Global and Stochastic Analysis
Vol. 2 No. 2, July-December 2015

ISSN 2248-9444
www.mukpubliactions.com

Forward-Backward SDEs driven by Lévy Processes and Application to Option Pricing

R.S. Pereira[a] **and E. Shamarova**[b]

Centro de Matemática da Universidade do Porto
Rua do Campo Alegre 687, 4169-007 Porto, Portugal
[a]manuelsapereira@gmail.com
[b]evelinas@fc.up.pt

Received by the Editorial Board on May 2, 2012

Abstract

Recent developments on financial markets have revealed the limits of Brownian motion pricing models when they are applied to actual markets. Lévy processes, that admit jumps over time, have been found more useful for applications. Thus, we suggest a Lévy model based on Forward-Backward Stochastic Differential Equations (FBSDEs) for option pricing in a Lévy-type market. We show the existence and uniqueness of a solution to FBSDEs driven by a Lévy process. This result is important from the mathematical point of view, and also, provides a much more realistic approach to option pricing.

Key words: Forward-backward stochastic differential equations; FBSDEs; Lévy processes; Partial integro-differential equation; Option pricing.

2010 Mathematics subject classification: 60J75 60H10 60H30 35R09 91G80

1 Introduction

Since the seminal contribution made by F. Black and M. Scholes [2], several methodologies to value contingent assets have been developed. From Plain

Vanilla options to complex instruments such as Collateral Debt Obligations or Baskets of Credit Default Swaps, there are models to price virtually any type of contingent asset. There are, indeed, successful attempts on providing general models which in theory could price any kind of contingent claim (see, for example [3]), given a payoff function. The idea behind these models is quite standard: A portfolio replicating the payoff function of the asset is devised and, under non-arbitrage conditions, the price of the asset at a certain instant of time is the price of this portfolio at that time.

However, in spite of all this diversity and sophistication, there is an assumption that is, up to recent times, rarely questioned. Specifically, we refer to the assumption that stock prices are continuous diffusion processes, presupposing thereby that the returns have normal distributions at any time. However, it is well known today that empirical distributions of stock prices returns tend to deviate from normal distributions, either due to skewness, kurtosis or even the existence of discontinuities (Eberlein et al. give evidence of this phenomenae in [4]). The recent developments have shown that the reliance on normal distribution can bring costly surprises, especially when extreme and disruptive events occur with a much higher frequency than the one estimated by models.

As such, we believe that no matter which historical status the normal distribution has acquired throughout the years, strong efforts should be undertaken in order to develop alternative models that incorporate assumptions adequated to the observed evidence on financial markets, such as asymmetry or skewness. We do not pretend that some definitive model can actually be developed, especially when market participants' main activities are currently shifting due to the conditions imposed on financial markets. Indeed, the recently introduced regulations on financial markets severely restraining the use of own's capital for trading purposes will force the financial players to find new ways of driving a profit. This adds another layer of uncertainty about the assumptions imposed on a model. We believe, however, that in spite of the inherent inability to prove that any present model can account for future market conditions, it is worth to attempt to correctly price financial claims in the present and near future market conditions, which, as it is clear now, is fundamental to the stability of markets.

Taking the above considerations into account, we propose to replicate contingent claims in Lévy-type markets, i.e. in markets with the stock-price dynamics described as $S_t = S_0 \, e^{X_t}$, where X_t is a Lévy-type stochastic integral defined in [1]. This allows the relaxation of conditions posed on the pricing process such as symmetry, non-skewness or continuity, imposed by the Brownian framework. The self-similiarity of the pricing process, appearing due to a Brownian motion, is also ruled out from the assumptions. We base our model on the study of Forward-Backward Stochastic Differential Equations (FBSDEs) driven by a Lévy process. FBSDEs combine equations with the initial and final conditions which allows one to search for a replicating portfolio. Specifically, we are concerned with the following fully coupled FBSDEs:

$$\begin{cases} X_t = x + \int_0^t f(s, X_s, Y_s, Z_s)\, ds + \sum_{i=1}^{\infty} \int_0^t \sigma_i(s, X_{s-}, Y_{s-})\, dH_s^{(i)}, \\ Y_t = h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s)\, ds - \sum_{i=1}^{\infty} \int_t^T Z_s^i\, dH_s^{(i)}, \end{cases} \tag{1.1}$$

where the stochastic integrals are written with respect to the orthogonalized Teugels martingales $\{H_t^{(i)}\}_{i=1}^{\infty}$ associated with a Lévy process L_t [10]. We are searching for an $\mathbb{R}^P \times \mathbb{R}^Q \times (\mathbb{R}^Q \times \ell_2)$-valued solution (X_t, Y_t, Z_t) on an arbitrary time interval $[0, T]$, which is square-integrable and adapted with respect to the filtration $\mathcal{F}_t$ generated by L_t. To the authors' knowledge, fully coupled FBSDEs of this type have not been studied before. Fully decoupled FBSDEs involving Lévy processes as drivers were studied by Otmani [6]. Backward SDEs driven by Teugels martingales were studied by Nualart and Schoutens [9]. Our method of solution to the FBSDEs could be compared to the Four Step Scheme [7]. The original four step scheme deals with FBSDEs driven by a Brownian motion, and the solution is obtained via the solution to a quasilinear PDE. Replacing the stochastic integral with respect to a Brownian motion with a stochastic integral with respect to the orthogonalized Teugels martingales leads to a partial integro-differential equation (PIDE). The solution to the PIDE is then used to obtain the solution to the FBSDEs.

The organization of the paper is as follows. In Section 2, we give some preliminaires on the martingales $\{H_t^{(i)}\}$. In Section 3, under certain assumptions, we obtain the existence and uniqueness result for the associated PIDE. Our main result is Theorem 3.7, where we obtain a solution to FBSDEs (1.1) via the solution to the PIDE and prove its uniqueness. In section 4, we apply the results of Section 3 to model hedging options for a large investor in a Lévy-type market. Previously, this problem was studied by Cvitanic and Ma [3] for a Brownian market model. Finally, we study conditions for the existence of replicating portfolios.

2 Preliminaires

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space, where $\{\mathcal{F}_t\}$, $t \in [0, T]$, is the filtration generated by a real-valued Lévy process L_t. Note that the Lévy measure ν of L_t always satisfies the condition

$$\int_{\mathbb{R}} (1 \wedge x^2)\, \nu(dx) < \infty.$$

We make the filtration $\mathcal{F}_t$ P-augmented, i.e. we add all P-null sets of $\mathcal{F}$ to each $\mathcal{F}_t$. Following Nualart and Schoutens [8] we introduce the orthogonalized

Teugels martingales $\{H_t^{(i)}\}_{i=1}^{\infty}$ associated with L_t. For this we assume that for every $\varepsilon > 0$, there exists a $\lambda > 0$ so that

$$\int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda|x|)\,\nu(dx) < \infty.$$

The latter assumption guaranties that

$$\int_{\mathbb{R}} |x|^i\,\nu(dx) < \infty \quad \text{for } i = 2, 3, \ldots.$$

It was shown in [10] that under the above assumptions one can introduce the power jump processes and the related Teugels maringales. Futhermore, it was shown that the strong orthogonalization procedure can be applied to the Teugels martinagles and that the orthonormalization of the Teugels martingales corresponds to the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $x^2\nu(dx) + a^2\delta_0(dx)$, where the parameter $a \in \mathbb{R}$ is defined in Lemma 2.1 below. As in [9], by $\{q_i(x)\}$ we denote the system of orthonormalized polynomials such that $q_{i-1}(x)$ corresponds to $H_t^{(i)}$. Also, we define the polynomial $p_i(x) = xq_{i-1}(x)$. We refer to [10] for details on the Teugels martingales and their orthogonalization procedure. In the following, Lemma 2.1 below will be usefull.

