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Bifurcations of Critical Orbits of $SO(2)$-invariant Fredholm Functionals at Critical Points with Double Resonances

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Abstract

In this paper we consider the problem of bifurcation of extremals of $SO(2)$-invariant (i.e., with circular symmetry) Fredholm functional near a steady-state point with a double-resonance (i.e., with two independent resonance relations). The main method of investigation is a variational modification of the Lyapunov-Schmidt reduction. It allows us to find a normal form of key functions of functionals. J. Mather's condition on a finite determinacy of a smooth map germ gives a simpler representation of the key function. Further bifurcational analysis of branching extremals reduces the problem to analysis of boundary and corner singularities via the secondary reduction.

Key words: Cycle, resonance, Lyapunov-Schmidt method, bifurcation, circular symmetry.

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Introduction

The problem we are dealing with, is close to multimode cyclogenesis problem in mechanical dynamic systems. The similar problem occurs also in the theory of phase transitions in crystals, in the nonlinear wave theory as well as in radiophysics, in economics, in population dynamics etc.

The theoretical environment of the approach suggested in this paper, is the theory of smooth $SO(2)$-equivariant (i.e., with circular symmetry) potential Fredholm equations in Banach spaces. The central point here is the modified Lyapunov-Schmidt method that allows one to realize approximate computation and analysis of critical orbits of the functional $V(x)$ via construction and analysis of the key function $W(\xi)$ on the space of key variables $\mathbb{R}^n$ [1] – [2]. A brief exposition of this theory is given in the first section of this article.

The transition to key function is presented in the form of Ritz approximation :

$$W(\xi) = V\left(\sum_{j=1}^{n} \xi_j e_j + \Phi(\xi)\right) = \inf_{x:\langle x, e_j\rangle = \xi_j \ \forall j} V(x)$$

(the inner product $\langle \cdot, \cdot \rangle$ is taken from a certain Hilbert space H). In the first section it is also shown that conditions of local finite determinacy of smooth function at singular points play important role in local investigation of key functions. We use one of the most often applied criterion of finite determinacy found by J. Mather (see [3] – [6]): $\mathfrak{M}^{r+1} \subset \mathfrak{M}^2 \cdot \mathfrak{A}(W)$, where $\mathfrak{M}^k$ is the the k-th power of maximal ideal $\mathfrak{M}$ in the ring of formal power series $\mathbb{R}[[\xi]]$ and $\mathfrak{A}(W)$ is the Jacobian ideal of the function W (at zero): $\mathfrak{A}(W)$ is generated by the components of $\operatorname{grad} W(\xi)$.

The authors are sure that Mather's conditions and their modifications give a good constructive basis for elaborating computational algorithms in concrete problems of bifurcational analysis.

In the second and the third sections a smooth functional (near zero) with circular symmetry ($SO(2)$-invariant functional) is considered. The action of the circle (i.e., the group $SO(2)$) is given by the orthogonal representation T of this group in the group of orthogonal operators in the Hilbert space H that contains the domain of the functional. This action generates the action of the circle on the kernel N of Frechét derivative of gradient of V at zero. It is assumed that: 1) the action of $SO(2)$ on N is smooth, 2) $N = N_1 \dot{+} N_2 \dot{+} \ldots \dot{+} N_m$, $\dim N_k = 2$, $T(N_k) = N_k$, and 3) the action of the circle on N_k is irreducible $\forall k$.

If N_k is identified with the complex plane $\mathbb{C}$, the induced action of the circle on N comes to the standard action of the circle on $\mathbb{C}^m$ (with a certain collection of indices p_k) $\{\varphi, z\} \mapsto \widetilde{z} = (e^{ip_1\varphi} z_1, \ldots, e^{ip_m\varphi} z_m)^\top$, $z = (z_1, \ldots, z_m)^\top$. It is assumed that $HOD(p_1, p_2, \ldots, p_m) = 1$. In this case one says that the *resonance of* $p_1 : p_2 : \cdots : p_m$ *type* takes place at the zero critical point.

The algebraic structure of the corresponding key function depends on the resonance linear combinations that are considered to be collections of integers $(l_1, l_2, \ldots l_m)$, for which the (resonance relation) $\sum\limits_{k=1}^{m} l_k p_k = 0$ holds.

There is a direct analogy between the resonances considered here, and the resonances in the theory of cycles bifurcations from multiple focus in the dynamical systems [7, 8]. As well as in the dynamic systems theory, the cases of (strong) resonances with orders no greater than 4, are the most complicated for investigation. This means that there exists a resonance collection of coefficients $(l_1, l_2, \dots l_m)$ such that

$$|l| \ := \ \sum_{k=1}^{m} |l_k| \ \leq \ 4.$$

Note that the definitions given above, are compatible with those from the dynamic systems theory and in fact are analogs of the latter.

In the case of a double resonance the key function takes the form

$$\frac{1}{2}\left(\sum_{k=1}^{3} \delta_k I_k\right) + \frac{1}{4}\left(\sum_{k=1}^{3} A_k I_k^2 + 2\sum_{k,j=1}^{3} B_{k,j} I_k I_j\right) + J + o(\|\xi\|^4),$$

where $I_k = \xi_{2k-1}^2 + \xi_{2k}^2$ are standard invariants and J is a linear combination of invariants with degree ≤ 4 (for the action under consideration), completing the standard invariants to the system of generating invariants (in the ring of formal power series). The generating invariants are defined in the third section.

