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The Cauchy Problem for the Second Order Semilinear Sobolev Type Equation

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Abstract

We prove a unique solvability of the Cauchy problem for a class of second order semilinear Sobolev type equations. We use ideas and techniques developed by G.A. Sviridyuk for the investigation of the Cauchy problem for a class of first order semilinear Sobolev type equations and by A.A. Zamyshlyaeva for the investigation of the high-order linear Sobolev type equations. We also use the theory of differential manifolds following, say, Lang's books. In the article we consider two cases. The first one is where an operator A at the highest time derivative is continuously invertible. In this case for any point from the tangent bundle of the original Banach space there exists a unique solution lying in this space as a trajectory. The second case, where the operator A is not continuously invertible, is of great interest for us. Here we use the phase space method. It consists in reducing a singular equation to a regular one which is defined on a subset of the original Banach space consisting of admissible initial values which is understood as a phase space. Under the condition of polynomial boundedness of operator pencil in the case where infinity is a removable singularity of its A-resolvent, a set, which is locally a phase space of the original equation, is constructed. In the last section the abstract theory is applied to an initial-boundary value problem for Boussinesque – Löve equation.

Key words: phase space, semilinear Sobolev type equation, relatively polynomially bounded operator pencil, Banach manifold.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ of class C^∞. We consider initial-boundary value problem

$$u(x,t) = 0, \quad (s,t) \in \partial\Omega \times \mathbb{R}, \tag{1.1}$$

$$u(x,0) = u_0, \quad u_t(x,0) = u_1, \tag{1.2}$$

for the Boussinesque – Löve equation

$$(\lambda - \Delta)u_{tt} = \alpha(\Delta - \lambda')u_t + \beta(\Delta - \lambda'')u + \Delta f(u), \tag{1.3}$$

given in the cylinder $\Omega \times \mathbb{R}$. Here $u(x,t)$ denotes the unknown function, $f(u)$ is a given nonlinear function, subscript t indicates partial derivative in t and Δ denotes the Laplace operator in $\mathbb{R}^n$. Equation (1.3) arises in a wide variety of physical systems. For example, equation (1.3) with $n = 1$ describes a continuum limit of a one-dimensional nonlinear lattice, shallow-water waves [1, p.403], longitudinal vibrations of an elastic rod provided inertia and external load and the parameter λ can take negative value.[1, p.403]

In suitable Banach spaces $\mathfrak{U}$ and $\mathfrak{F}$ problem (1.1) – (1.3) can be reduced to the operator differential equation

$$A\ddot{u} = B_1\dot{u} + B_0 u + N(u) \tag{1.4}$$

with initial conditions

$$u(0) = u_0, \dot{u}(0) = u_1 \tag{1.5}$$

where $A, B_0, B_1 \in \mathcal{L}(\mathfrak{U}; \mathfrak{F})$ and $N \in C^\infty(\mathfrak{U}; \mathfrak{F})$.

The problem (1.4), (1.5) with noninvertible operator A having nontrivial kernel, is of great interest for us. Such equations are commonly referred to as Sobolev type equations. It is well known that the Cauchy problem for the Sobolev type equation may not solvable for arbitrary initial values. To our opinion the most fruitful (taking into account the existing applications) way for investigation of this equation is the phase space method [2] which was suggested by G.A. Sviridyuk in the study of the first order semilinear Sobolev type equations. Later this method was developed in [3, 4, 5, 6]. Essence of the method consists in reducing the singular equation to a regular one given on a subset of the original Banach space consisting of admissible initial values that is understood as a phase space of given equation.

Our purpose is to propagate ideas of the phase space method to the case of second order equation. In this paper we use the relatively polynomially bounded operator pencil theory [3], which was developed for investigation of the higher order Sobolev type equations, and the theory of differential manifolds following [7].

Besides introduction and references, the article includes three sections. The first section presents the relatively polynomially bounded operator pencil theory [3]. The second section contains the main results on the solvability of abstract problem (1.4), (1.5). In the last section the abstract results are applied to problem (1.1) – (1.3).

All problems are considered in the real Banach spaces, but for spectral problem we introduce their natural complexification. By expression $T\mathfrak{M}$ we denote the tangent bundle of the set $\mathfrak{M}$. All contours are oriented counterclockwise motion and they limit a domain lying to the left of the contour. Symbol $\mathbb{I}$ denotes unit operator.