Lemma 2.1. *The process $H_t^{(i)}$ can be represented as follows:*

$$H_t^{(i)} = q_{i-1}(0)B^{\lambda}(t) + \int_{\mathbb{R}} p_i(x)\tilde{N}(t, dx),$$

where $B^{\lambda}(t) = \sum_{i=1}^{N} \lambda_i B_i(t)$ with $\lambda^T\lambda = a$, $\lambda_i \in \mathbb{R}$, $\{B_i(t)\}_{i=1}^{N}$ are independent real-valued Brownian motions, and $\tilde{N}(t, A)$ is the compensated Poisson random measure that corresponds to the Poisson point process ΔL_t.

Proof. We will use the representation below for $H_t^{(i)}$ obtained in [9]:

$$H_t^{(i)} = q_{i-1}(0)L_t + \sum_{0<s\leqslant t} \tilde{p}_i(\Delta L_s) - tE\Big[\sum_{0<s\leqslant 1} \tilde{p}(\Delta L_s)\Big] - tq_{i-1}(0)E[L_1],$$

where $\tilde{p}_i(x) = p_i(x) - xq_{i-1}(0)$, and E is the expectation with respect to P. Taking into account that $L_t = L_t^c + \sum_{0\leqslant s\leqslant t} \Delta L_s$, where L_t^c is the continuous part of L_t, we obtain:

$$\begin{aligned} H_t^{(i)} &= q_{i-1}(0)L_t^c + \sum_{0<s\leqslant t} p_i(\Delta L_s) - tE\Big[\sum_{0<s\leqslant 1} \tilde{p}_i(\Delta L_s)\Big] - tq_{i-1}(0)E[L_1] \\ &= q_{i-1}(0)\Big[L_t^c - E[L_t^c]\Big] + \sum_{0<s\leqslant t} p_i(\Delta L_s) - E\Big[\sum_{0<s\leqslant t} p_i(\Delta L_s)\Big] \\ &= q_{i-1}(0)B^{\lambda}(t) + \int_{\mathbb{R}} p_i(x)\tilde{N}(t, dx). \end{aligned}$$

□

In the sequence, the following lemma will be frequently applied:

Lemma 2.2. *It holds that*

$$\int_{\mathbb{R}} p_i(x)p_j(x)\nu(dx) = \delta_{ij} - a^2 q_{i-1}(0)q_{j-1}(0).$$

Proof. The proof is a straightforward corollary of the orthonormality of $q_{i-1}(x)$ with respect to the measure $x^2\nu(dx) + a^2\delta_0(dx)$. □

We will need an analog of Lemma 5 from [9] which was proved in the latter article for a pure-jump L_t. We obtain this result for the case when L_t has both the continuous and the pure-jump parts.

Lemma 2.3. *Let $h : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}^n$ be a random function satisfying*

$$E\int_0^T |h(s,y)|^2\nu(dy) < \infty. \tag{2.1}$$

Then, for each $t \in [0,T]$,

$$\sum_{t<s\leqslant T} h(s,\Delta L_s) = \sum_{i=1}^{\infty}\int_t^T \int_{\mathbb{R}} \nu(dy)h(s,y)p_i(x)\, dH_s^{(i)} + \int_t^T \int_{\mathbb{R}} h(s,y)\nu(dy)ds.$$

Proof. Note that

$$M_t = \sum_{0\leqslant s\leqslant t} h(s,\Delta L_s) - \int_0^t \int_{\mathbb{R}} h(s,y)\nu(dy)ds = \int_0^t \int_{\mathbb{R}} h(s,x)\,\tilde{N}(ds,dx) \tag{2.2}$$

is a square integrable martingal, i.e. $\sup_{t\in[0,T]} E|M_t|^2 < \infty$, by (2.1). By the predictable representation theorem [8], there exist predictable processes φ_i with $E\Big[\int_0^T \sum_{i=1}^{\infty} |\varphi_i|^2\Big] < \infty$ and such that $M_t = \sum_{i=1}^{\infty}\int_0^t \varphi_i(s)dH_s^{(i)}$. Since $\big\langle H^{(i)}, H^{(j)}\big\rangle_t = t\,\delta_{ij}$ [10], then

$$\big\langle M, H^{(i)}\big\rangle_t = \int_0^t \varphi_i(s)ds.$$

On the other hand, by (2.2) and Lemma 2.1,

$$\begin{aligned}
&\langle M, H^{(i)}\rangle_t \\
&= \Big\langle \int_0^t \int_{\mathbb{R}} h(s,x)\tilde{N}(ds,dx), q_{i-1}(0)B_t^{\lambda} + \int_0^t \int_{\mathbb{R}} p_i(x)\tilde{N}(dt,dx)\Big\rangle_t \\
&= \int_0^t \int_{\mathbb{R}} h(s,x)p_i(x)\nu(dx)ds.
\end{aligned}$$

This implies that $\varphi_i(s) = \int_{\mathbb{R}} h(s,y)p_i(y)\nu(dy)$, and therefore,

$$\sum_{0\leqslant s\leqslant t} h(s,\Delta L_s) - \int_0^t \int_{\mathbb{R}} h(s,y)\nu(dy)ds = \sum_{i=1}^{\infty}\int_0^t \int_{\mathbb{R}} h(s,y)p_i(y)\nu(dy)dH_s^{(i)}.$$

□

3 FBSDEs and the associated PIDE

3.1 Problem Formulation and Assumptions

Consider the FBSDEs:

$$\begin{cases} X_t = x + \int_0^t f(s, X_s, Y_s, Z_s)\, ds + \int_0^t \sigma(s, X_{s-}, Y_{s-})\, dH_s, \\ Y_t = h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s)\, ds - \int_t^T Z_s\, dH_s, \\ t \in [0, T], \end{cases} \tag{3.1}$$

where

$$\begin{aligned} &f : [0, T] \times \mathbb{R}^P \times \mathbb{R}^Q \times (\mathbb{R}^Q \times \ell_2) \to \mathbb{R}^P, \\ &\sigma : [0, T] \times \mathbb{R}^P \times \mathbb{R}^Q \to \mathbb{R}^P \times \ell_2, \\ &g : [0, T] \times \mathbb{R}^P \times \mathbb{R}^Q \times (\mathbb{R}^Q \times \ell_2) \to \mathbb{R}^Q, \\ &h : \mathbb{R}^P \to \mathbb{R}^Q \end{aligned}$$

are Borel-measurable functions. Here, the stochastic integrals

$$\int_0^t \sigma(s, X_{s-}, Y_{s-})\, dH_s \quad \text{and} \quad \int_t^T Z_s\, dH_s$$

are shorthand notation for

$$\sum_{i=1}^{\infty} \int_0^t \sigma_i(s, X_{s-}, Y_{s-})\, dH_s^{(i)} \quad \text{and} \quad \sum_{i=1}^{\infty} \int_t^T Z_s^i\, dH_s^{(i)}$$

respectively, where $Z_s = \{Z_s^i\}_{i=1}^{\infty}$, $\sigma = \{\sigma_i\}_{i=1}^{\infty}$, $\sigma_i : [0, T] \times \mathbb{R}^P \times \mathbb{R}^Q \to \mathbb{R}^P$. The solution to FBSDEs (3.1), when exists, will be an $\mathbb{R}^P \times \mathbb{R}^Q \times (\mathbb{R}^Q \times \ell_2)$-valued $\mathcal{F}_t$-adapted triple (X_t, Y_t, Z_t) satisfying

$$E \int_0^T \Big(|X_t|^2 + |Y_t|^2 + \sum_{i=1}^{\infty} |Z_t^i|^2 \Big)\, dt < \infty,$$

and verifying (3.1) P-a.s.. The latter includes the existence of the stochastic integrals in (3.1). Implicitly, we are assuming that X_t and Y_t have left limits, and that Z_t is $\mathcal{F}_t$-predictable. So in fact, we are searching for càdlàg (X_t, Y_t), which will guarantee the existence of X_{t-} and Y_{t-}, and predictable Z_t.