After transition to polar coordinates $z_k = r_k e^{i\varphi_k}$ we obtain the key function in the form

$$\mathcal{W}_\delta = \sum_{j=1}^{3} r_k^4 + \sum_{j<k} a_{j,k} r_j^2 r_k^2 + b_1 r_1^2 r_2 r_3 + b_2 r_1 r_2^2 x_3 + b_3 r_1 r_2 r_3^2 +$$
$$+ c_{1,2} r_1^3 r_2 + c_{1,3} r_1^3 r_3 + c_{2,1} r_2^3 r_1 + c_{2,3} r_2^3 r_3 + c_{3,2} r_3^3 r_2 + c_{3,1} r_3^3 r_1 +$$
$$+ d_{1,2} r_1^2 r_2 + d_{1,3} r_1^2 r_3 + d_{2,1} r_2^2 r_1 + d_{2,3} r_2^2 r_3 + d_{3,2} r_2^3 r_2 + d_{3,1} r_2^3 r_1 +$$
$$+ \delta_1 r_1^2 + \delta_2 r_2^2 + \delta_3 r_3^2 + O(\|r\|^4),$$

where $\{a_{j,k}, b_l, c_{j,k}, d_{j,k},\}$ are structural parameters (that, roughly speaking, depend on φ), and $\{\delta_j\}$ are small parameters.

The points, stationary with respect to the angular variables, are regular with respect to them and so we can eliminate those variables by transition to the function

$$\mathcal{U}(r) := \mathrm{extr}_\varphi \, \mathcal{W}_\delta(r, \varphi).$$

The principal information on the bifurcations at the generating critical point is contained in the function $\mathcal{U}(r)$ (of three variables). A special type of singularity of the (secondary) reduced key function corresponds to every case of double resonance. In the fourth section we present a theorem on normal forms of the corresponding singularities.

In the fifth section we consider some examples of bifurcational analysis of the normalized key functions.

1 Nonlinear Ritz approximations, Lyapunov-Schmidt reduction and key functions.

Let V be a functional on Banach space E. A classical Ritz approximation of V in E is a function :

$$W(\xi) := V\left(\sum_{j=1}^{n} \xi_j e_j\right), \qquad \xi = (\xi_1, \ \dots \ , \xi_n)^\top,$$

where $e_1, \ \dots \ , e_n$ is some set of linear independent vectors in E (the approximation's basis). A Ritz approximation of extremal of V is a point $\bar{x} = \sum\limits_{j=1}^{n} \bar{\xi}_j e_j$ corresponding to the extremal $\bar{\xi} = (\bar{\xi}_1, \ \dots \ , \bar{\xi}_n)^\top$ of the function W. In applications the exactness of Ritz approximations is growing up by increasing the number of basis vectors.

In the general case we consider a nonlinear approximation determined by the function :

$$W(\xi) = V\left(\sum_{j=1}^{n} \xi_j e_j + \Phi(\xi)\right), \tag{1.1}$$

where Φ is a smooth map from $N := Lin(e_1, ..., e_n)$ to $N^\perp$ (orthogonal complement to N, e.g., in the space of square integrable functions). By using such approximations, in many problems one can reach arbitrarily high exactness of approximation for a priori fixed approximation basis and so for a priori limited number of approximating system's degrees of freedom.

Various methods for "finite truncation" of variational problems are well-known at the moment. The reduction based on the representation (1.1) is one of them. The other important ones are Lyapunov-Schmidt method, Morse-Bott method and many others [2].

It is convenient to describe a reduction scheme starting from the abstract operator equation:

$$f(x) = 0, \tag{1.2}$$

where f is a potential Fredholm mapping of zero index from the Banach space E to the Banach space F. The fact that the index of Fredholm mapping equals zero means that the Frechet derivative $\frac{\partial f(x)}{\partial x}$, $\forall x \in E$, has a finite-dimensional kernel $Ker\, \frac{\partial f(x)}{\partial x}$ and a finite-dimensional cokernel $Coker\, \frac{\partial f(x)}{\partial x} := F/Im\, \frac{\partial f(x)}{\partial x}$ such that $dim\ Ker\, \frac{\partial f(x)}{\partial x} = dim\ Coker\, \frac{\partial f(x)}{\partial x}$. The fact that f is a potential mapping, means that equation (1.2) is equivalent to the extremal problem:

$$V(x) \longrightarrow \inf, \quad x \in E,$$

where V is a smooth functional (the potential of the mapping f) on E. In other words,

$$\langle f(x), h\rangle \equiv \pm\frac{\partial V}{\partial x}(x)h, \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ is a scalar product in some Hilbert space H, in which the spaces E and F are continuously and densely embedded. Usually it is also supposed that E is continuously embedded into F. In this case we say that V has the gradient realization in the triple of spaces $\{E, F, H\}$ and use the notation $f = \mathrm{grad\,V} = \nabla\,\mathrm{V}$.