2 Relatively polynomially bounded operator pencil

By $\overrightarrow{B}$ we denote the operator pencil of operators B_1, B_0 [3, 8].

Definition 1. *The sets* $\rho^A(\overrightarrow{B}) = \{\mu \in \mathbb{C} : (\mu^2 A - \mu B_1 - B_0)^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})\}$ *and* $\sigma^A(\overrightarrow{B}) = \overline{\mathbb{C}} \setminus \rho^A(\overrightarrow{B})$ *are called A-resolvent set and A-spectrum of pencil* $\overrightarrow{B}$*, respectively.*

Definition 2. *The operator-function* $R^A_\mu(\overrightarrow{B}) = (\mu^2 A - \mu B_1 - B_0)^{-1}$ *with domain* $\rho^A(\overrightarrow{B})$ *is called A-resolvent of pencil* $\overrightarrow{B}$*.*

Definition 3. *The operator pencil* $\vec{B}$ *is called polinomially bounded relative to operator A (or polinomially A-bounded) if* $\exists a \in \mathbb{R}_+ \quad \forall \mu \in \mathbb{C} \quad (|\mu| > a) \Rightarrow (R^A_\mu(\vec{B}) \in \mathcal{L}(\mathfrak{F}; \mathfrak{U}))$*.*

Remark 1. *If there exists operator* $A^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$*, the pencil* $\vec{B}$ *is A-bounded. If* A, B_1 *are null operators and* $B_0^{-1} \in \mathcal{L}(\mathfrak{F}, \mathfrak{U})$ *exists, the pencil* $\vec{B}$ *is A-bounded.*

Introduce the additional condition

$$\int_\gamma R^A_\mu(\vec{B}) d\mu \equiv \mathbb{O} \tag{2.1}$$

where contour $\gamma = \{\mu \in \mathbb{C} : |\mu| = r > a\}$.

Remark 2. *If operator* $A^{-1} \in \mathcal{L}(\mathfrak{F}; \mathfrak{U})$ *exists, condition* (2.1) *is fulfilled.*

Lemma 1. [3] *Let the pencil $\vec{B}$ be polynomially A-bounded and condition (2.1) be fulfilled. Then the operators*

$$P = \frac{1}{2\pi i}\int\limits_{\gamma} R^A_\mu(\vec{B})\mu A d\mu, \quad Q = \frac{1}{2\pi i}\int\limits_{\gamma} \mu A R^A_\mu(\vec{B}) d\mu$$

are projectors in spaces $\mathfrak{U}$ and $\mathfrak{F}$ respectively.

Introduce the notation $\mathfrak{U}^0 = \ker P$, $\mathfrak{F}^0 = \ker Q$, $\mathfrak{U}^1 = \text{im } P$ and $\mathfrak{F}^1 = \text{im } Q$. By lemma 1 $\mathfrak{U} = \mathfrak{U}^0 \oplus \mathfrak{U}^1$ and $\mathfrak{F} = \mathfrak{F}^0 \oplus \mathfrak{F}^1$. By symbols A^k (B^k_l) we denote the restrictions of operators A (B_l) to $\mathfrak{U}^k$, $k = 0, 1;\ l = 0, 1$.

Theorem 2. [3] *Let operator pencil $\vec{B}$ be polynomially A-bounded and condition (2.1) be fulfilled. Then*
(i) $A^k \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k),\ k = 0, 1$;
(ii) $B^k_l \in \mathcal{L}(\mathfrak{U}^k; \mathfrak{F}^k),\ k = 0, 1,\ l = 0, 1$;
(iii) operator $(A^1)^{-1} \in \mathcal{L}(\mathfrak{F}^1; \mathfrak{U}^1)$ *exists;*
(iv) operator $(B^0_0)^{-1} \in \mathcal{L}(\mathfrak{F}^0; \mathfrak{U}^0)$ *exists.*

By theorem 2 we can construct operators $H_0 = (B^0_0)^{-1}A^0 \in \mathcal{L}(\mathfrak{U}^\circ)$ and $H_1 = (B^0_0)^{-1}B^0_1 \in \mathcal{L}(\mathfrak{U}^\circ)$.