We associate to (3.1) the following final value problem for a PIDE:

$$\begin{cases} \partial_t \theta(t, x) + f^k(t, x, \theta(t, x), \theta^{(1)}(t, x))\, \partial_k \theta(t, x) + \beta^{kl}(t, x, \theta(t, x))\, \partial_{kl}^2\, \theta(t, x) \\ - \int_{\mathbb{R}} \big[\theta\big(t, x + \delta(t, x, \theta(t, x), y)\big) - \theta(t, x) - \partial_k \theta(t, x) \delta^k(t, x, \theta(t, x), y)\big] \nu(dy) \\ + g(t, x, \theta(t, x), \theta^{(1)}(t, x)) = 0, \\ \theta(T, x) = h(x) \end{cases} \tag{3.2}$$

with $\theta^{(1)} : [0,T] \times \mathbb{R}^P \to \mathbb{R}^Q \times \ell_2$,

$$\theta_i^{(1)}(t,x) = \int_{\mathbb{R}} [\theta\big(t, x + \delta(t,x,\theta(t,x),y)\big) - \theta(t,x)] p_i(y)\, \nu(dy) \\ + c_i^k(t,x,\theta(t,x))\, \partial_k \theta(t,x). \quad (3.3)$$

The connection between β^{kl}, δ, c_i^k and the coefficients of FBSDEs (3.1) is the following:

$$\delta(t,x,y,y') = \sum_{i=1}^{\infty} \sigma_i(t,x,y) p_i(y'), \quad (3.4)$$

$$\beta^{kl}(t,x,y) = \frac{a^2}{2} \Big(\sum_{i=1}^{\infty} \sigma_i^k(t,x,y) q_{i-1}(0) \Big) \Big(\sum_{j=1}^{\infty} \sigma_j^l(t,x,y) q_{j-1}(0) \Big), \quad (3.5)$$

$$c_i^k(t,x,y) = \sigma_i^k(t,x,y) - \int_{\mathbb{R}} \delta^k(t,x,y,y') p_i(y')\, \nu(dy'). \quad (3.6)$$

To guarantee the existence of the above functions we will make the assumption:

A0 $\quad \sum_{i=1}^{\infty} q_{i-1}(0)^2 < \infty$.

Since $\sigma^k = \{\sigma_i^k\}_{i=1}^{\infty}$ takes values in ℓ_2, A0 immediately guarantees the convergence of the both multipliers in (3.5). The convergence of the series in (3.4) is understood in $L_2(\nu(dy'))$ for each fixed (t,x,y). Moreover, it holds that

$$\int_{\mathbb{R}} \Big| \sum_{i=1}^{\infty} \sigma_i(t,x,y) p_i(y') \Big|^2 \nu(dy') \\ = \|\sigma(t,x,y)\|^2_{\mathbb{R}^P \times \ell_2} - a^2 \Big| \sum_{i=1}^{\infty} \sigma_i(t,x,y)\, q_{i-1}(0) \Big|^2. \quad (3.7)$$

Indeed, applying Lemma 2.2 for each fixed N, we obtain:

$$\int_{\mathbb{R}} \Big| \sum_{i=1}^{N} \sigma_i(t,x,y) p_i(y') \Big|^2 \nu(dy') = \sum_{i,j=1}^{N} (\sigma_i, \sigma_j) \int_{\mathbb{R}} p_i(y') p_j(y') \nu(dy') \\ = \sum_{i=1}^{N} |\sigma_i|^2 - a^2 \Big| \sum_{i=1}^{N} \sigma_i q_{i-1}(0) \Big|^2.$$

Now letting N tend to infinity, we obtain (3.7).

Lemma 3.1. *The following assertions hold:*

1. $c_i^k(t,x,y) = a^2\, q_{i-1}(0) \sum_{j=1}^{\infty} \sigma_j^k(s,x,y) q_{j-1}(0)$.

2. *For each* k, $c^k = \{c_i^k\}_{i=1}^\infty$ *takes values in* ℓ_2.

3. *For each* (s,x,y), $\left\{ \int_{\mathbb{R}} \delta(s,x,y,y')p_i(y')\,\nu(dy') \right\}_{i=1}^\infty \in \ell_2$.

Proof. Define $\delta_N(s,x,y,y') = \sum_{j=1}^N \sigma_j(s,x,y)p_j(y')$. By what was proved, for each (s,x,y), $\delta_N(s,x,y,\,\cdot\,) \to \delta(s,x,y,\,\cdot\,)$ in $L_2(\nu(dy'))$, and therefore, for each i, and for each (s,x,y),

$$\int_{\mathbb{R}} \delta_N(s,x,y,y')p_i(y')\nu(dy') \to \int_{\mathbb{R}} \delta(s,x,y,y')p_i(y')\nu(dy')$$

as $N \to \infty$. On the other hand, by Lemma 2.2,

$$\int_{\mathbb{R}} \delta_N(s,x,y,y')p_i(y')\nu(dy') = \sigma_i(s,x,y) - a^2\, q_{i-1}(0) \sum_{j=1}^N \sigma_j(s,x,y)q_{j-1}(0).$$

Comparing the last two relations, we obtain that

$$\int_{\mathbb{R}} \delta(s,x,y,y')p_i(y')\nu(dy') = \sigma_i(s,x,y) - a^2\, q_{i-1}(0) \sum_{j=1}^\infty \sigma_j(s,x,y)q_{j-1}(0) \tag{3.8}$$

which proves Assertion 1. Assertion 2 is implied by Assumption A0 and Assertion 1. Finally, (3.6) implies Assertion 3. □

The heuristic argument behind PIDE (3.2) assumes the connection $Y_t = \theta(t, X_t)$ between the solution processes X_t and Y_t to (3.1) via a $C^{1,2}$-function θ. Itô's formula applied to $\theta(t, X_t)$ at points t and T leads to another BSDE which has to be the same as the given BSDE in (3.1). Thus we "guess" PIDE (3.2) by equating the drift and stochastic terms of these two BSDEs.

