}$$

$$\rho_j \dot{\theta}_j = -K\rho_j^2 \theta_j \delta v_j + \rho_j^2 (\delta v_j)^2 + \rho_j^2 (\delta\omega_j)^2 + \omega_j^2 + D\rho_j \delta(\rho\delta\theta)_j, \tag{4.6}$$

where $j = \frac{1}{2}, \ldots, N - \frac{1}{2}$ and $k = 1, \ldots, N-1$. The operator δ is defined by

$$\delta g_l = \frac{g_{l+\frac{1}{2}} - g_{l-\frac{1}{2}}}{h}, \tag{4.7}$$

for $l = j$ or $l = k$. For $k \in \{1, \ldots, N\}$ and $j \in \{\frac{1}{2}, \ldots, N - \frac{1}{2}\}$, the functions ρ_k, θ_k and v_j, ω_j are defined by

$$\rho_k = \rho_{k-\frac{1}{2}}, \quad \theta_k = \theta_{k-\frac{1}{2}} \quad \text{and} \quad v_j = v_{j+\frac{1}{2}}, \quad \omega_j = \omega_{j+\frac{1}{2}}. \tag{4.8}$$

In accordance with the boundary conditions (2.36), we have

$$v_0(t) = v_N(t) = 0, \ \omega_0(t) = \omega_N(t) = 0, \ \delta\theta_0(t) = \delta\theta_N(t) = 0. \tag{4.9}$$

The initial conditions are defined in accordance with the given initial conditions (2.37) as:

$$(\rho_j, \theta_j)(0) = \left(\frac{1}{h}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \rho_0(x)dx, \frac{1}{h}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \theta_0(x)dx\right), \tag{4.10}$$

$j \in \{\frac{1}{2}, \ldots, N - \frac{1}{2}\}$,

$$(v_k, \omega_k)(0) = \left(\frac{1}{h} \int_{(k-1)h}^{kh} v_0(x)dx, \frac{1}{h} \int_{(k-1)h}^{kh} \omega_0(x)dx \right), \tag{4.11}$$

$k \in \{1, \ldots, N-1\}$ and

$$v_0(0) = v_N(0) = 0 \quad \omega_0(0) = \omega_N(0) = 0, \quad \delta\theta_0(0) = \delta\theta_N(0) = 0. \tag{4.12}$$

From the basic theory of differential equations, it is known that there exists a smooth solution of the Cauchy problem (4.3)-(4.6), (4.9) with the initial conditions (4.10)-(4.12) locally on some time interval $[0, T[$, $T > 0$. Moreover, it has been shown that its solution is globally defined on $[0, \infty[$, i.e., that $T_{max} = \infty$. This was achieved by showing, for fixed $h > 0$, the boundedness of the mass density, the velocity, the microrotation velocity and the temperature, as well as the lower boundedness of the density and the temperature away from zero.

Using the solution of the Cauchy problem (4.3)-(4.6), (4.9)-(4.12), for $t \geq 0$ the following approximate functions are constructed.

For each fixed N, $x \in]\frac{1}{N}[xN], \frac{1}{N}([xN]+1)]$, we have

$$v^N(x,t) = v_{[xN]}(t) + (xN - [xN])(v_{[xN]+1}(t) - v_{[xN]}(t)), \tag{4.13}$$

$$\omega^N(x,t) = \omega_{[xN]}(t) + (xN - [xN])(\omega_{[xN]+1}(t) - \omega_{[xN]}(t)) \tag{4.14}$$

and for $x \in]\frac{1}{N}([xN + \frac{1}{2}] - \frac{1}{2}), \frac{1}{N}([xN + \frac{1}{2}] + \frac{1}{2})]$, we have

$$\begin{aligned} &\rho^{N-\frac{1}{2}}(x,t) = \\ &\rho_{[xN+\frac{1}{2}]-\frac{1}{2}}(t) + (xN - ([xN + \tfrac{1}{2}] - \tfrac{1}{2}))(\rho_{[xN+\frac{1}{2}]+\frac{1}{2}}(t) - \rho_{[xN+\frac{1}{2}]-\frac{1}{2}}(t)), \end{aligned} \tag{4.15}$$

$$\begin{aligned} &\theta^{N-\frac{1}{2}}(x,t) = \\ &\theta_{[xN+\frac{1}{2}]-\frac{1}{2}}(t) + (xN - ([xN + \tfrac{1}{2}] - \tfrac{1}{2}))(\theta_{[xN+\frac{1}{2}]+\frac{1}{2}}(t) - \theta_{[xN+\frac{1}{2}]-\frac{1}{2}}(t)). \end{aligned} \tag{4.16}$$

