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On Solvability of Stochastic Differential Inclusions with Current Velocities. II

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Abstract

An existence of solution theorem is obtained for stochastic differential inclusions given in terms of the so-called current velocities (direct analogs of ordinary velocity of deterministic systems) and quadratic mean derivatives (giving information on the diffusion coefficient) on the flat n-dimensional torus. The set-valued current velocity part has a smooth selector and the set-valued quadratic part takes values in symmetric $(2, 0)$ tensor fields with given (constant) determinant. The values of current velocity parts are closed and bounded. The right-hand side of quadratic part is upper semi-continuous, its values are closed, bounded and satisfy some additional hypotheses that replace the convexity condition.

Key words: Mean derivatives; stochastic differential inclusions; group $SL(n)$

Mathematics Subject Classification 2010: 58J65 60H10

Introduction

In [2] an existence theorem for differential inclusions with current velocities having single-valued part for quadratic mean derivative, was obtained under some very strong conditions. Then in [8] in some sense the opposite problem was considered, i.e., the current velocity part was single-valued and smooth while the quadratic part was set-valued and took values in the symmetric $(2, 0)$-tensors with unit determinant.

In this paper we deal with the case where both current velocity and quadratic parts are set-valued. We assume that the current velocity part has a smooth selector (some conditions, under which this happens, are obtained, say, in [1, 4]). For the set-valued quadratic right-hand side we assume that it takes values in the symmetric $(2,0)$-tensors with constant (equal to some $C > 0$) determinant. The values of current velocity parts are closed and bounded. The right-hand side of quadratic part is upper semi-continuous and its values are closed, bounded and satisfy some additional hypotheses that replace the convexity condition.

To avoid some technical difficulties we consider the inclusions on a flat n-dimensional torus $\mathcal{T}^n$. This means that the torus is considered as a quotient space of $\mathbb{R}^n$ relative to the integral lattice and that the Riemannian metric on $\mathcal{T}^n$ is inherited from the Euclidean metric in $\mathbb{R}^n$. Everywhere below we use the operations of addition and subtraction of points and integration in $\mathcal{T}^n$ as in $\mathbb{R}^n$ modulo factorization relative to the integral lattice. The construction and notation of stochastic integrals and stochastic differential equations on $\mathcal{T}^n$ are the same as in $\mathbb{R}^n$ because of the use of Euclidean metric.

The detailed exposition of preliminary notions and facts used in the paper, can be found in [7].

For convenience, here we repeat some basic definitions and constructions from [8].

Everywhere in the paper we use Einstein's convention of summation relative to a shared upper and lower index (see, e.g., [7]).

1 Preliminaries on mean derivatives

Consider the n-dimensional flat torus $\mathcal{T}^n$. We shall deal with stochastic processes in $\mathcal{T}^n$ given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$, $t \in [0, T] \subset \mathbb{R}$.

Denote by $\mathcal{P}_t^\xi$ the sub-σ-algebra of $\mathcal{F}$ generated by preimages of Borel sets from ƀ $\mathcal{T}^n$ by all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \le s \le t$; $\mathcal{P}_t^\xi$ is called the "past" for $\xi(t)$.

Denote by $\mathcal{N}_t^\xi$ the sub-σ-algebra of $\mathcal{F}$ generated by preimages of Borel sets from $\mathcal{T}^n$ by the mapping $\xi(t) : \Omega \to \mathcal{T}^n$; $\mathcal{N}_t^\xi$ is called the "present" for $\xi(t)$.

The sub-σ-algebras $\mathcal{P}_t^\xi$ and $\mathcal{N}_t^\xi$ for all t are supposed to be complete, i.e., containing all sets of probability zero. Obviously $\mathcal{N}_t^\xi$ is a sub-σ-algebra in $\mathcal{P}_t^\xi$.

For the sake of convenience we denote by E_t^ξ the conditional expectation $E(\cdot|\mathcal{N}_t^\xi)$ with respect to $\mathcal{N}_t^\xi$ for $\xi(t)$.

As in [9, 10, 11], we introduce the following notions of forward and backward mean derivatives.

Definition 1.1. *(i) The forward mean derivative $D\xi(t)$ of $\xi(t)$ at the time* *instant t is an L_1 random element of the form*

$$D\xi(t)=\lim_{\triangle t\to+0}E_t^{\xi}(\frac{\xi(t+\triangle t)-\xi(t)}{\triangle t}),\qquad (1.1)$$

where the limit is supposed to exist in $L_1(\Omega,\mathcal{F},\mathsf{P})$ and $\triangle t\to+0$ means that $\triangle t$ tends to 0 and $\triangle t>0$.