3.2 Solvability of the PIDE

We solve Problem (3.2) for a particular case when the functions $f(t,x,y,z)$ and $g(t,x,y,z)$ do not depend on z, and for a short time duration T. Thus, we are dealing with the following final value problem for a PIDE:

$$\begin{cases} \partial_t\theta(t,x) = -[A(t,\theta(t,\cdot))\theta](x) + g(t,x,\theta(t,x)), \\ \theta(T,x) = h(x), \end{cases} \tag{3.9}$$

where $A(t,\rho(t,\cdot))$ is a partial integro-differential operator given by

$$\begin{aligned} [A(t,\rho(t,\cdot))\theta](x) &= f^k(t,x,\rho(t,x))\,\partial_k\theta(t,x) + \beta^{kl}(t,x,\rho(t,x))\,\partial^2_{kl}\,\theta(t,x) \\ &+ \int_{\mathbb{R}} \big[\theta\big(t, x + \delta(t,x,\rho(t,x),y)\big) - \theta(s,x) - \partial_k\theta(t,x)\delta^k(t,x,\rho(t,x),y)\big]\,\nu(dy). \end{aligned} \tag{3.10}$$

with the domain $D(A(t,\rho(t,\cdot))) = \mathrm{C}_b^2(\mathbb{R}^P \to \mathbb{R}^Q)$, the space of bounded continuous functions $\mathbb{R}^P \to \mathbb{R}^Q$ whose first and second order derivatives are also bounded. We assume the following:

A1 Functions f, g, σ, and h are bounded and have bounded spatial derivatives of the first and the second order.

Lemma 3.2. *Let A0 and A1 be fulfilled. Then $A(t,\rho(t,\cdot))$, defined by* (3.10)*, is a generator of a strongly continuous semigroup on* $\mathrm{C}_b(\mathbb{R}^P \to \mathbb{R}^Q)$.

Proof. Note that by Assertion 1 of Lemma 3.1 and by (3.5), functions c^k and β^{kl} are bounded and Lipschitz in the spatial variables. This implies that the SDE

$$dX_s^k = f^k(t, X_s, \rho(t, X_s)) + \sum_{i=1}^{\infty} \sigma_i^k(t, X_{s-}, \rho(t, X_{s-}))dH_s^{(i)}, \tag{3.11}$$

with $X_t = x$, has a pathwise unique càdlàg adapted solution on $[t,T]$. The existence and uniqueness of a solution to an SDE of type (3.11) will be proved in Paragraph 3.3. Now application of Itô's formula to $\varphi(X_s)$, where φ is twice continuously differetiable, shows that the operator (3.10) is the generator of the solution to SDE (3.11), and therefore, it generates a strongly continuous semigroup on $\mathrm{C}_b(\mathbb{R}^P \to \mathbb{R}^Q)$. □

The common method to deal with problems of type (3.9) is to fix a $\mathrm{C}_b^{1,2}$-function $\rho(t,x)$, and consider the following non-autonomous evolution equation:

$$\begin{cases} \partial_t \theta(t,x) = -[A(t,\rho(t,\cdot))\theta](x) - g(t,x,\rho(t,x)), \\ \theta(T,x) = h(x). \end{cases} \tag{3.12}$$

By Assumption A1 and the results of [11] and [5], there exists a backward propagator $U(s,t,\rho)$, $0 \leqslant s \leqslant t \leqslant T$, so that

$$\theta(t,x) = [U(t,T,\rho)h](x) + \int_t^T [U(t,s,\rho)g(s,\,\cdot\,,\rho(s,\,\cdot\,))](x)\,ds.$$

We organize the map

$$\Phi : \mathrm{C}_b([0,T]\times\mathbb{R}^P \to \mathbb{R}^Q) \to \mathrm{C}_b([0,T]\times\mathbb{R}^P \to \mathbb{R}^Q),\ \rho \mapsto \theta, \tag{3.13}$$

and prove the existence of a fixed point.

Define $E = \mathrm{C}_b(\mathbb{R}^P \to \mathbb{R}^Q)$ and $D = \mathrm{C}_b^2(\mathbb{R}^P \to \mathbb{R}^Q)$.

Lemma 3.3. *Let Assumptions A0 and A1 hold. Then, there exists a constant $K > 0$ that does not depend on s, t, ρ, and ρ', so that for any function $\varphi \in D$,*

$$\sup_{s\in[t,T]} \|U(t,s,\rho)\varphi - U(t,s,\rho')\varphi\|_E \leqslant K\,T \sup_{s\in[t,T]} \|\rho(s,x) - \rho'(s,x)\|_E\, \|\varphi\|_D.$$

Proof. We have:

$$\big(U(t,s,\rho') - U(t,s,\rho)\big)\varphi = U(t,r,\rho')U(r,s,\rho)\varphi\big|_{r=t}^{s}$$
$$= \int_t^s dr\, U(t,r,\rho')\big(A(r,\rho'(r,\,\cdot\,)) - A(r,\rho(r,\,\cdot\,))\big)U(r,s,\rho)\varphi.$$

This implies:

$$\sup_{s\in[t,T]} \|U(t,s,\rho')\varphi - U(t,s,\rho)\varphi\|_E \leqslant T \sup_{s\in[t,T]} \|U(t,s,\rho')\|_{\mathcal{L}(E)}$$
$$\times \sup_{\substack{r,s\in[t,T],\\ r\leqslant s}} \|U(r,s,\rho)\|_{\mathcal{L}(D)} \sup_{s\in[t,T]} \|A(s,\rho(s,\,\cdot\,)) - A(s,\rho'(s,\,\cdot\,))\|_{\mathcal{L}(D,E)}\, \|\varphi\|_D. \tag{3.14}$$

Taking into account that

$$\|\theta(s,\,\cdot\,)\|_D = \sup_{x\in\mathbb{R}^P} |\theta(s,x)| + \sup_{x\in\mathbb{R}^P} |\nabla\theta(s,x)| + \sup_{x\in\mathbb{R}^P} |\nabla\nabla\theta(s,x)|,$$

and applying (3.5), (3.10), and Lemma 3.1, we obtain that there exists a constant $\bar{K} > 0$ which does not depend on s, ρ, and ρ', so that

$$\sup_{\|\theta\|_D\leqslant 1} \sup_{x\in\mathbb{R}^P} \|A(s,\rho(s,x))\theta - A(s,\rho'(s,x))\theta\|_{\mathcal{L}(D,E)}$$
$$\leqslant \bar{K} \sup_{x\in\mathbb{R}^P} \Big[|f(s,x,\rho(s,x)) - f(s,x,\rho'(s,x))|$$
$$+ \|\sigma(s,x,\rho(s,x)) - \sigma(s,x,\rho'(s,x))\|_{\mathbb{R}^P\times\ell_2}$$
$$+ \sup_{x'\in\mathbb{R}^P} |\nabla\nabla\theta(t,x')| \Big(\int_{\mathbb{R}} |\delta(s,x,\rho(t,x),y) - \delta(s,x,\rho'(t,x),y)|^2 \nu(dy) \Big)^{\frac{1}{2}} \Big]. \tag{3.15}$$

By (3.7), the last summand in (3.15) is smaller than

$$\|\sigma(s,x,\rho(s,x)) - \sigma(s,x,\rho'(s,x))\|_{\mathbb{R}^P\times\ell_2}$$

up to a multiplicative constant. Therefore, modifying the constant $\bar{K}$, if necessary, by Assumption A1, we obtain that

$$\sup_{\|\theta\|_D\leqslant 1} \|A(s,\rho(s,\,\cdot\,))\theta - A(s,\rho'(s,\,\cdot\,))\theta\|_E \leqslant \bar{K}\|\rho(s,\,\cdot\,) - \rho'(s,\,\cdot\,)\|_E$$

where $\bar{K}$ does not depend on s, ρ, and ρ'. Now by (3.14), there exists a constant $K > 0$, so that

$$\sup_{s\in[t,T]} \|U(t,s,\rho)\varphi - U(t,s,\rho')\varphi\|_E \leqslant K\,T \sup_{s\in[t,T]} \|\rho(s,\,\cdot\,) - \rho'(s,\,\cdot\,)\|_E\, \|\varphi\|_D.$$