For $x \in [0, \frac{1}{2N}]$ we take

$$\rho^{N-\frac{1}{2}}(x,t) = \rho_{\frac{1}{2}}(t), \quad \theta^{N-\frac{1}{2}}(x,t) = \theta_{\frac{1}{2}}(t)$$

and for $x \in]1 - \frac{1}{2N}, 1]$

$$\rho^{N-\frac{1}{2}}(x,t) = \rho_{N-\frac{1}{2}}(t), \quad \theta^{N-\frac{1}{2}}(x,t) = \theta_{N-\frac{1}{2}}(t).$$

The corresponding step functions are also introduced:

$$(v_h, \omega_h)(x,t) = (v_{[xN]}, \omega_{[xN]})(t), \tag{4.17}$$

$x \in]\frac{1}{N}[xN], \frac{1}{N}([xN]+1)]$,

$$(\rho_{h-\frac{1}{2}}, \theta_{h-\frac{1}{2}})(x,t) = (\rho_{[xN+\frac{1}{2}]-\frac{1}{2}}, \theta_{[xN+\frac{1}{2}]-\frac{1}{2}})(t), \tag{4.18}$$

$x \in]\frac{1}{N}([xN + \frac{1}{2}] - \frac{1}{2}), \frac{1}{N}([xN + \frac{1}{2}] + \frac{1}{2})]$,

$$(\rho_{h-\frac{1}{2}}, \theta_{h-\frac{1}{2}})(x,t) = (\rho_{\frac{1}{2}}, \theta_{\frac{1}{2}})(t), \;\; x \in [0, \tfrac{1}{2N}], \tag{4.19}$$

$$(\rho_{h-\frac{1}{2}}, \theta_{h-\frac{1}{2}})(x,t) = (\rho_{N-\frac{1}{2}}, \theta_{N-\frac{1}{2}})(t), \;\; x \in]1 - \tfrac{1}{2N}, 1]. \tag{4.20}$$

The main result is given in the next theorem.

Theorem 4.1. *Let the initial functions ρ_0, v_0, ω_0 and θ_0 satisfy the same conditions as in Theorem 3.2. Then there exist subsequences of approximate solutions (still denoted) $\{(\rho^{N-\frac{1}{2}}, v^N, \omega^N, \theta^{N-\frac{1}{2}})\}$ and $\{(\rho_{h-\frac{1}{2}}, v_h, \omega_h, \theta_{h-\frac{1}{2}})\}$ in the domain Q_T (for each $T \in \mathbf{R}^+$) such that, as $N \to \infty$ (or $h \to 0$),*

$$(\rho^{N-\frac{1}{2}}, v^N, \omega^N, \theta^{N-\frac{1}{2}}) \to (\rho, v, \omega, \theta) \tag{4.21}$$

*strongly in $(C(\overline{Q}_T))^4$, *weakly in $(L^\infty(0,T;H^1(]0,1[)))^4$ and weakly in $(H^1(Q_T))^4$,*

$$(v^N, \omega^N, \theta^{N-1}) \to (v, \omega, \theta) \tag{4.22}$$

weakly in $(L^2(0,T;H^2(]0,1[)))^3$,

$$(\rho_{h-\frac{1}{2}}, v_h, \omega_h, \theta_{h-\frac{1}{2}}) \to (\rho, v, \omega, \theta) \tag{4.23}$$

strongly in $(L^\infty(0,T;L^2(]0,1[)))^4$.

The function $(\rho, v, \omega, \theta)$ satisfies equations (2.32)-(2.35) a.e. in Q_T, conditions (2.36)-(2.37) in the sense of traces and ρ and θ have the properties

$$\inf_{Q_T} \rho > 0, \quad \inf_{Q_T} \theta > 0. \tag{4.24}$$

The proof of Theorem 4.1 is based on a careful examination of a priori estimates and limit procedure.

Let us also mention that in [15] the numerical solution to the considered problem is also obtained, whereby the described convergent finite difference scheme is used. The considered problem was analyzed numerically also in [16] and [4] where Faedo-Galerkin approximations were used to obtain numerical solution. These two approaches were compared in [2], where is concluded that the finite difference scheme outperforms the Faedo-Galerkin method in the numerical simulations for the compressible micropolar fluid flow problems.

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