(ii) The backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at t is the L_1-random element

$$D_*\xi(t)=\lim_{\Delta t\to+0}E_t^{\xi}(\frac{\xi(t)-\xi(t-\Delta t)}{\Delta t})\qquad (1.2)$$

where (as well as in (i)) the limit is assumed to exist in $L^1(\Omega,\mathcal{F},\mathsf{P})$ and $\Delta t\to+0$ means that $\Delta t\to 0$ and $\Delta t>0$.

As usual in the machinery of conditional expectation (see, e.g., [12]), there exist Borel measurable vector fields $a^{\xi}(t,m)$ and $a_*^{\xi}(t,m)$ such that $D\xi(t)=a^{\xi}(t,\xi(t))$ and $D_*\xi(t)=a_*^{\xi}(t,\xi(t))$, respectively (see [9, 10, 11]).

Definition 1.2. *The derivative $D_S=\frac{1}{2}(D+D_*)$ is called the symmetric mean derivative. The derivative $D_A=\frac{1}{2}(D-D_*)$ is called the antisymmetric mean derivative.*

Consider the vectors

$$v^{\xi}(t,x)=\frac{1}{2}(a^{\xi}(t,x)+a_*^{\xi}(t,x))$$

and

$$u^{\xi}(t,x)=\frac{1}{2}(a^{\xi}(t,x)-a_*^{\xi}(t,x)).$$

Definition 1.3. *$v^{\xi}(t)=v^{\xi}(t,\xi(t))=D_S\xi(t)$ is called the current velocity of the process $\xi(t)$; $u^{\xi}(t)=u^{\xi}(t,\xi(t))=D_A\xi(t)$ is called the osmotic velocity of the process $\xi(t)$.*

The physical meaning of current velocity is a direct analog of the ordinary velocity of a deterministic process. The osmotic velocity measures how fast the randomness increases. This interpretation becomes clear from the following features of v^{ξ} and u^{ξ} (see [11]).

Consider an autonomous smooth field of non-degenerate linear operators $A(x):\mathbb{R}^n\to\mathbb{R}^n$, $x\in\mathcal{T}^n$. Suppose that $\xi(t)$ is a diffusion type process whose diffusion integrand is $A(\xi(t))$. Then its diffusion coefficient $A(x)A^*(x)$ is a smooth field of symmetric positive definite $(2,0)$-tensors with matrices $\alpha(x)=(\alpha^{ij}(x))$. Since all those matrices are non-degenerate, the field of inverse matrices (α_{ij}) exists and is smooth and at any x the matrix $(\alpha_{ij})(x)$ is symmetric and positive definite. Thus it defines a new Riemannian metric (symmetric positive definite $(0,2)$-tensor field) $\alpha(\cdot,\cdot)=\alpha_{ij}dx^i dx^j$ on $\mathbb{R}^n$. Consider the Riemannian volume form of this Riemannian metric $\Lambda_\alpha=\sqrt{\det(\alpha_{ij})}dx^1\wedge dx^2\wedge\dots\wedge dx^n$.

Denote by $\rho^\xi(t,x)$ the probability density of $\xi(t)$ with respect to the volume form $dt\wedge\Lambda_\alpha=\sqrt{\det(\alpha_{ij})}dt\wedge dx^1\wedge dx^2\wedge\dots\wedge dx^n$ on $[0,T]\times\mathcal{T}^n$, i.e., for any continuous bounded function $f:[0,T]\times\mathcal{T}^n\to\mathbb{R}$ the relation

$$\int_0^T E(f(t,\xi(t)))dt=\int_0^T(\int_\Omega f(t,\xi(t))d\mathsf{P})dt=\int_{[0,T]\times\mathbb{R}^n} f(t,x)\rho^\xi(t,x)dt\wedge\Lambda_\alpha \quad (1.3)$$

holds. Then

$$u^\xi(t,x)=\frac{1}{2}Grad\log\rho^\xi(t,x)=Grad\log\sqrt{\rho^\xi(t,x)}, \quad (1.4)$$

where $Grad$ denotes the gradient with respect to the Riemannian metric $\alpha(\cdot,\cdot)$.