Let us show that K does not depend on t, s, ρ, and ρ'. By Itô's formula, for $s \in [t, T]$ and for $\varphi \in D$,

$$[U(t,s,\rho)\varphi](x) = E[\varphi(X_s)|X_t = x], \tag{3.16}$$

where X_s is the solution to

$$dX_s^k = f^k(s, X_s, \rho(s, X_s)) + \sum_{i=1}^{\infty} \sigma_i^k(s, X_{s-}, \rho(s, X_{s-}))dH_s^{(i)}.$$

Moreover, by the results of [11] (p. 102), $U(t,s,\rho)$ maps D into D, and (3.16) implies that $U(t,s,\rho) \in \mathcal{L}(D)$ so that the norm $\|U(t,s,\rho)\|_{\mathcal{L}(D)}$ is bounded uniformly in ρ. Next, since for each $\varphi \in D$, $U(t,s,\rho)\varphi \in D$ is continuous in t and s, then it is bounded uniformly in t and s. Therefore, $\|U(t,s,\rho)\|_{\mathcal{L}(D)}$ is bounded uniformly in t, s, and ρ. This implies the statement of the lemma. □

Theorem 3.4. *Let Assumptions A0 and A1 hold. Then, there exists a $T_0 > 0$ so that for all $T \in (0, T_0]$, Problem (3.9) has a unique solution on $[0, T]$.*

Proof. Consider the equation:

$$\theta(t,x) = [U(t,T,\theta)h](x) + \int_t^T [U(t,s,\theta)\, g(s,\,\cdot\,,\theta(s,\,\cdot\,))](x)\, ds. \tag{3.17}$$

The proof of the existence and uniqueness of a solution to (3.17) is equivalent to the existence of a unique fixed point of map (3.13) in the space E. For a sufficiently small time interval $[0, T]$, the latter is implied by Assumption A1 and Lemma 3.3. Now let θ be the solution to (3.17) on $[0, T]$. Consider the equation

$$\bar{\theta}(t,x) = [U(t,T,\theta)h](x) + \int_t^T [U(t,s,\theta)\, g(s,\,\cdot\,,\bar{\theta}(s,\,\cdot\,))](x)\, ds \tag{3.18}$$

in the space D. Since $\|U(t,s,\theta)\|_{\mathcal{L}(D)}$ is bounded, and $g(s,x,y)$ is Lipschitz in y whose Lipschitz constant does not depend on s and x, the fixed point argument implies the existence of a unique solution $\bar{\theta} \in D$ to (3.18). Clearly, $\bar{\theta}$ is also a unique solution to (3.18) in E. Hence $\bar{\theta} = \theta$, and therefore, $\theta \in D$. This implies that θ is the unique solution to Problem (3.9). □

3.3 Existence and Uniqueness Theorem for the FBSDEs

In Paragraph 3.2 we found some conditions under which there exists a unique solution to PIDE (3.2). However, this solution may exist under more general assumptions. Thus, we prove the existence and uniqueness of a solution to FBSDEs (3.1) assuming the existence and uniqueness of a solution to PIDE (3.2). Specifically, we will assume the following:

A2 Functions f, g, and σ possess bounded first order derivatives in all spatial variables.

A3 Assumption A0 is fulfilled and Final value problem (3.2) has a unique solution θ which belongs to the class $\mathrm{C}_b^{1,2}([0,T]\times\mathbb{R}^P\to\mathbb{R}^Q)$.

A4 There exists a constant $K>0$ which does not depend on (t,x,y,z), such that $\sum_{i=1}^\infty \bigl|\frac{\partial}{\partial z_i} f(t,x,y,\{z_i\}_{i=1}^\infty)\bigr|\bigl(\int_{\mathbb{R}}|p_i(y)|^2\nu(dy)\bigr)^{\frac12} < K$.

Lemma 3.5. *Assume A2, A3, and A4 hold. Then the function* $f(t,\bar x,\bar y,\,\cdot\,)\circ\theta^{(1)}(t,x)$*, where* $\theta^{(1)}(t,x)$ *is given by* (3.3)*, is Lipschitz in* x *for all* $(t,\bar x,\bar y)$*, and the Lipschitz constant does not depend on* $(t,\bar x,\bar y)$*.*

Proof. Note that by Assertions 1 and 2 of Lemma 3.1, the function $c^k=\{c_i^k\}_{i=1}^\infty$ is Lipschitz in two spatial variables as an ℓ_2-valued function. By A3, θ and $\partial_k\theta$ are Lipschitz. Therefore, the last summand in (3.3) is Lipschitz in x, and moreover, its Lipschitz constant does not depend on t by boundedness of the both multipliers. Let us prove that the map

$$\mathbb{R}^P\to\mathbb{R}^Q,\quad x\mapsto\int_{\mathbb{R}}\theta(t,\bar x+\delta(t,x,\rho(t,x),y))\,p_i(y)\nu(dy) \tag{3.19}$$

is Lipschitz, where $\bar x$ and t are fixed. Let $x_1,x_2\in\mathbb{R}^P$, and let $\rho_1=\rho(t,x_1)$ and $\rho_2=\rho(t,x_2)$, where t is fixed. We have:

$$\begin{aligned}
&\Bigl|\int_{\mathbb{R}}[\theta(t,\bar x+\delta(t,x_1,\rho_1,y)-\theta(t,\bar x+\delta(t,x_2,\rho_2,y)]p_i(y)\nu(dy)\Bigr|\\
&\qquad\leqslant\max_{x\in\mathbb{R}^P}|\nabla\theta(t,x)|\int_{\mathbb{R}}|\delta(t,x_1,\rho_1,y)-\delta(t,x_2,\rho_2,y)|\,|p_i(y)|\nu(dy)\\
&\leqslant\max_{x\in\mathbb{R}^P}|\nabla\theta(t,x)|\Bigl(\int_{\mathbb{R}}|\delta(t,x_1,\rho_1,y)-\delta(t,x_2,\rho_2,y)|^2\nu(dy)\int_{\mathbb{R}}|p_i(y)|^2\nu(dy)\Bigr)^{\frac12}\\
&\quad\leqslant K\max_{x\in\mathbb{R}^P}|\nabla\theta(t,x)|\Bigl(\int_{\mathbb{R}}|p_i(y)|^2\nu(dy)\Bigr)^{\frac12}\|\sigma(t,x_1,\rho_1)-\sigma(t,x_2,\rho_2)\|_{\mathbb{R}^P\times\ell_2}.
\end{aligned} \tag{3.20}$$

Now the Lipschitzness of map (3.19) and the boundedness of the gradient of θ imply that the map

$$\Phi:\mathbb{R}^P\to\mathbb{R}^Q\times\ell_2,\ x\mapsto\int_{\mathbb{R}}\theta(t,x+\delta(t,x,\theta(t,x),y))-\theta(t,x))p_i(y)\,\nu(dy)$$

is also Lipschitz. Argument (3.20) implies that the Lipschitz constant of Φ has the form $\tilde K\Bigl(\int_{\mathbb{R}}|p_i(y)|^2\nu(dy)\Bigr)^{\frac12}$ where $\tilde K$ is a constant that does not depend on i. Now A4 implies the statement of the lemma.