To (1.3) we also add the condition

$$\left\langle \frac{\partial f}{\partial x}(x)h, h\right\rangle > 0 \qquad \forall (x,h) \in E\times(\tilde{E}\setminus 0), \tag{1.4}$$

where $\tilde{E} = E\cap N^{\perp}$, $N = Lin(e_1, \dots, e_n)$, $N^{\perp}$ is the orthogonal complement to N and $e_1, \dots, e_n$ is an orthonormal in H system of vectors in E. Under these hypotheses we can define the key function of Lyapunov-Schmidt reduction as:

$$W(\xi) := \inf_{x:\langle x,e_j\rangle=\xi_j\ \forall j} V(x), \qquad \xi = (\xi_1, \dots, \xi_n)^{\top}, \tag{1.5}$$

The function W and the functional V have similar behavior in a neighborhood of the origin. If $f : E \longrightarrow F$ is a proper mapping (i.e., the pre-images of compact subsets are compact), the key function is well-posed globally on the entire $\mathbb{R}^n$ [2]. We can weaken the condition that f is proper, replacing it by the properness condition for the fiber-wise mapping $\tilde{f}_\xi$

$$\tilde{f}_\xi : \tilde{E} \longrightarrow \tilde{F}, \tag{1.6}$$

at every ξ (where $\tilde{F} = F\cap N^{\perp}$,):

$$\tilde{f}_\xi(v) := P_{\tilde{F}}\left(f(l(\xi)+v)\right) = f(l(\xi)+v) - \sum_{j=1}^{n}\langle e_j, f(l(\xi)+v)\rangle e_j,$$

$$l(\xi) = \sum_{j=1}^{n}\xi_j e_j.$$

If condition (1.4) is satisfied, the equation

$$\tilde{f}_\xi(v) = q \tag{1.7}$$

has a unique solution for each ξ and q. It follows from the implicit function theorem that the solution depends smoothly on ξ and q. The left-hand side of (1.5) admits the presentation

$$W(\xi) \equiv V(l(\xi) + \Phi(\xi))$$

where $\Phi(\xi)$ is a solution of (1.6) for q=0. For the key equation

$$\theta(\xi) = 0, \qquad \xi \in \mathbb{R}^n,$$

where

$$\theta(\xi) = (\theta_1(\xi), \dots, \theta_n(\xi))^\top, \qquad \theta_j(\xi) = \langle f(l(\xi) + \Phi(\xi)), e_j \rangle,$$

we have

$$\theta(\xi) = \text{grad } W(\xi).$$

Now we formulate one of the most important theorem of bifurcational analysis [2]. *If mapping (1.6) is proper and condition (1.4) is satisfied, then the marginal mapping $\varphi : \xi \mapsto l(\xi) + \Phi(\xi)$, (where $\Phi(\xi)$ is a solution of (1.7) for $q = 0$) is a one-to-one correspondence between critical points of key function (1.5) and extremals of V. Moreover, under these conditions the local singularity rings*[1] *of V at the point $\varphi(\xi)$ are isomorphic to the local singularity rings of (1.6) at the point ξ so that the corresponding to each other single critical points have equal Morse indices.*

The same methods are also applicable for problems with continuous group symmetries (an equivariant version) [2].

For local investigation of key functions an important role is played by the conditions of local finite determinacy of a smooth mapping at a critical point. The following condition found by Mather (see [3] – [6]) is in use very much often. Recall that a smooth function W is called strongly $r-$determined at the point a if any function U such that U and W have the same Taylor polynomial of degree r at the point a, is locally strongly smoothly equivalent to W, i.e., there is exist a mapping $\varphi : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a)$ such that its Jacobi matrix at a is the unit matrix and $W(\varphi(\xi)) = U(\xi)$ in a certain neighborhood of a. By Mather's sufficient condition, a function V is strongly $r-$determined at the origin if the following condition holds:

$$\mathfrak{M}^{r+1} \subset \mathfrak{M}^2 \cdot \mathfrak{A}(W).$$

Here $\mathfrak{M}^k$ is a k-th power of maximal $\mathfrak{M}$ ideal of the ring of formal power series $\mathbb{R}[[\xi]]$, $\mathfrak{A}(W)$ is a Jacobian ideal of W (at zero). In other words, $\mathfrak{A}(W)$ is generated by a gradient components of W [3, 6]. It is not hard to prove the following natural generalization:

$$\mathfrak{U} \subset \mathfrak{M}^2 \cdot \mathfrak{A}(W), \tag{1.8}$$

[1] The local singularity ring of a functional V at the critical point a is defined as the quotient-ring of the ring of germs of smooth functionals at a with respect to the ideal generated by functionals of the form $\alpha(f(x))$, where α is an arbitrary smooth functional given on an arbitrary neighborhood of origin in the space F ($f = \text{grad}_H V$).

where $\mathfrak{U}$ is any finite-defective ideal of $\mathbb{R}[[\xi]]$. If condition (1.8) holds, it is possible to "remove" (after a diffeomorphic change of coordinates) the "tail" of Taylor series included in $\mathfrak{U}$.