For $v^\xi(t,x)$ and $\rho^\xi(t,x)$ the so called equation of continuity

$$\frac{\partial\rho^\xi(t,x)}{\partial t}=-Div(v^\xi(t,x)\rho^\xi(t,x)) \quad (1.5)$$

holds, where Div denotes divergence with respect to the Riemannian metric $\alpha(\cdot,\cdot)$.

Following [2] we introduce the differential operator D_2 that differentiates an L_1 random process $\xi(t)$, $t\in[0,T]$ according to the rule

$$D_2\xi(t)=\lim_{\triangle t\to+0}E_t^\xi\left(\frac{(\xi(t+\triangle t)-\xi(t))(\xi(t+\triangle t)-\xi(t))^*}{\triangle t}\right), \quad (1.6)$$

where $(\xi(t+\triangle t)-\xi(t))$ is considered as a column vector (vector in $\mathbb{R}^n$), $(\xi(t+\triangle t)-\xi(t))^*$ is a row vector (transposed, or conjugate vector) and the limit is supposed to exists in $L_1(\Omega,\mathcal{F},\mathsf{P})$. We emphasize that the matrix product of a column on the left and a row on the right is a matrix with rank 1 but after passing to limit and taking conditional expectation $D_2\xi(t)$ becomes a symmetric semi-positive definite matrix function on $[0,T]\times\mathbb{R}^n$ (in many cases positive definite). We call D_2 the quadratic mean derivative. It takes values in the set $(2,0)$-tensors having symmetric positive semi-definite matrices.

As mentioned above, the notion of current velocity is analogous to ordinary velocity for a non-random process. Thus, from the physical point of view, it is an important problem to study equations and inclusions with current velocities.

Let $v(t,m)$ be a vector field and $\alpha(t,m)$ be a symmetric positive semi-definite $(2,0)$-tensor field on $\mathcal{T}^n$. The system

$$\begin{cases} D_S\xi(t) &= v(t,\xi(t)) \\ D_2\xi(t) &= \alpha(t,\xi(t)) \end{cases} \quad (1.7)$$

is called *the first order differential equation with current velocities.*

Note that equation (1.7) on the flat torus $\mathcal{T}^n$ can be considered as an equation on $\mathbb{R}^n$ periodic in space variables.

Definition 1.4. *We say that (1.7) on $\mathcal{T}^n$ has a solution on $[0,T]$ with initial condition $\xi(0)=\xi_0$ if there exists a probability space $(\Omega,\mathcal{F},\mathsf{P})$ and a process $\xi(t)$ given on $(\Omega,\mathcal{F},\mathsf{P})$ and taking values in $\mathcal{T}^n$ such that $\xi(0)=\xi_0$ and for almost all $t\in[0,T]$ equation (1.7) is satisfied P-a.s. by $\xi(t)$.*

Theorem 1.1. *Let $v:[0,T]\times\mathcal{T}^n\to\mathbb{R}^n$ be smooth and $\alpha:\mathcal{T}^n\to \mathrm{S}_+(n)$ be smooth and autonomous (so it determines the Riemannian metric $\alpha(\cdot,\cdot)$ on $\mathcal{T}^n$, introduced above). Let ξ_0 be a random element with values in $\mathcal{T}^n$ whose probability density ρ_0 with respect to the volume form Λ_α of $\alpha(\cdot,\cdot)$ on $\mathcal{T}^n$ (see above) is smooth and nowhere equal to zero. Then for the initial condition $\xi(0)=\xi_0$ equation* (1.7) *has a solution that is well-defined on the entire interval $t\in[0,T]$.*

Theorem 1.1 is a simple corollary to [2, Theorem 4.1] (see also [7, Theorem 8.50]). Here we use the fact that on the compact manifold $\mathcal{T}^n$ the right-hand sides of (1.7) are uniformly bounded and so the hypothesis of [2, Theorem 4.1] is fulfilled.

Introduce $p_0=\log\rho_0$ and consider $p(t,m)=\log\rho^{\xi}(t,m)$ where $\rho^{\xi}(t,m)$ is the density (1.3) corresponding to the solution $\xi(t)$ of (1.7). It is shown in the proof of [2, Theorem 4.1] (see also [7, Theorem 8.50]) that $p(t,m)$ is well-posed and takes the form

$$p(t,m)=p_0(g_{-t}(m))-\int_0^t(\mathrm{Div}\ v)(s,g_s(g_{-t}(m))\ ds \tag{1.8}$$

where Div is the divergence with respect to $\alpha(\cdot,\cdot)$ and g_t is the flow of smooth vector field $v(t,m)$.