Definition 4. *Define the family of operators $\{K^1_q, K^2_q\}$ by:*

$$K^1_1 = H_0,\ K^2_1 = -H_1,$$
$$K^1_{q+1} = K^2_q H_0,\ K^2_{q+1} = K^1_q - K^2_q H_1,\ q = 1, 2, \ldots.$$

Definition 5. *The point ∞ is called*
(i) a removable singularity of A-resolvent of the pencil $\vec{B}$ if $K^1_1 \equiv \mathbb{O}$ and $K^2_1 \equiv \mathbb{O}$;
(ii) a pole of order $p \in \mathbb{N}$ of A-resolvent of the pencil $\vec{B}$ if $K^1_p \not\equiv \mathbb{O}$ and $K^2_p \not\equiv \mathbb{O}$, but $K^1_{p+1} \equiv \mathbb{O}$, $K^2_{p+1} \equiv \mathbb{O}$;
(iii) an essential singularity of A-resolvent of the pencil $\vec{B}$ if $K^2_k \not\equiv \mathbb{O}$ for all $k \in \mathbb{N}$.

Remark 3. *Below a removable singularity of A-resolvent of the pencil $\vec{B}$ is called a pole of order 0 for convenience.*

Definition 6. *If operator pencil $\vec{B}$ is polynomially A-bounded and the point ∞ is a pole of order $p \in \{0\} \cup \mathbb{N}$ of A-resolvent of the pencil $\vec{B}$, operator pencil $\vec{B}$ is called (A, p)-bounded.*

3 The abstract problem

Definition 7. *If a vector-function $u \in C^\infty((-\tau,\tau);\mathfrak{U})$, $\tau \in \mathbb{R}_+$ satisfies equation (1.4), it is called a solution of this equation. If a vector-function satisfies condition (1.5) then it is called a solution of problem (1.4), (1.5).*

If operator $A^{-1} \in \mathcal{L}(\mathfrak{F};\mathfrak{U})$ exists, equation (1.4) can be trivially reduced to the equivalent equation

$$\ddot{u} = F(u,\dot{u}) \tag{3.1}$$

where the operator F is from C^∞-class by construction. The existence of a unique solution of problem (1.4), (1.5) for all $(u_0,u_1) \in T\mathfrak{U} = \mathfrak{U} \times \mathfrak{U}$ follows from classical results [7, p. 104].

If $\ker A \neq \{0\}$, there arises the problem of constructing a phase space [2, p.99] for equation (3).

Definition 8. *The set $\mathfrak{P}$ is called a phase space of equation (1.4) if*
(i) for all $(u_0,u_1) \in T\mathfrak{P}$ there exists a unique solution of problem (1.4), (1.5);
(ii) a solution $u = u(t)$ of the equation (1.4) lies in $\mathfrak{P}$ as a trajectory, i.e. $u(t) \in \mathfrak{P}$ for all $t \in (-\tau,\tau)$.

Let $\ker A \neq \{0\}$ and operator pencil $\vec{B}$ be $(A,0)$-bounded. Then by theorem 2 equation (1.4) can be reduced to equivalent system of equations

$$\begin{aligned} 0 &= (I-Q)(B_0+N)(u^0+u^1), \\ \ddot{u}^1 &= A_1^{-1}QB_1(\dot{u}^0+\dot{u}^1) + A_1^{-1}Q(B_0+N)(u^0+u^1), \end{aligned} \tag{3.2}$$

where $u^1 = Pu, u^0 = (I-P)u$.

Now consider the set $\mathfrak{M} = \{u \in \mathfrak{U} : (I-Q)(B_0u + N(u)) = 0\}$. Let u_0 be a point of the set $\mathfrak{M}$. Introduce the notation $u_0^1 = Pu_0 \in \mathfrak{U}^1$.