Among possible smooth deformations the so-called *versal* and *miniversal* ones play important role in the general theory of deformation of singularities. The reason for this is that the versal deformations contain information about all metamorphoses (transformation of lever curves, sewing or splitting of singular points, various bifurcational effects, etc.) that can occur under arbitrary deformation of a function.

A smooth deformation $U(\cdot, \lambda)$ of the singularity of W at zero is called *versal* if the quotient-classes of the functions $\frac{\partial U}{\partial \lambda_k}(x, 0)$ form a system of linear generators in the local ring (treated as a linear space) of the singularity of W at the origin (see [3] – [6]). System of functions $\left\{\frac{\partial U}{\partial \lambda_k}(x, 0),\ k = 1, 2, \ldots, \mu\right\}$ is called the initial velocities of deformation.

A smooth deformation $U(x, \lambda)$ is called *miniversal* if the quotient-classes of the initial velocities of deformation generate a basis of the local ring of singularity of W at origin. If we remove the monomial of degree zero we obtain a restricted miniversal deformation [3]. The number of parameters contained in the restricted miniversal deformation, equals the codimension of singularity.

Using versal deformations we can introduce some bifurcational diagrams. The most important of them are caustics, discriminant sets and Maxwell sets.

The caustic $\Sigma(W)$ of a function W is the set of all values λ close to zero and such that $W(\cdot, \lambda)$ has a degenerate critical point in a small neighborhood of zero. A discriminant set $Dskr$ is a set of all (small) values λ such that $W(\cdot, \lambda)$ has a critical point at the zero-level surface in a neighborhood of origin. A Maxwell set is a set of all values λ such that $W(\cdot, \lambda)$ has the same values at a pair of different critical points near the origin. Analogously we can give the same definitions for Fredholm functional [2].

The geometrical structure of these sets is not subjected to change after transition to the key function $W(\xi, \delta) = \inf\limits_{x:\ p(x)=\xi} V(x, \lambda)$ (here p is a reducing submersion [2]) .

In practice one looks for the function W in the polynomial form:

$$W_0(\xi) + \sum_{k \in K} \alpha_k(\lambda)\xi^k$$

where K is a finite subset of $\mathbf{Z}^n_+$, $W_0(\xi)$ is a polynomial normal form of singularity, $\{\xi^k,\ k \in K\}$ is a basis of the local ring of singularity at zero. If the deformation is versal, the mapping

$$\pi :\ \lambda \longmapsto \alpha = (\alpha_k(\lambda))_{k \in K}$$

is a submersion at zero [3]. Hence

$$\Sigma(V) = \pi^{-1}(\Sigma(\tilde{W})), \tag{1.9}$$

where

$$\tilde{W}(\xi,\alpha) = W_0(\xi) + \sum_{k\in K} \alpha_k \xi^k, \qquad \alpha = \{\alpha_k\}_{k\in K},$$

$\Sigma(V)$ and $\Sigma(\tilde{W})$ are caustics.

From (1.9) it follows that $\Sigma(V)$ is equal to Cartesian product of $\Sigma(W)$ and a disc (infinite-dimensional if the initial parameter space is infinite-dimensional).

For finding the exact location of caustic in the space of controlling parameters one needs at least to determine the character of dependence of α on λ, and it is a very difficult computational problem.

Note that the above formulas form the base for elaborating the computational algorithms in concrete bifurcational problems.

2 Functionals with circular symmetry.

Let $V : E \to \mathbb{R}$ be a Fredholm functional with gradient $f(x)$ in $\{E, F, H\}$. Let also T be a representation of group $SO(2)$ in the group $O(H)$ of orthogonal operators $H \to H$ such that $T_g(E) \subset E$, $T_g(F) \subset F$, $\forall g \in SO(2)$ (continuity of $SO(2)$ action on E is not required) and the functional V is invariant under the action of $SO(2)$ on E

$$V(T_g x) = V(x) \qquad \forall\, x \in E,\; g \in SO(2).$$

Suppose that the reduction $p : E \to \mathbb{R}^n$ is equivariant. This means that there exists a representation T of group $SO(2)$ on $\mathbb{R}^n$ such that $p(T_g(x)) = T_g(p(x))$ $\forall\, x \in E,\; g \in G$ (i.e., the reducing map is equivariant).

Everywhere below here we assume that the action of $SO(2)$ on $\mathbb{R}^n$ is smooth and that the action of the circle on $\mathbb{R}^n$ has no nonzero fixed points. This yields that $n = 2\,l$. Therefore $\mathbb{R}^n$ may be decomposed into a direct sum of two-dimensional subspaces:

$$N^1 \dot{+} N^2 \dot{+} \ldots \dot{+} N^m,$$

where every subspace is invariant under the $SO(2)$ action and restriction of $SO(2)$ action to every subset is irreducible.