2 Some technical constructions

Everywhere below we denote by $S_+(n)$ the set of symmetric positive definite $n\times n$ matrices.

In [2], on the basis of Hauss decomposition (see [14]), every matrix $\alpha\in S_+(n)$ is represented in the form $\alpha=\zeta\delta\zeta^*$ where ζ is a lower-triangle matrix with units on the diagonal, ζ^* is its transposed matrix, i.e., an upper-triangle matrix with units on the diagonal, and δ is a diagonal matrix whose angular minors (note that they all a positive) coincide with those of α. Denote the diagonal elements of δ by $\delta_1,\dots\delta_n$. Then the matrix $A=\zeta\sqrt{\delta}$ where $\sqrt{\delta}$ is the diagonal matrix with $\sqrt{\delta_1},\dots,\sqrt{\delta_n}$ on the diagonal, is such that $\alpha=AA^*$. If we deal with a continuous (smooth, measurable) field $\alpha(t,m)$, $t\in\mathbb{R}$ and $m\in\mathcal{T}^n$, of the above matrices, the corresponding matrices $A(t,m)$ are also continuous (smooth, measurable, respectively).

Denote by $\mathsf{T}_-(n)$ the set of lower-triangle $n\times n$ matrices with zeros on the diagonal that is obviously a linear subspace in $\mathbb{R}^{n^2}$, the linear space of all

$n \times n$ matrices. It is evident that the matrix ζ introduced above, belongs to the linear submanifold $\mathsf{T}_-(n) + I$ in $\mathbb{R}^{n^2}$ where I is the unit $n \times n$ matrix. Denote by $\mathsf{T} : S_+(n) \to \mathsf{T}_-(n)$ the smooth mapping that sends $\alpha \in S_+(n)$ to

$$\mathsf{T}\alpha = \zeta - I \in \mathsf{T}_-(n). \tag{2.1}$$

Now specify some $C > 0$ and denote by S_{LC} the set of matrices from $S_+(n)$ having determinants equal to C. In particular, this means that $\delta_1 \cdot ... \cdot \delta_n = C$ and $\sqrt{\delta_1} \cdot ... \cdot \sqrt{\delta_n} = \sqrt{C}$ where the dot denotes multiplication.

Denote by $\mathsf{L}_0(n)$ the linear subspace in $\mathbb{R}^n$ consisting of vectors $X = (X^1, \dots, X^n)$ such that $X^1 + ... + X^n = 0$.

Introduce the smooth mapping $\mathsf{L_C} : S_{LC} \to \mathsf{L}_0$, that sends a symmetric matrix $\alpha \in S_{LC}$ to

$$\mathsf{L_C}(\alpha) = (\log \frac{\sqrt{\delta_1}}{\sqrt{C}}, ..., \log \frac{\sqrt{\delta_1}}{\sqrt{C}}) \in \mathsf{L}_0(n). \tag{2.2}$$

Note that $\mathsf{T}_-(n)$ and $\mathsf{L}_0(n)$ are linear spaces and so the notion of convex set is well-posed in them.

Lemma 2.1. *For every smooth autonomous* $(2, 0)$*-tensor field* $\alpha(m)$ *on flat torus* $\mathcal{T}^n$ *with values in* S_{LC}*:*

(i) The volume form Λ_α *of the corresponding Riemannian metric* $\alpha(\cdot, \cdot)$ *(see above) equals* $\sqrt{C}\Lambda_E$ *where* Λ_E *is the volume form of the Euclidean metric on* $\mathcal{T}^n$ *inherited from* $\mathbb{R}^n$ *after factorization with respect to the integral lattice.*

(ii) For every smooth vector field $v(t, m)$ *on* $\mathcal{T}^n$ *its divergence* $\mathrm{Div}\, v$ *with respect to* Λ_α *coincides with ordinary divergence* $\mathrm{div}\, v$ *(i.e., with respect to* Λ_E*).*

(iii) For every random element having values in $\mathcal{T}^n$*, its distribution with respect to* Λ_α *equals the distribution with respect to* Λ_E *divided by* $\sqrt{C}$*.*

Proof. Indeed, $\Lambda_\alpha = \sqrt{det(\alpha_{ij})} dq^1 \wedge \cdots \wedge dq^n$ and since $det(\alpha_{ij}) = C$, we obtain that it equals $\sqrt{C}\Lambda_E = Cdq^1 \wedge \cdots \wedge dq^n$.