Definition 9. *The set $\mathfrak{M}$ is called a Banach C^k-manifold at point u_0 if there exist neighborhoods $\mathcal{O} \subset \mathfrak{M}$ and $\mathcal{O}^1 \subset \mathfrak{U}^1$ of points u_0 and u_0^1, respectively, and C^k diffeomorphism $\delta : \mathcal{O}^1 \to \mathcal{O}$ such that δ^{-1} is a restriction of projector P to $\mathcal{O}$. The set $\mathfrak{M}$ is called a Banach C^k-manifold simulated by the space $\mathfrak{U}^1$ if it is a Banach C^k-manifold at any point. Connected C^k-manifold is simple if any atlas is equivalent to the atlas consisting of only one chart.*

Let $\mathfrak{M} \neq \varnothing$ (i.e. there exists a point $u_0 \in \mathfrak{M}$) and the following condition is fulfilled:

$$(I-Q)(B_0 + N'_{u_0}) : \mathfrak{U}^0 \to \mathfrak{F}^0 \text{ is a toplinear isomorphism.} \tag{3.3}$$

Lemma 3. *If condition (3.3) is fulfilled then the set $\mathfrak{M}$ is a C^∞-manifold at the point u_0.*

Proof. According to implicit function theorem [9, p.107] there exist neighborhoods $\mathcal{O}^0 \subset \mathfrak{U}^0$ and $\mathcal{O}^1 \subset \mathfrak{U}^1$ of points $u_0^0 = (I-P)u_0, u_0^1 = Pu_0$, respectively, and the operator $B \in C^\infty(\mathcal{O}^1; \mathcal{O}^0)$ such that $u_0^0 = B\big(u_0^1\big)$. Construct the operator $\delta = I + B : \mathcal{O}^1 \to \mathfrak{M}$, $\delta\big(u_0^1\big) = u_0$. Then the operator δ^{-1} together with the set $\mathcal{O}^1$ forms a chart in $\mathfrak{M}$ and it is a restriction of P on $\delta[\mathcal{O}^1] = \mathcal{O} \subset \mathfrak{M}$. □

Apply the second order Frechet derivative $\delta''_{(u^1,v^1)}$ to the second equation of system (3.2). Then, since

$$\delta''_{(u^1,v^1)}\ddot{u}^1 = \frac{d^2}{dt^2}\big(\delta(u^1)\big) \text{ and } \delta(u^1) = u,$$

we obtain the equation of form (3.1) defined on $\mathcal{O}$ where

$$F(u,\dot{u}) = \delta''_{(u^1,v^1)}A_1^{-1}QB_1\dot{u} + \delta''_{(u^1,v^1)}A_1^{-1}(B_0+N)(u).$$

By lemma 3 and the regular equations theory [7, p.104] we obtain

Theorem 4. *Let the operator pencil $\vec{B}$ be $(A,0)$-bounded, condition (2.1) be fulfilled and $N \in C^\infty(\mathfrak{U};\mathfrak{F})$. Then for any pair $(u_0,u_1) \in T\mathfrak{M}$ under condition (3.3) there exists a unique solution of problem (1.4) – (1.5) lying in $\mathfrak{M}$ as a trajectory.*

4 Applications

Now let us turn to the Boussinesque – Löve equation (1.3) with boundary condition (1.1). In order to reduce the problem (1.1), (1.3) to equation (1.4) we set

$$\mathfrak{U} = \{u \in W_2^{l+2}(\Omega) : u(x) = 0, x \in \partial\Omega\}, \quad \mathfrak{F} = W_2^l(\Omega),$$

where $W_2^l(\Omega)$ is a Sobolev space and the operators A, B_1, B_0 are defined by formulas $A = \lambda - \Delta$, $B_1 = \alpha(\Delta - \lambda')$ and $B_0 = \beta(\Delta - \lambda'')$. For any $l \in \{0\} \cup \mathbb{N}$ the operators $A, B_1, B_0 \in \mathcal{L}(\mathfrak{U};\mathfrak{F})$ [3].

Denote by $\{\lambda_k\}(\sigma(\Delta))$ the eigenvalues of Dirichlet problem for the Laplace operator Δ in non-increasing manner taking into account their multiplicity. Denote by $\{\varphi_k\}$ the corresponding orthonormal (in the sense of the scalar product of $L^2(\Omega)$) eigenfunctions. Since $\{\varphi_k\} \subset C^\infty(\Omega)$,

$$\mu^2 A - \mu B_1 - B_0 = \sum_{k=1}^{\infty}[(\lambda - \lambda_k)\mu^2 + \alpha(\lambda' - \lambda_k)\mu + \beta(\lambda'' - \lambda_k)] < \varphi_k, \cdot > \varphi_k$$

where $< \cdot, \cdot >$ is the scalar product in $L^2(\Omega)$.