If we identify every subset N_k with complex plane $\mathbb{C}$, the action of circle on N induced by the restriction, is reduced to standard action of circle on complex space $\mathbb{C}^m$ with some set of parameters p_k:

$$\{\varphi, z\} \mapsto \widetilde{z} = (e^{ip_1\varphi}\, z_1, \ldots, e^{ip_l\varphi}\, z_m)^\top, \qquad z = (z_1, \ldots, z_m)^\top.$$

Hence the key function is invariant under the action of group $SO(2)$

$$\{\exp(i\varphi), z\} \longmapsto (\exp(i\, p_1\varphi)z_1,\, \dots\, , \exp(i\, p_m\varphi)z_m)^\top.$$

If the $GCD(p_1, p_2, \dots, p_m) = 1$, we say that a resonance of type $p_1 : p_2 : \dots : p_m$ occurs at zero critical point. Algebraic structure of key function depends on the resonance type and moreover, it depends on resonant linear combinations. The resonant linear combinations are nontrivial sets of integers $(l_1, l_2, \dots\, l_m)$ such that

$$\sum_{k=1}^{m} l_k p_k = 0.$$

There is a direct analogy between the resonances we have described, and the resonances occurring in cycle bifurcation from multiple focus in dynamical systems described in [7, 8].

As well as in dynamical systems theory, the cases of (strong) resonances with orders no greater than 4, are the most complicated for investigation. This means that there exists a set of coefficients such that

$$|l| \ := \ \sum_{k=1}^{m} |l_k| \ \leq \ 4.$$

Denote by $\mathcal{R}$ the set of all resonant sets. It is a subgroup of $\mathbb{Z}^m$. Every basis of $\mathcal{R}$ is called *basis system of resonant relations.*

In particular, if $m = 2$, the basis system consist of one vector (l_1, l_2). In the case $m = 3$ basis is a pair of vectors (l_1, l_2, l_3), (n_1, n_2, n_3) .

Note that these definitions are consistent with definitions from the theory of resonances in dynamical systems.

The resonance $p : q$ is called strong if there exists a set of nonzero integers n_1, n_2 such that it satisfies the resonant relation $n_1p+n_2q = 0$ and $|n_1|+|n_2| \leq 4.$. The value $|n_1| + |n_2|$ is called the order of resonant relation. Minimal order of resonant relation is called the order of resonance. Resonant relations of order ≤ 4 are called strong, and the others are called weak. Thus, the resonance is called strong if there exists a strong resonant relation for this resonance. Otherwise the resonance is called weak.

In the same way, we may define strong and weak resonances in general case (i.e., for multiple resonances).

Remark 2.1. *In concrete problems of bifurcational analysis an orthogonal action of Lie group may not be smooth, but nevertheless this action induces a smooth action on the space of key parameters. By this reason we use the weaker condition of smooth action of $SO(2)$ on the space of key parameters instead of the condition of smooth orthogonal action of Lie group on the Hilbert space H.*

3 Basis invariants in case of double resonances.

In the case of double resonance the key function has the form:

$$\frac{1}{2}\left(\sum_{k=1}^{3}\delta_k I_k\right)+\frac{1}{4}\left(\sum_{k=1}^{3}A_k I_k^2+2\sum_{k,j=1}^{3}B_{k,j}I_k I_j\right)+J+o(\|\xi\|^4),$$

where $I_k=\xi_{2k-1}^2+\xi_{2k}^2$ is a standard invariant, J is a linear combination of invariants of degree ≤ 4 completing the standard invariants to the system of generating invariants.

In order to describe the invariants for some double resonances, we use the complex form of real polynomial:

$$U(\xi_1,\xi_2,\xi_3,\xi_4,\xi_3,\xi_6)=\widetilde{U}(z_1,\overline{z}_1,z_2,\overline{z}_2,z_3,\overline{z}_3),$$

$$z_k:=\xi_{2k-1}+\xi_{2k}\,i\,,\qquad \overline{z}_k:=\xi_{2k-1}-\xi_{2k}\,i\,.$$

Besides the standard invariant $I_k=|z_k|^2,\ k=1,2,3$ (invariant of degree 2), in all cases the invariants of the circle action are the following polynomials of degrees 3 end 4: $\bar{z}_1^2z_2,\ \bar{z}_1\bar{z}_2z_3,\ \bar{z}_1\bar{z}_3z_2^2,\ \bar{z}_1^3z_3$ in the case $1:2:3$; $\bar{z}_1^2z_2,\ \bar{z}_2^2z_3,\bar{z}_1^2\bar{z}_2z_3$ in the case $1:2:4$; $\bar{z}_1^2z_2,\ \bar{z}_2^3z_3$ in the case $1:2:6$; $\bar{z}_1^3z_2,\ \bar{z}_2^2z_3$ in the case $1:3:6$; $\bar{z}_1^3z_2,\bar{z}_2^3z_3$ in the case $1:3:9$; $\bar{z}_1^3z_2,\bar{z}_1\bar{z}_2z_3$ in the case $1:3:4$, etc.

The last set of invariants arises for the series of resonances $q:p:q+p$, $p+q\geq 5$. For the series $q:p:2q+p,\ \ p+q\geq 5$, we have invariant $\bar{z}_1^2\bar{z}_2z_3$, and for $q:p:q+2p$ invariant is $\bar{z}_1\bar{z}_2^2z_3$. For the series $q:p:2p$ we have invariant $\bar{z}_2^2z_3$, and for $q:p:3p$ invariant is $\bar{z}_2^3z_3$.