Recall that the divergence $\mathrm{Div}\, v$ is found from the equality

$$\mathcal{L}_v \Lambda_\alpha = (\mathrm{Div} v)\Lambda_\alpha$$

where $\mathcal{L}_v$ is the Lie derivative by v (see details, e.g., in [7]). Recall also that $\mathcal{L}_v \Lambda_\alpha = d(v \rfloor \Lambda_\alpha)$ where $\rfloor$ denotes the internal multiplication of vectors and differential forms. Since C is constant, $d(v \rfloor \Lambda_\alpha) = \frac{\partial v^i}{\partial q^i} \sqrt{C}\Lambda_E = \frac{\partial v^i}{\partial q^i} \Lambda_\alpha$. Hence $\mathrm{Div} v = \frac{\partial v^i}{\partial q^i} = \mathrm{div} v$.

Assertion (iii) follows from (i). $\square$

3 The main result

Let $\mathbf{v}(t,m)$ be a set-valued vector field and $\boldsymbol{\alpha}(t,m)$ a set-valued symmetric positive semi-definite $(2,0)$-tensor field on $\mathcal{T}^n$. The system of the form

$$\begin{cases} D_S\xi(t) \in \mathbf{v}(t,\xi(t)), \\ D_2\xi(t) \in \boldsymbol{\alpha}(t,\xi(t)). \end{cases} \tag{3.1}$$

is called a first order differential inclusion with current velocities. The notion of solution of (3.1) is quite analogous to that from Definition 1.4.

Below we suppose that the set-valued field $\boldsymbol{\alpha}$ satisfies the following condition:

Condition 3.1. *(i) The set-valued $(2,0)$-tensor field $\boldsymbol{\alpha}$ on $\mathcal{T}^n$ takes values in S_{LC}; it is autonomous and upper semicontinuous.*

(ii) The values of $\boldsymbol{\alpha}$ are closed and uniformly bounded.

(iii) For every $m \in \mathcal{T}^n$ the set $\mathsf{T}(\boldsymbol{\alpha}(m))$ (see (2.1)) is convex in $\mathsf{T}_-(n)$ and the set $\mathsf{L}_C(\boldsymbol{\alpha}(m))$ (see (2.2)) is convex in $\mathsf{L}_0(n)$.

For $\mathbf{v}(t,m)$ we suppose that it has a smooth single-valued selector denoted by $v(t,m)$. Recall that some conditions, under which a set-valued mapping has a smooth selector, are obtained in [1, 4].

Theorem 3.1. *Let $\mathbf{v}(t,m)$ be a set-valued vector field on $\mathcal{T}^n$ having smooth single-valued selector $v(t,m)$ for $t \in [0,T]$. Let also $\boldsymbol{\alpha}(m)$ be a set-valued $(2,0)$-tensor field that satisfies Condition 3.1. Consider a random ξ_0 element with values in $\mathcal{T}^n$ whose probability density with respect to the volume form Λ_E equals $\sqrt{C}\rho_0$ where ρ_0 is smooth and nowhere equal to zero. Then for the initial condition $\xi(0) = \xi_0$ inclusion (3.1) has a solution that is well-defined on the entire interval $t \in [0,T]$.*

Proof. Specify a sequence of positive numbers $\varepsilon_k \to 0$. Since the mappings T and L_C are smooth, the set-valued mappings $\mathsf{T}\boldsymbol{\alpha}$ with values in $\mathsf{T}_-(n)$ and $\mathsf{L}_C\boldsymbol{\alpha}$ with values in $\mathsf{L}_0(n)$ are upper semicontinuous since such is $\boldsymbol{\alpha}$. By Condition 3.1 their values are convex, closed and uniformly bounded. Then by [3, Theorem 2] (see also [7, Theorem 4.11]) there exist the sequences of single-valued continuous ε_k-approximations that point-wise converge to Borel measurable selectors of $\mathsf{T}\boldsymbol{\alpha}$ and $\mathsf{L}_C\boldsymbol{\alpha}$, respectively. Without loss of generality those approximations can be supposed as smooth. Thus there exists a sequence α_k of single-valued smooth uniformly bounded $(2,0)$-tensor fields with values in S_{LC} that point-wise converges to a Borel measurable selector $\alpha(m)$ of $\boldsymbol{\alpha}(m)$. The components of $\alpha_k(m)$ will be denoted as α_k^{ij}.