Lemma 5. *[3] Let one of the following conditions be fulfilled:*

(i) $\lambda \notin \sigma(\Delta)$;

(ii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda \neq \lambda')$;

(iii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda'')$.

Then the pencil $\vec{B}$ *is polynomially A-bounded.*

Let conditions (i) or (iii) of lemma 5 be fulfilled. Then condition (2.1) takes place. If $(\lambda \in \sigma(\Delta)) \wedge (\lambda \neq \lambda')$, i.e. condition (ii) of lemma 5 is fulfilled, (2.1) doesn't take place, therefore we eliminate it from further consideration. The A-spectrum of pencil $\vec{B}$ consists of solutions $\mu_k^{1,2}$ of the equation

$$(\lambda - \lambda_k)\mu^2 + \alpha(\lambda' - \lambda_k)\mu + \beta(\lambda'' - \lambda_k) = 0, \; k \in \mathbb{N}.$$

In further consideration we need regularity theorem [10, p. 282] that in our case takes the form

Lemma 6. *Let* f *be* $C^\infty(\mathbb{R})$ *and* $l + 2 > n/2$ *then* $N : u \to \Delta f(u)$ *belongs to* $C^\infty(\mathfrak{U}; \mathfrak{F})$ *class.*

Thus reduction of problem (1.1) – (1.3) to abstract problem (1.4) – (1.5) is completed. By lemma 1 we construct projector P:

$$P = \begin{cases} \mathbb{I} \text{ if (i) of lemma 6 is fulfilled,} \\ \mathbb{I} - \sum\limits_{\lambda = \lambda_k} \langle \cdot, \varphi_k \rangle \varphi_k \text{ if (iii) of lemma 6 is fulfilled.} \end{cases}$$

Projector Q has the same form but it is given on the space $\mathfrak{F}$.

Specify $l > n/2 - 2$ and construct the set

$$\mathfrak{M} = \begin{cases} \mathfrak{U} \text{ if (i) of lemma 6 is fulfilled,} \\ \{u \in \mathfrak{U} : \langle Mu + N(u), \varphi_l \rangle = 0\} \text{ if (iii) of lemma 6 is fulfilled} \end{cases}$$

and space

$$\mathfrak{U}^1 = \begin{cases} \mathfrak{U} \text{ if (i) of lemma 6 is fulfilled,} \\ \{u \in \mathfrak{U} : \langle u, \varphi_l \rangle = 0\} \text{ if (iii) of lemma 6 is fulfilled.} \end{cases}$$

In the first case, where $\lambda \notin \sigma(\Delta)$, since the set $\mathfrak{M}$ coincides with space $\mathfrak{U}$, it is C^∞-manifold. In the second case, where $\lambda \in \sigma(\Delta)$, if condition (iii) of lemma 6 and condition (3.3) are fulfilled, by lemma 3, $\mathfrak{M}$ is a Banach C^∞-manifold at the point $u_0 \in \mathfrak{M}$.

Theorem 7. *(i) For all* $\lambda \notin \sigma(\Delta)$, $l > n/2 - 2$, $u_0, u_1 \in \mathfrak{U}$ *and* $\tau > 0$ *there exists a unique solution* $u \in C^\infty\big((-\tau, \tau), \mathfrak{U}\big)$ *of problem (1.1) – (1.3).*

(ii) Let $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda'')$, $l > n/2 - 2$, $(u_0, u_1) \in T\mathfrak{M}$ *and condition (3.3) be fulfilled. Then for* $\tau > 0$ *there exists a unique solution* $u \in C^\infty\big((-\tau, \tau), \mathfrak{M}\big)$ *of problem (1.1) – (1.3).*

By (i) of theorem 6 the set $\mathfrak{M} = \mathfrak{U}$ is a phase space of equation (1.3) in the case $\lambda \notin \sigma(\Delta)$. When condition (iii) of lemma 6 is fulfilled any solution of problem (1.1) – (1.3) lies in $\mathfrak{M}$ as a trajectory.

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