If we use the polar coordinates $z_k=r_ke^{i\varphi_k}$, we obtain that the non-normalized key function has the form:

$$\mathcal{W}_\delta=\sum_{j=1}^{3}r_k^4+\sum_{j<k}a_{j,k}r_j^2r_k^2+b_1r_1^2r_2r_3+b_2r_1r_2^2x_3+b_3r_1r_2r_3^2+$$

$$+c_{1,2}r_1^3r_2+c_{1,3}r_1^3r_3+c_{2,1}r_2^3r_1+c_{2,3}r_2^3r_3+c_{3,2}r_3^3r_2+c_{3,1}r_3^3r_1+$$

$$+d_{1,2}r_1^2r_2+d_{1,3}r_1^2r_3+d_{2,1}r_2^2r_1+d_{2,3}r_2^2r_3+d_{3,2}r_2^3r_2+d_{3,1}r_2^3r_1+$$

$$+\delta_1r_1^2+\delta_2r_2^2+\delta_3r_3^2+O(\|r\|^4),$$

where $\{a_{j,k},b_l,c_{j,k},d_{j,k},\}$ are structural parameters (that, roughly speaking, depend on φ), and $\{\delta_j\}$ are small parameters.

4 The reduced key function.

One can easily see that the points stationary in angular variables, are regular in these variables and so they can be "removed", i.e., it is possible to organize the secondary reducing transition form the function $W_\delta(r,\varphi)$ to the function $U(r)$ that eliminates the angular variables:

$$\mathcal{U}(r) := \mathrm{extr}_\varphi \, \mathcal{W}_\delta(r,\varphi).$$

The main information on bifurcations of functional at a critical point is contained in this function. A certain type of singularity of the (secondary) reduced key function corresponds to every case of resonance described above. Below we obtain the normal forms of various singularities.

First let us recall the notation: $\mathfrak{A}$ is a Jacobian ideal, generated by the gradient's components of V: $\mathfrak{A} = \langle \frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3} \rangle$, $\mathfrak{M}$ is the maximal ideal of $\mathbb{R}[[\xi]]$: $\mathfrak{M} = \langle x_1, x_3, x_3 \rangle$.

Theorem 4.1. *Let the key function $\mathcal{U}$ be obtained by reduction of angular variables φ_k at the point of minimum with double strong resonance of order 3. Then $\mathcal{U}$ can be represented in the general position after coordinate changing and redefining the parameters in the following form*

$$\mathcal{U} = \widetilde{\mathcal{U}} + \frac{1}{2}\left(\delta_1 x_1^2 + \delta_2 x_2^2 + \delta_3 x_3^2\right),$$

where $\widetilde{\mathcal{U}}$ is a normal form of singularity of the key function that depends on the resonance type. In the table below some normal form are listed (in the right column we indicate the corresponding type of resonance):

$x_2^4 + x_3^4 + x_1x_2x_3 + x_1^2x_2$,	$\{1:2:3\}$;	
$x_3^4 + a\,x_1^2x_3^2 + x_1^2x_2 + x_2^2x_3$,	$\{1:2:4\}$,	$a \neq 0$;
$x_1^4 + x_2^4 + x_3^4 + x_1x_2x_3$,	$\{p:q:p+q\}$,	$p+q \geq 5$;
$x_2^4 + x_3^4 + a\,x_2^2x_3^2 + x_1^2x_2$,	$\{p:2p:q\}$,	$p+q \geq 5,\ a \neq 0$.

Proof. The proof goes in standard way. First, we compute the Taylor polynomial of degree 4 that approximates the key function $\widetilde{W}$ (on the space of key variables $\mathbb{R}^6$). Then we introduce the polar coordinates in three planes invariant with respect to the induced action of the circle. After that we find angular critical points of the Taylor polynomial in polar coordinates and substitute them into $\widetilde{W}$. Finally, we obtain the reduced main part of the key function in the form of polynomial of degree 4 with respect to 3 radial variables. The normalization process of the key function is based on Mather's theorem mentioned above.

For example, in case of $W = x_3^4 + c\,x_1^2x_3^2 + x_1^2x_3 + x_2^2x_3$ we have

$$\mathrm{grad}\, W = \ (f_1,\ f_2,\ f_3)^\top \ =$$

$$= \left(2x_1x_2 + 2c\, x_1x_3^2,\quad x_1^2 + 2x_2x_3,\quad x_2^2 + 4x_3^3 + 2c\, x_1^2x_3\right)^\top .$$

Note that the following inclusions hold:

$$\{x_1x_2,\ x_2^2\} \in \mathfrak{A} \qquad (mod\ \ \mathfrak{M}^3),$$

$$x_1^3 \in \mathfrak{M}\cdot\mathfrak{A} \qquad (mod\ \ \mathfrak{M}^4),$$

$$\{\ x_1x_3^3,\ x_3^4\} \in \mathfrak{M}\cdot\mathfrak{A} \qquad (mod\ \ \mathfrak{M}^5).$$

Here $Q = \mathbb{R}[[x_1, x_2, x_3]]/\mathfrak{A}$ is the local ring of singularity of W at zero [3].