Construct the Riemannian metrics $\alpha_k(\cdot,\cdot)$ from tensor fields $\alpha_k(m)$.

Consider the sequence of equations

$$\left\{\begin{array}{lcl} D_S\xi(t) & = & v(t,\xi(t)) \\ D_2\xi(t) & = & \alpha_k(t,\xi(t)) \end{array}\right. . \tag{3.2}$$

Note that by Lemma 2.1 for those equations we can consider the same initial value ξ_0 since its densities with respect to all $\alpha_k(\cdot,\cdot)$ coincide with ρ_0. All equations (3.2) satisfy the conditions of Theorem 1.1, so there exist solutions $\xi_k(t)$ of those equations.

From Lemma 2.1 it follows that the functions $p(t,m)$ defined by (1.8) for all $\xi_k(t)$ coincide (in particular this means that the densities $\rho(t,m)$ coincide as well).

For a solution $\xi_k(t)$ the osmotic velocity takes the form

$$u_k(t,m) = \frac{1}{2} Grad_k p(t,m)$$

where $Grad_k$ is the gradient calculated with respect to $\alpha_k(\cdot,\cdot)$. One can easily show by the definition of gradient that the coordinate presentation of $Grad_k p(t,m)$ has the form $(Grad_k p(t,m))^i = \alpha_k^{ij}\frac{\partial p}{\partial q^j}$. Thus, from the hypothesis and from Condition 3.1 it follows that all $u_k(t,m)$ are smooth and uniformly bounded. Since the components α_k^{ij} point-wise converge to the components α^{ij} of $\alpha(m)$, the vectors $u_k(t,m)$ point-wise converge to $u(t,m)$ with components $u^i = \frac{1}{2}\alpha^{ij}\frac{\partial p}{\partial q^j}$.

Introduce the vector fields $a_k(t,m) = v(t,m) + u_k(t,m)$, denote its point-wise limit by $a(t,m)$. As it is mentioned in Section , every $\alpha_k(m)$ can be represented as $\alpha_k(m) = A_k(m)A_k^*(m)$. By construction the sequence $A_k(m)$ point-wise converge to the Borel-measurable field $A(m)$ such that $\alpha(m) = A(m)A^*(m)$.

Consider the sequence of Itô type stochastic differential equations

$$\xi_k(t) = \xi_0 + \int_0^t a_k(s,\xi(s))ds + \int_0^t A_k(s,\xi(s)dw(s) \tag{3.3}$$

on $\mathcal{T}^n$. Since the coefficients of (3.3) for all k are smooth and bounded, all the equations have unique strong solutions well-defined on the entire interval $[0,T]$. On the Banach manifold $C^0([0,T],\mathcal{T}^n)$ of continuous curves in $\mathcal{T}^n$ introduce the σ-algebra $\mathcal{C}$ generated by cylinder sets and denote by μ_k the measure on $(C^0([0,T],\mathcal{T}^n),\mathcal{C})$ generated by the solution $\xi_k(t)$ of (3.3). Introduce also the family of complete sub-σ-algebras $\mathcal{P}_t$ generated by cylinder sets with bases in $[0,t]$, $t\in[0,T]$.

Since equations (3.3) can be considered as the ones in $\mathbb{R}^n$ with space-periodic coefficients, one can apply [6, Corollary III.2] and show that the set $\{\mu_k\}$ of measures on $(C^0([0,T],\mathcal{T}^n),\mathcal{C})$ is weakly compact. Hence we can select a sub-sequence that weakly converges to a certain measure μ. Without loss or generality we can suppose that the sequence μ_k weakly converges to μ. Consider the coordinate process $\xi(t)$ on the probability space $(C^0([0,T],\mathcal{T}^n),\mathcal{C},\mu)$, i.e., for every elementary event $x(\cdot)\in C^0([0,T],\mathcal{T}^n)$, by definition $\xi(t,x(\cdot)) = x(t)$. Note that $\mathcal{P}_t$ is the "past" for $\xi(t)$. As usual, $\mathcal{N}_t^\xi$ is a sub-σ-algebra of $\mathcal{P}_t$.

By construction, $D_S\xi_k(t) = v(t,\xi_k(t))$ for all k. This means that for every bounded continuous real-valued function f on $C^0([0,T],\mathcal{T}^n)$ measurable with respect to $\mathcal{N}_t^\xi$, the equality

$$\lim_{\Delta t\to 0}\int_{C^0([0,T],\mathcal{T}^n)}[x(t+\Delta t)-x(t-\Delta t)-v(t,x(t))]f(x(\cdot))d\mu_k = 0$$

holds for all k.