□

Proposition 3.6. *Assume A2, A3, and A4. Then, the SDE*

$$\begin{cases} dX_t = f(s,X_s,\theta(s,X_s),\theta^{(1)}(s,X_{s-}))ds + \sum_{i=1}^{\infty}\sigma_i(s,X_{s-},\theta(s,X_{s-}))dH_s^{(i)}, \\ X_0 = x, \end{cases} \tag{3.21}$$

where θ is the solution to (3.2) *and $\theta^{(1)}$ is defined by* (3.3)*, has a pathwise unique càdlàg adapted solution.*

Proof. We will show that

$$\Psi(X)_t = x + \int_0^t f(s,X_s,\theta(s,X_s),\theta^{(1)}(s,X_{s-}))ds + \int_0^t \sigma(s,X_{s-},\theta(s,X_{s-}))dH_s$$

is a contraction map in the Banach space S with the norm $\|\Phi\|_S^2 = E\sup_{t\in[0,T]}|\Phi_t|^2$. Take two points X_s and X'_s from S. For simplicity of notation, let $\sigma_s = \sigma(s,X_{s-},\theta(s,X_{s-}))$ and $\sigma'_s = \sigma(s,X'_{s-},\theta(s,X'_{s-}))$. To estimate the difference of the stochastic integrals with the integrands σ_s and σ'_s with respect to the $\|\cdot\|_S$-norm, we apply the Burkholder–Davis–Gundy inequality to the martingale $\int_0^t(\sigma_s - \sigma'_s)dH_s$. We obtain that there exists a constant $C > 0$ such that

$$\begin{aligned} E\sup_{r\in[0,t]}\Big|\int_0^r(\sigma_s-\sigma'_s)dH_s\Big|^2 &\leqslant CE\Big[\int_0^{\bullet}(\sigma_s-\sigma'_s)dH_s\Big]_t \\ = CE\Big(\Big\langle\int_0^{\bullet}(\sigma_s-\sigma'_s)dH_s\Big\rangle_t + U_t\Big) &= CE\sum_{i,j=1}^{\infty}\int_0^t(\sigma_i-\sigma'_i,\sigma_j-\sigma'_j)d\langle H_i,H_j\rangle_s \\ &= CE\int_0^t\|\sigma_s-\sigma'_s\|^2_{\mathbb{R}^P\times\ell_2}ds \end{aligned}$$

where $[\,\cdot\,]_t$ and $\langle\,\cdot\,\rangle_t$ are the quadratic variation and the predictable quadratic variation, respectively. Moreover, we applied the identity $\langle H_i,H_j\rangle_s = \delta_{ij}s$ and the decomposition $[M]_t = \langle M\rangle_t + U_t$ for the quadratic variation of a square integrable martingale (i.e. a martingale M_t with $\sup_t|M_t|^2 < \infty$) into the sum of the predictable quadratic variation and a uniformly integrable martingale U_t starting at zero. Next, we note that the functions $x \mapsto f(s,x,\theta(t,x),\theta^1(t,x))$ and $x \mapsto \sigma(t,x,\theta(t,x))$ are Lipschitz whose Lipschitz constants do not depend on t. This and the above stochastic integral estimate imply that there exist a constant $K > 0$ such that

$$E\sup_{s\in[0,t]}|\Psi(X)_s-\Psi(X')_s|^2 \leqslant KE\int_0^t|X_s-X'_s|^2ds \leqslant KE\int_0^t\sup_{r\in[0,s]}|X_r-X'_r|^2ds.$$

Iterating this $n-1$ times we obtain:

$$E\sup_{s\in[0,t]}|\Psi^n(X)_s-\Psi^n(X)'_s| \leqslant \frac{K^nt^n}{n!}E\sup_{s\in[0,t]}|X_s-X'_s|^2.$$

Choosing n sufficienty large so that $\frac{K^n T^n}{n!} < 1$, we obtain that Ψ^n is a contraction, and thus, Ψ is a contraction as well. By the Banach fixed point theorem, the map Ψ has a unique fixed point in the space S. Clearly, this fixed point is a unique solution to (3.21). Setting $X^{(0)} = x$, and then, sucessively, $X^{(n)} = \Psi(X^{(n-1)})$, we can choose càdlàg modifications for each $X^{(n)}$. Since the $X^{(n)}$'s converge to the solution X in the norm of S, X will be also càdlàg a.s.. This càdlàg solution is unique in the space S, and therefore, pathwise unique. □

Introduce the space $\mathcal{S}$ of $\mathcal{F}_t$-predictable $\mathbb{R}^Q \times \ell_2$-valued stochastic processes with the norm $\|\Phi\|^2_{\mathcal{S}} = E\int_0^T \|\Phi_s\|^2_{\mathbb{R}^Q \times \ell_2} ds$. Now we formulate our main result.

Theorem 3.7. *Suppose A2, A3, and A4 hold. Let X_t be the càdlàg adapted solution to* (3.21)*. Then, the triple (X_t, Y_t, Z_t), where $Y_t = \theta(t, X_t)$, $Z_t = \theta^{(1)}(t, X_{t-})$ with $\theta^{(1)}$ given by* (3.3)*, is a solution to FBSDEs* (3.1)*. Moreover, the pair of càdlàg solution processes (X_t, Y_t) is pathwise unique. The solution process Z_t is unique in the space $\mathcal{S}$.*

Proof. It suffices to prove that the triple (X_t, Y_t, Z_t) defined in the statement of the theorem verifies the BSDE in (3.1). Application of Itô's formula to $\theta(t, X_t)$ gives:

$$\begin{aligned}\theta(T, X_t) - \theta(t, X_t) = &\int_t^T \partial_s\theta(s, X_{s-})ds + \int_t^T \partial_k\theta(s, X_{s-})dX_s^k \\ &+ \frac{1}{2}\int_t^T \partial^2_{kl}\theta(s, X_{s-})d[(X^c)^k, (X^c)^l]_s \\ &+ \sum_{t<s\leqslant T} [\theta(s, X_s) - \theta(s, X_{s-}) - \Delta X_s^k \partial_k\theta(s, X_{s-})], \end{aligned} \quad (3.22)$$

where X_s^c is the continuous part of X_s. Using the representation for $H_s^{(i)}$ from Lemma 2.1 we obtain that

$$d[(X^c)^k, (X^c)^l]_s = 2\beta^{kl}(s, X_s, \theta(s, X_s))ds,$$

where β^{kl} is given by (3.5). The forward SDE in (3.1), the relation $\Delta H_s^{(i)} = p_i(\Delta L_s)$, obtained in [9], and representation (3.4) for the function δ imply:

$$\Delta X_s = \sum_{i=1}^{\infty} \sigma_i(s, X_{s-}, Y_{s-})\Delta H_s^{(i)} = \delta(s, X_{s-}, Y_{s-}, \Delta L_s). \quad (3.23)$$

Next, one can rewrite the last term in (3.22) as

$$\sum_{t<s\leqslant T} \Big[\theta(s, X_{s-} + \delta(s, X_{s-}, Y_{s-}, \Delta L_s)) - \theta(s, X_{s-}) \\ - \delta^k(s, X_{s-}, Y_{s-}, \Delta L_s)\, \partial_k\theta(s, X_{s-})\Big],$$

where δ^k is the kth component of δ. Define the random function

$$h(s,y) = \theta(s, X_{s-} + \delta(s, X_{s-}, \theta(s, X_{s-}), y)) - \theta(s, X_{s-}) \\ - \delta^k(s, X_{s-}, \theta(s, X_{s-}), y)\, \partial_k \theta(s, X_{s-}). \tag{3.24}$$