The first two inclusions easily follow from the algebraic structure of gradient. The last inclusion follows from the relations

$$x_1f_3 = 4x_1x_3^3 + x_1x_2^2 + 2c\, x_1^3x_3, \quad x_3f_3 - x_1f_1 = 4x_3^4 + x_2^2x_3 - 2x_1^2x_2.$$

In addition we have:

$$\{x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_1^3x_3, x_1^2x_2x_3, x_1x_2^2x_3, x_2^3x_3, x_1x_2x_3^2, x_2^2x_3^2\} \subset \mathfrak{M}^2\cdot\mathfrak{A},$$

$$\mathfrak{M}^5 \subset \mathfrak{M}^2\cdot\mathfrak{A}.$$

The monomials

$$1, x_1, x_2, x_3, x_1^2, x_2^2, x_3^2, x_1x_3, x_1^2x_3, x_1x_3^2, x_1^2x_3^2$$

form a basis of local ring Q of singularity of W at zero.

Thus, the bounded miniversal deformation of the Key function in the case of resonance 1:2:4 has the following form:

$$x_1^2x_2 + x_2^2x_3 + x_3^4 + c\, x_1^2x_3^2 + \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 +$$

$$+\delta_1x_1^2 + \delta_2x_2^2 + \delta_3x_3^2 + \lambda_4x_1x_3 + \lambda_5x_1x_3^2 + \lambda_6x_1^2x_3.$$

The quasi-symmetric deformation takes the form:

$$x_1^2x_3 + x_2^2x_3 + x_3^4 + c\, x_1^2x_3^2 + \delta_1x_1^2 + \delta_2x_2^2 + \delta_3x_3^2.$$

Here we mean the symmetry with respect to the transformation group generated by the following actions:

$$\begin{pmatrix} x_1\\ x_2\\ x_3 \end{pmatrix} \to \begin{pmatrix} -x_1\\ x_2\\ x_3 \end{pmatrix};\ \begin{pmatrix} x_1\\ x_2\\ x_3\\ c_1 \end{pmatrix} \to \begin{pmatrix} x_1\\ -x_2\\ x_3\\ -c_1 \end{pmatrix};\ \begin{pmatrix} x_1\\ x_2\\ x_3\\ c_2 \end{pmatrix} \to \begin{pmatrix} x_1\\ x_2\\ -x_3\\ -c_2 \end{pmatrix}.$$

where c_1 is a coefficient of the monomial $x_1^2x_2 :\ c_1 = 1,\quad c_2$ is a coefficient of the monomial $x_2^2x_3 :\ c_2 = 1.$

In the other cases the arguments are analogous. □

5 Examples of bifurcational analysis of the normalized key functions.

5.1 The case of min-singularity with resonance 1:2:4

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After the change of coordinates $x_1^2 = y_1,\ x_2 = y_2,\ x_3 = y_3$ in the reduced and normalized key function we obtain the function

$$\overline{U}_\delta(y_1, y_2, y_3) = y_1 y_2 + y_2^2 y_3 + y_3^4 + c\, y_1 y_3^2 + \delta_1 y_1 + \delta_2 y_2^2 + \delta_3 y_3^2, \tag{5.1}$$

given on the half-space $y_1 \geq 0$ (the function with boundary singularity [3]).

The caustic can be represented in the form $\Sigma = \Sigma_{0,1,1}^{int} \cup \Sigma_{0,1,1}^{ext} \cup \Sigma_{1,1,1}$, where the component $\Sigma_{0,1,1}^{int}$ corresponds to the degeneration along the boundary y=0, $\Sigma_{0,1,1}^{ext}$ corresponds to the degeneration along the normal to the boundary, $\Sigma_{1,1,1}$ corresponds to the degeneration at internal critical points. Now consider every component in detail.

The description of component $\Sigma_{0,1,1}^{int}$ (looking after internal degeneration – degeneration along the boundary) is reduced to investigation of its restriction to the boundary

$$\overline{U}_\delta(0, y_2, y_3) = y_2^2 y_3 + y_3^4 + \delta_2 y_2^2 + \delta_3 y_3^2.$$

At boundary critical points (where the partial derivatives in y_2 and y_3 are equal to zero) we have:

$$\begin{cases} \frac{\partial \overline{U}}{\partial y_2} = 2y_2 y_3 + 2\delta_2 y_2 = 0; \\ \frac{\partial \overline{U}}{\partial y_3} = y_2^2 + 4y_3^3 + 2\delta_3 y_3 = 0. \end{cases}$$

Thus we obtain 3 classes of extremals:

1. $y_2 = 0, y_3 = 0;$
2. $y_2 = 0, y_3^2 = -\frac{\delta_3}{2};$
3. $y_3 = -\delta_2, y_2^2 = 2\delta_2\delta_3 + 4\delta_2^3.$

Since for the Hessian matrix we have

$$H_{0,1,1} = \begin{pmatrix} 2y_3 + 2\delta_2 & 2y_2 \\ 2y_2 & 12y_3^2 + 2\delta_3 \end{pmatrix} = 2\begin{pmatrix} y_3 + \delta_2 & y_2 \\ y_2 & 6y_3^2 + \delta_3 \end{pmatrix},$$

for the Hessian determinant we obtain

$$h_{0,1,1}^{int} = 4\, det \begin{pmatrix} y_3 + \delta_2 & y_2 \\ y_2 & 6y_3^2 + \delta_3 \end{pmatrix},$$