Specify an arbitrary $\varepsilon>0$. Since μ_k weakly converges to μ, there exists $K(\varepsilon)$ such that for all $k>K(\varepsilon)$

$$\begin{aligned}\|\int_{C^0([0,T],\mathcal{T}^n)}&[x(t+\Delta t)-x(t-\Delta t)-v(t,x(t))]f(x(\cdot))d\mu_k\\ -\int_{C^0([0,T],\mathcal{T}^n)}&[x(t+\Delta t)-x(t-\Delta t)-v(t,x(t))]f(x(\cdot))d\mu\| < \varepsilon.\end{aligned}$$

Hence,

$$\|\lim_{\Delta t\to 0}\int_{C^0([0,T],\mathcal{T}^n)}[x(t+\Delta t)-x(t-\Delta t)-v(t,x(t))]f(x(\cdot))d\mu\| < \varepsilon.$$

Since ε is an arbitrary positive number and f is an arbitrary function, measurable with respect to $\mathcal{N}_t^\xi$, this means that

$$D_S\xi(t) = v(t,\xi(t)). \tag{3.4}$$

By construction, for every $\xi_k(t)$ its quadratic derivative equals $\alpha_k(\xi_k(t))$. This means that for each $f(x(\cdot))$ as above we obtain the equality

$$\lim_{\Delta t\to 0}\int_{C^0([0,T],\mathcal{T}^n)}[(x(t+\Delta t)-x(t))(x(t+\Delta t)-x(t))^*-\alpha_k(x(t))]f(x(\cdot))d\mu_k = 0.$$

Since $\alpha_k(t,m)$ tends to $\alpha(t,m)$ as $k\to\infty$ point-wise, it tends a.s. with respect to all μ_k and with respect to μ. Specify $\delta>0$. By Egorov's theorem (see, e.g., [13]) for any i there exists a subset $\tilde{K}_\delta^i\subset C^0([0,T],\mathcal{T}^n)$ such that $(\mu_i)(\tilde{K}_\delta^i)>1-\delta$, and the sequence $\alpha_k(x(t))$ converges to $\alpha(x(t))$ uniformly on $\tilde{K}^i_\delta$. Introduce $\tilde{K}_\delta = \bigcup\limits_{i=0}^{\infty} \tilde{K}^i_\delta$. The sequence $\alpha_k(x(t))$ converges to $\alpha(x(t))$ uniformly on $\tilde{K}_\delta$ and $\mu_i(\tilde{K}_\delta) > 1-\delta$ for all i and $\mu(\tilde{K}_\delta) > 1-\delta$.

Note that $\alpha(x(t))$ is continuous on a set of full measure μ on $C^0([0,T],\mathcal{T}^n)$. Indeed, consider a sequence $\delta_i \to 0$ and the corresponding sequence $\tilde{K}_{\delta_i}$. By the above construction $\alpha(x(t))$ is a uniform limit of continuous functions on each $\tilde{K}_{\delta_i}$. Thus it is continuous on each $\tilde{K}_{\delta_i}$ and so, on every finite union $\bigcup\limits_{i=1}^{n} \tilde{K}_{\delta_i}$. Evidently $\lim\limits_{n\to\infty} \mu(\bigcup\limits_{i=1}^{n} \tilde{K}_{\delta_i}) = 1$.

Because of the above uniform convergence on $\tilde{K}_\delta$ for all k and boundedness of $f(x(\cdot))$ we get that for k large enough

$$\| \int_{\tilde{K}_\delta} [\alpha_k(x(t)) - \alpha(x(t))] f(x(\cdot)) d\mu_k \| < \delta.$$

Since $f(x(\cdot))$ is bounded, there is some $\Xi > 0$ such that $|f(x(\cdot))| < \Xi$ for all $x(\cdot)$. Recall that all $\alpha_k(m)$ and $\alpha(m)$ are uniformly bounded, i.e., their norms are not greater than a certain $Q > 0$. Then, since $\mu_k(C^0([0,T],\mathcal{T}^n)\backslash \tilde{K}_\delta) < \delta$ for all k large enough, we obtain that

$$\| \int_{C^0([0,T],\mathcal{T}^n)\backslash \tilde{K}_\delta} [\alpha_k(x(t)) - \alpha(x(t))] f(x(\cdot)) d\mu_k \| < 2\delta Q \Xi$$

for all k large enough. Since δ is an arbitrary positive number, we obtain that

$$\lim_{k\to\infty} \int_{C^0([0,T],\mathcal{T}^n)} [\alpha_k(x(t)) - \alpha(x(t))] f(x(\cdot)) d\mu_k = 0.$$