Note that for each fixed $s \in [0,T]$ and $\omega \in \Omega$, the function h satisfies condition (2.1). Indeed, the mean value theorem, e.g. in the integral form, can be applied to the difference of the first two terms in (3.24). By boundedness of the partial derivatives $\partial_k \theta$, it suffices to verify that

$$E \int_0^T |\delta(s, X_{s-}, \theta(s, X_{s-}), y)|^2 \, ds\, \nu(dy) < \infty.$$

The latter holds by Assumption A0 and formula (3.7). Now Lemma 2.3 implies:

$$\begin{aligned}
&\sum_{t<s\leqslant T} h(s, \Delta L_s) \\
&= \sum_{i=1}^{\infty} \int_t^T \int_{\mathbb{R}} \Big[\theta(s, X_{s-} + \delta(s, X_{s-}, \theta(s, X_{s-}), y)) - \theta(s, X_{s-}) \\
&- \delta^k(s, X_{s-}, \theta(s, X_{s-}), y)\, \partial_k \theta(s, X_{s-})\Big] p_i(y) \nu(dy) dH_s^{(i)} \\
&+ \int_t^T \int_{\mathbb{R}} \Big[\theta(s, X_{s-} + \delta(s, X_{s-}, \theta(s, X_{s-}), y)) - \theta(s, X_{s-}) \\
&- \delta^k(s, X_{s-}, \theta(s, X_{s-}), y)\, \partial_k \theta(s, X_{s-})\Big] \nu(dy)\, ds.
\end{aligned}$$

Substituting this into (3.22), replacing dX_s^k with the right-hand side of (3.21), and taking into acccount that $Y_t = \theta(t, X_t)$ and that $\theta(T, X_T) = h(X_T)$ by (3.2), we obtain:

$$\begin{aligned}
Y_t = h(X_T) - \int_t^T &\Big[\partial_s \theta(s, X_{s-}) + \partial_k \theta(s, X_{s-}) f^k(s, X_{s-}, \theta(s, X_{s-}), \theta^{(1)}(s, X_{s-})) \\
&+ \frac{1}{2} \partial_{kl} \theta(s, X_{s-}) \beta^{kl}(s, X_{s-}, \theta(s, X_{s-})) \\
&+ \int_{\mathbb{R}} \big[\theta(s, X_{s-} + \delta(s, X_{s-}, \theta(s, X_{s-}), y)) \\
&- \theta(s, X_{s-}) - \delta^k(s, X_{s-}, \theta(s, X_{s-}), y)\, \partial_k \theta(s, X_{s-})\big] \nu(dy)\Big] ds \\
&- \int_t^T \sum_{i=1}^{\infty} \Big[\int_{\mathbb{R}} \big[\theta(s, X_{s-} + \delta(s, X_{s-}, \theta(s, X_{s-}), y)) - \theta(s, X_{s-}) \\
&- \partial_k \theta(s, X_{s-}) c_i^k(X_{s-}, y)\big] p_i(y) \nu(dy)\Big]\, dH_s^{(i)}.
\end{aligned}$$

Clearly, in the first three summands under the ds-integral sign one can equivalently write X_s or X_{s-}. This is true since X_s has càdlàg paths, and therefore, X_s and X_{s-} can differ only at a countable number of points. Now taking into account PIDE (3.2), we note that the integrand in the drift term is $-g(s, X_{s-}, \theta(s, X_{s-}), \theta^{(1)}(s, X_{s-}))$ which is $-g(s, X_{s-}, Y_{s-}, Z_s)$ by the definitions of Y_s and Z_s, or, it can be replaced by $-g(s, X_s, Y_s, Z_s)$ since X_s and Y_s have càdlàg paths. Finally, by (3.3) and the definition of Z_s, the integrand in the stochastic term is Z_s. Consequently,

$$Y_t = h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s) - \int_t^T Z_s \, dH_s,$$

which implies that (X_s, Y_s, Z_s) is a solution.

Let us prove the uniqueness. Let (X_s, Y_s, Z_s) be an arbitrary solution to (3.1). Let $\tilde{Y}_s = \theta(s, X_s)$, and $\tilde{Z}_s = \theta^{(1)}(s, X_{s-})$, where θ is the solution to (3.2), and $\theta^{(1)}$ is defined by (3.3). By the above argument, $(X_s, \tilde{Y}_s, \tilde{Z}_s)$ verifies the BSDE in (3.1). Applying Itô's product formula to $|\tilde{Y}_t - Y_t|^2$ and taking into consideration that $\tilde{Y}_T = Y_T$, we obtain:

$$|\tilde{Y}_t - Y_t|^2 = -2 \int_t^T \Big(\tilde{Y}_{s-} - Y_{s-}, d(\tilde{Y}_s - Y_s) \Big) + [\tilde{Y} - Y]_t - [\tilde{Y} - Y]_T.$$

Taking the expectations in the above relation gives:

$$\begin{aligned} &E|\tilde{Y}_s - Y_s|^2 + E \int_t^T \|\tilde{Z}_s - Z_s\|^2_{\mathbb{R}^Q \times \ell_2} ds \\ &= 2E \int_t^T \big(\tilde{Y}_s - Y_s, g(s, X_s, \tilde{Y}_s, \tilde{Z}_s) - g(s, X_s, Y_s, Z_s) \big) \, ds. \end{aligned}$$

By A2, there exists a constant $C > 0$ such that

$$\begin{aligned} E|\tilde{Y}_t - Y_t|^2 + E \int_t^T &\|\tilde{Z}_s - Z_s\|^2_{\mathbb{R}^Q \times \ell_2} \, ds \\ &\leqslant CE \int_t^T |\tilde{Y}_s - Y_s| \big(|\tilde{Y}_s - Y_s| + \|\tilde{Z}_s - Z_s\|_{\mathbb{R}^Q \times \ell_2} \big) \, ds. \end{aligned}$$

Now using the standard estimates and applying Gronwall's inequality, we obtain that $E|\tilde{Y}_t - Y_t|^2 + cE \int_t^T \|\tilde{Z}_s - Z_s\|^2_{\mathbb{R}^Q \times \ell_2} ds = 0$ for some constant $c > 0$. The latter relation holds for all $t \in [0, T]$. This proves that $\tilde{Y}_t$ is a modification of Y_t and that $\|\tilde{Z} - Z\|_{\mathcal{S}} = 0$. This implies the uniqueness result follows. □

4 Option Pricing with a Large Investor in Lévy-type Markets

Usually, when modeling financial assets it is assumed that all investors are price takers whose individual buy and sell decisions do not influence the price of assets. Cvitanic and Ma [3] have already developed a model for hedging options in the presence of a large investor in a Brownian market. However, observation of real data suggests that patterns, like skewness, kurtosis, or the occurence of jumps are sufficiently significant (see, e.g., Eberlein and Keller [4]) to deserve to be accounted in a realistic model of option pricing. Furthermore, the graphs of the evolution of stock prices at different time-scales are sufficiently different from the self-similarity of a Brownian motion. Thus, we develop a Lévy-FBSDE option pricing model. We believe that such a model conveys a much more realistic approach to option pricing in the presence of the already mentioned empirical market characteristics. We assume the existence of a Large investor, whose wealth and strategy may induce distortions of the price process.