Hence, for the first class of extremals ($y_2 = 0, y_3 = 0$) we have

$$h^{int}_{0,1,1} = 4\delta_2\delta_3 = 0 \Longrightarrow (\delta_2 = 0) \vee (\delta_3 = 0).$$

For the second class ($y_2 = 0,\ y_3^2 = -\frac{\delta_3}{2}$)

$$h^{int}_{0,1,1} = 4 \begin{vmatrix} \delta_2 \pm \sqrt{-\frac{\delta_3}{2}} & 0 \\ 0 & \delta_3 \end{vmatrix} = 0 \Longrightarrow (\delta_3 = 0) \vee (\delta_3 = -2\delta_2^2),$$

and for the third class ($y_3 = -\delta_2,\ y_2^2 = 2\delta_2\delta_3 + 4\delta_2^3$)

$$h^{int}_{0,1,1} = 4 \begin{vmatrix} 0 & y_2 \\ y_2 & 0 \end{vmatrix} = 4y_2^2 = 8(\delta_2\delta_3 + 2\delta_2^3) = 0 \Longleftrightarrow (\delta_2 = 0) \vee (\delta_3 = -2\delta_2^2).$$

The component $\Sigma^{ext}_{0,1,1}$ looking for external degeneration (degeneration along the normal) is determined by consideration of boundary critical points of the function $\overline{U}_\delta$, at which the derivative in y_1 turns to zero (external degeneration) :

$$h^{ext}_{0,1,1} = \frac{\partial \overline{U}_\delta}{\partial y_1}(0, y_2, y_3) = y_2 + c\, y_3^2 + \delta_1 = 0.$$

Hence for the first and the second classes we have $\delta_1 = 0$ and for the third class $2\delta_2\delta_3 + 4\delta_2^3 - \delta_1^4 - c^2\delta_2^4 - 2c\delta_1\delta_2^2 = 0.$

One can easily describe the component $\Sigma^{ext}_{1,1,1}$ looking after the degeneration outside the boundary,by the reduction to the one-dimensional singularity. Indeed, function (5.1) has one-dimensional degeneration at the origin (this function is regular with respect to y_1, y_2). Thus, we can reduce it to the function of one variable

$$R(y_3) := \text{extr}_{\ y_1,\, y_2}\ \overline{U}_\delta(y).$$

Since

$$\frac{\partial \overline{U}_\delta}{\partial y_1} = \delta_1 + y_2 + c\, y_3^2, \qquad \frac{\partial \overline{U}_\delta}{\partial y_2} = y_1 + 2\delta_2 y_2 + 2y_2 y_3,$$

for the reduced function we obtain the following presentation

$$R(y_3) = c^2 y_3^5 + \left(1 + \delta_2\, c^2\right) y_3^4 + 2c\, \delta_1 y_3^3 + (2c\, \delta_1\delta_2 + \delta_3)\, y_3^2 + \delta_1^2 y_3 + const,$$

This presentation allows us to obtain computer images of $\Sigma^{ext}_{1,1,1}$ and to list the sets of critical points.

5.2 The case of *min*-singularity with resonance $p : 2p : q,\ p + q \geq 5,\ a \neq 0$ [10].

After the change of coordinates $x_1^2 = y_1,\ x_2 = y_2,\ x_3^2 = y_3$ (for the reduced and normalized key function) we obtain the function with corner singularity [11] (see also [2])

$$\widetilde{U}_\delta = y_2^4 + y_3^2 + a y_2^2 y_3 + y_1 y_2 + \delta_1 y_1 + \delta_2 y_2^2 + \delta_3 y_3^2.$$

At critical points the partial derivatives are equal to zero. Hence, the caustic is described by the following system of equations:

$$\begin{cases} \frac{\partial \widetilde{U}_\delta}{\partial y_1} = \delta_1 + y_1 = 0, \\ \frac{\partial \widetilde{U}_\delta}{\partial y_2} = 4y_2^3 + 2ay_2y_3 + 2\delta_2 y_2 = 0, \\ \frac{\partial \widetilde{U}_\delta}{\partial y_1} = 2y_3 + ay_2^2 + \delta_3 = 0. \end{cases}$$

Thus, the caustic takes the form:

$$\delta_2 = \frac{1}{2}(\delta_1^2(4+a^2) + a\delta_3)$$

5.3 The case of min-singularity with weak resonance $p:q:r,\ |p|+|q|+|r| \geq 5$.

After the change of coordinates $x_1^2 = y_1,\ x_2^2 = y_2,\ x_3^2 = y_3$ (for the reduced and normalized key function) we obtain the function with principal part (after eliminating monomials over the diagonal, see [3])

$$\widehat{U}_\delta(y_1, y_2, y_3) = y_1^4 + y_2^4 + y_3^4 + ay_1y_2 + by_1y_3 + cy_2y_3 + \delta_1 y_1 + \delta_2 y_2 + \delta_3 y_3$$

given on the positive octant $\{y_1 \geq 0, y_2 \geq 0, y_3 \geq 0\}$ (the function with corner singularity [11], see also [2]). The analysis of this function is described in [12].

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