The function $\alpha(x(t))$ is μ-a.s. continuous and bounded on $C^0([0,T],\mathcal{T}^n)$ (see above). Since in addition the measures μ_k weakly converge to μ, by Lemma from [5, Section VI.1] we obtain that

$$\lim_{k\to\infty} \int_{C^0([0,T],\mathcal{T}^n)} \alpha(x(t)) f(x(\cdot)) d\mu_k = \int_{C^0([0,T],\mathcal{T}^n)} \alpha(x(t)) f(x(\cdot)) d\mu.$$

Obviously

$$\lim_{k\to\infty} \int_{C^0([0,T],\mathcal{T}^n)} [(x(t+\Delta t) - x(t))(x(t+\Delta t) - x(t))^*] f(x(\cdot)) d\mu_k$$

$$= \int_{C^0([0,T],\mathcal{T}^n)} [(x(t+\Delta t) - x(t))(x(t+\Delta t) - x(t))^*] f(x(\cdot)) d\mu.$$

Thus

$$\lim_{\Delta t\to 0} \int_{C^0([0,T],\mathcal{T}^n)} [(x(t+\Delta t) - x(t))(x(t+\Delta t) - x(t))^* - \alpha(x(t))] f(x(\cdot)) d\mu = 0.$$

Since $f(x(\cdot))$ is an arbitrary bounded continuous function, measurable with respect to $\mathcal{N}_t^{\xi}$, this means that $D_2\xi(t) = \alpha(\xi(t))$. But by construction $\alpha(\xi(t)) \in \boldsymbol{\alpha}(\xi(t))$ μ-a.s.

Together with (3.4) this means that $\xi(t)$ is a solution of inclusion (3.1) we are looking for. □

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References

[1] S.M. Aseev, "Existence of a differentiable single-valued branch of the set-valued mapping", Nekotorye voprosy prikladnoy matematiki i programmnogo obespecheniya EVM, Moscow, pp. 36 – 39, 1982 (in Russian)

[2] S.V. Azarina and Yu.E. Gliklikh, "Differential inclusions with mean derivatives", Dynamic Sysytems and Applications, Vol. 16, No. 1, pp. 49–72, 2007.

[3] S.V. Azarina, Yu.E. Gliklikh and A.V. Obukhovskii, "Solvability of Langevin differential inclusions with set-valued diffusion terms on Riemannian manifolds", Applicable Analysis, Vol. 86, No. 9, pp. 1105–1116, 2007

[4] M.Sh. Farber, "On smooth selectors of the intersection of set-valued mappings", Izvestiya Akademii Nauk Azerbaijanskoy SSR, No. 6, pp. 23 – 28, 1979 (in Russian)

[5] I.I. Gihman and A.V. Skorohod, "Theory of Stochastic Processes, Vol. 1", Springer-Verlag New York, 1974.

[6] I.I. Gihman and A.V. Skorohod, "Theory of Stochastic Processes, Vol. 3", Springer-Verlag New York, 1979

[7] Yu.E. Gliklikh, "Global and Stochastic Analysis with Applications to Mathematical Physics", Springer-Verlag London, 2011.

[8] Yu.E. Gliklikh and A.V. Makarova, "On solvability of stochastic differential inclusions with current velocities", Applicable Analysis, DOI: 10.1080/00036811.2011.579565, pp. 1 – 9, 2011.

[9] E. Nelson, "Derivation of the Schrödinger equation from Newtonian mechanics", Phys. Reviews, Vol. 150, No. 4, pp. 1079-1085, 1966.

[10] E. Nelson, "Dynamical theory of Brownian motion", Princeton University Press, 1967

[11] E. Nelson, "Quantum fluctuations", Princeton University Press, 1985

[12] K.R. Parthasarathy, "Introduction to Probability and Measure", Springer-Verlag New York, 1978.

[13] Y. Yosida, "Functional Analysis", Springer-Verlag, Berlin et. al., 1965

[14] D.P. Zhelobenko, "Compact Lie groups and their representations", Amer. Math. Soc., Providence, RI, 1973