Let $\mathcal{M}$ be a Lévy-type Market, i.e. a market whose stock price dynamics S_t obeys the equation $S_t = S_0 e^{X_t}$, where X_t is a Lévy-type stochastic integral [1]. The market consists of d risky assets and a money market account. For the price process $P_0(t)$ of the money market account, we assume that its evolution is given by the following equation

$$\begin{aligned} &dP_0(t) = P_0(t)\, r(t, W(t), Z(t))\, dt, \;\; 0 \leqslant t \leqslant T, \\ &P_0(0) = 1, \end{aligned}$$

where $W(t)$ is the *wealth process*, and $Z(t)$ is a portfolio-related process in a way that will be explained later. For the risky assets, we add the stochastic component represented by the volatility matrix σ taking values in $\mathbb{R}^d \times \ell_2$. We postulate that the evolution of the d-dimensional risky asset price process $P(t) = \{P_i(t)\}_{i=1}^d$ is given by the following SDE:

$$\begin{aligned} &dP_i(t) = f_i(t, P(t), W(t), Z(t))\, dt + \sum_{j=1}^{\infty} \sigma_j^i(t, P(t), W(t))\, dH_t^{(j)}, \\ &P_i(0) = p_i, \;\; p_i \geqslant 0, \;\; 1 \leqslant i \leqslant d, \quad t \in [0, T]. \end{aligned} \tag{4.1}$$

We derive the BSDE for the wealth process as in [3]. For the convinience of the reader we repeat this derivation:

$$dW(t) = \sum_{i=1}^{d} \alpha_i(t)\, dP_i(t) + \frac{W(t) - \sum_{i=1}^{d} \alpha_i(t) P_i(t)}{P_0(t)}\, dP_0(t),$$

where $\alpha_i(t)$ is the *portfolio process*. Substituting $dP_i(t)$ with the right-hand sides of (4.1), we obtain:

$$
\begin{aligned}
dW(t) = \sum_{i=1}^{d} \alpha_i(t)\big[f_i(t, P(t), W(t), Z(t))dt + \sum_{j=1}^{\infty} \sigma_j^i(t, P(t), W(t))\, dH^{(j)}(t)\big] \\
+ (W(t) - \sum_{i=1}^{d} \alpha_i P_i(t))\, r(t, W(t), Z(t))\, dt \\
= g(t, P(t), W(t), Z(t), \alpha(t))dt + \sum_{i=1}^{\infty} Z_i(t) dH^{(i)}(t), \quad (4.2)
\end{aligned}
$$

where

$$
\begin{aligned}
& g(t, \pi, w, z, a) = \sum_{i=1}^{d} a_i f_i(t, \pi, w, z) + (w - \sum_{i=1}^{d} a_i \pi_i)\, r(t, w, z), \\
& a = \{a_i\}_{i=1}^d,\ \pi = \{\pi_i\}_{i=1}^d; \\
& Z_i(t) = \sum_{j=1}^{d} \alpha_j(t) \sigma_i^j(t, P(t), W(t)), \quad i = 1, 2, \dots. \qquad (4.3)
\end{aligned}
$$

As we are assuming the absence of risk for the money market account, the evolution of its price depends totaly on the interest rate the investor is earning. On the other hand, to describe the evolution of the risky assets we use an SDE with the stochastic term given as a sum of stochastic integrals with respect to $H^{(j)}$'s. This adds explanative power to the model, as it affords the isolation of the individual contributions of each $H^{(j)}$. Now, to guarantee that the stock price has positive components we will rewrite (4.1) for $Q_i(t) = \log P_i(t)$ using Itô's formula. For simplicity of notation, we will use the same symbols f, g, σ, and h for the coefficients of the FBSDEs which we obtain after rewriting SDE (4.1) with respect to $Q(t) = \{Q_i(t)\}_{i=1}^d$ and substituting $P_i(t) = \exp\{Q_i(t)\}$:

$$
Q(t) = q + \int_0^t f(s, Q(s), W(s), Z(s))\, ds + \int_0^t \sigma(s, Q(s), W(s))\, dH(s), \quad (4.4)
$$

where $q = \{\log p_i\}_{i=1}^d$. Due to relation (4.3), we exclude the dependence on $\alpha(t)$ in (4.2). BSDE (4.2) takes the form:

$$
W(t) = h(Q(T)) + \int_t^T g(s, Q(s), W(s), Z(s))\, ds - \int_t^T Z(s)\, dH(s). \quad (4.5)
$$

Theorem 3.7 and relation (4.3) imply the following result.

Theorem 4.1. *Assume A2, A3 and A4. Then, FBSDEs (4.4-4.5) has a unique solution $(Q(t), W(t), Z(t))$ such that the pair $(Q(t), W(t))$ is càdlàg. Furthermore, if for some $\mathbb{R}^d$-valued stochastic process $\{\alpha_j(t)\}_{j=1}^d$, $\alpha_j(t) \geqslant 0$, relation* (4.3) *holds in the space $\mathcal{S}$, then $\{\alpha_j(t)\}_{j=1}^d$ is a replicating portfolio.*

It is evident that Lévy-type markets pose theoretical questions that had never been raised in the Brownian motion framework. We conclude by reenforcing the idea that the impossibility of replicating every potential contingent claim is rather an expected characteristic due to the complexity of the price formation in Lévy-type markets, than a drawback of our model.

Acknowledgements. The research of E. Shamarova was funded by the European Regional Development Fund through the program COMPETE and by the Portuguese Government through the FCT (Fundação para a Ciência e a Tecnologia) under the project PEst-C/MAT/UI0144/2011. R. S. Pereira was supported by the Ph.D. grant SFRH7BD/51172/2010 of FCT. The authors thank Wolfgang Polasek for useful comments.

References

[1] D. Applebaum, "Lévy Processes and Stochastic Calculus", Cambridge University Press, 2009.

[2] F. Black and M. Scholes, "The pricing of options and corporate liabilities", J. Political Economy, Vol. 81, pp. 637–659, 1973.

[3] J. Cvitanic and J. Ma, "Hedging options for a large investor and forward-backward SDEs", The Annals of Applied Probability, Vol 6, No. 2, pp. 370–398, 1996.

[4] E. Eberlein and U. Keller, "Hyperbolic distributions in finance", Bernoulli, Vol. 1, No. 3, pp. 281–299, 1995.

[5] A. Gulisashvili and I.A. van Casteren, "Non-autonomous Kato Classes and Feynman-Kac Propagators", World Scientific, 2006.

[6] M. el Otmani, "Backward stochastic differential equations associated with Lévy processes and partial integro-differential equations", Communications on Stochastic Analysis, Vol. 2, No. 2, pp. 277–288, 2008.

[7] P. Protter, J. Ma and J. Yong, "Solving forward-backward stochastic differential equations explicitly – a Four Step Scheme", Probability Theory and Related Fields, Vol. 98, pp. 339–359, 1994.

[8] D. Nualart and W. Schoutens, "Chaotic and predictable representations for Lévy processes", Stochastic Processes and their Applications, Vol. 90, pp. 109–122, 2000.

[9] D. Nualart and W. Schoutens, "BSDEs and Feynman-Kac formula for Lévy processes with applications in finance", Bernoulli, Vol. 7, pp. 761–776, 2001.

[10] W. Schoutens, "Stochastic processes and orthogonal polynomials", Springer, 2000.

[11] H. Tanabe, "Equations of evolution", Pitman, London-San Francisco-Melbourne, 1979.