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Using Lyapunov Functions in Proofs of Existence of Solutions to Stochastic Equations

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Abstract

The method of Lyapunov functions is used to prove new existence theorems for stochastic equations in infinite dimensions. Existence of strong and generalized solutions is proved. Martingale solutions are discussed. Examples of application of the theorems are described. One of them is the stochastic equation of Navier–Stokes type.

Key words: Stochastic equation; strong solution; weak solution; Brownian motion with drift; Navie–Stokes equation; Lyapunov function; tight family of measures.

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1 Introduction

Theorems on existence of solutions to stochastic differential equations were proved many years ago. Now we see that they can not be applied to some equations used for modeling the real phenomena (see, e.g., [All07]). Therefore, new theorems are needed. We present one of such new theorems in this paper. Its proof is a generalization of reasoning that was used to prove the existence of solutions to the stochastic Navier–Stokes equation in [BT73], [VF80], [Cru89], and [FG95] (see [Gli11] for another approach). First steps of such generalization are presented in [BM11]. We make one step more. It is a general method to obtain the estimates required to apply the results of the previous steps. This method uses the Lyapunov functions. It was used to investigate finite-dimensional stochastic systems in [Has69]. Its infinite-dimensional generalization was presented in [Kir93] and [Kir94]. We describe an improvement of this generalization that can be used to continue the steps of [BM11].

We use the following notation.

H is a real separable Hilbert space. Its inner product is denoted by $(\cdot,\cdot)_H$.

Ξ is a normed space with the norm $\|\cdot\|_\Xi$. We assume that $\Xi \subset H$. The case where $\Xi = H$ is allowed.

Y is a normed space with the norm $\|\cdot\|_Y$. We assume that $H \subset Y$. The case where $Y = H$ is allowed.

The Wiener process we use here takes values in a Hilbert space W. We assume that the topological duals Y^* and W^* w.r.t. $(\cdot,\cdot)_H$ are such that the space $U = \Xi \cap W^* \cap Y^*$ is dense in H. Therefore, an orthonormal basis $e_1,\ e_2,\ \dots$ exists in H such that $e_k \in U$ for every $k = 1,\ 2,\ \dots$.

If W is finite-dimensional, then we assume that $W \subset \Xi$ and the Wiener process is simply

$$w(\cdot) = \sum_{k=1}^{\mathrm{dimW}} w_k(\cdot) e_k$$

where $w_k(\cdot)$ are independent standard Wiener processes in $\mathbb{R}$.

If W is infinite-dimensional, then we should assume that $W \supset H$ and that the natural embedding $H \subset W$ is a Hilbert–Schmidt operator. This implies that $W \neq H$.

By Sazonov theorem, $\forall t \in [0,\ \infty)$ and $\forall x \in W$ the map

$$p \in W^* \mapsto e^{i(p,x)_H - t\|p\|_H^2/2}$$

is the characteristic functional of a probability measure $P(t,x,dw)$ on the σ-algebra $\mathcal{B}(W)$ of the Borel subsets of W. The measures $P(t,x,dw)$ with various t and x form a family of Markov transition probabilities.

$C(\mathbb{R}_+;W)$ is the space of continuous curves $\mathbb{R}_+ \mapsto W$. Its topology is defined by the seminorms $p_k(w) = \max_{0\le t\le k} \|w(t)\|_W$ $(k = 1,2,\dots)$, or, equivalently, by the metric (quasinorm) $\rho(\cdot,\cdot)$ such that for any $w_1,\ w_2 \in C(\mathbb{R}_+;W)$

$$\rho(w_1,w_2) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{p_k(w_1-w_2)}{1+p_k(w_1-w_2)}\,.$$

The initial σ-algebra of measurable subsets of $C(\mathbb{R}_+;W)$ is the cylinder σ-algebra $\mathcal{C}$.

The probability measure on $\mathcal{C}$ is generated by the transition probabilities $P(t, x, dy)$ under the initial condition $w(0) = 0$. That means that for any $n \in \mathbb{N}$, $t_1 < t_2 < \cdots < t_n$ from $[0, \infty)$ and $G_1, \ldots, G_n \in \mathcal{B}(W)$

$$\mathcal{P}_w\big\{w(\cdot) \in C(\mathbb{R}_+; W) :\ w(t_1) \in G_1, \ldots ; w(t_n) \in G_n)\big\} =$$

$$\int_{G_1} P(t_1, 0, dw_1) \int_{G_2} P(t_2 - t_1, w_1, dw_2) \ldots \int_{G_n} P(t_n - t_{n-1}, w_{n-1}, dw_n).$$

By the Prokhorov theorem (see, e.g., [VTC85, Sec. I.3.5]) the measure $\mathcal{P}_w$ has a unique extension to the σ-algebra $\mathcal{B}(C(\mathbb{R}_+; W))$ of the Borel subsets of $C(\mathbb{R}_+; W)$.

Direct calculations prove that the characteristic functional of $\mathcal{P}_w$ is

$$\int\limits_{C(\mathbb{R}_+;W)} \exp\left[i \int\limits_0^\infty (w(t), d\nu(t))_H \right] d\mathcal{P}_w(w) = \exp\left\{ -\frac{1}{2} \int\limits_0^\infty \|\nu(t, \infty)\|_H^2 \, dt \right\},$$

where ν is any W^*-valued measure on $[0, \infty)$ with compact support.

The triple $(C(\mathbb{R}_+; W),\ \mathcal{B}(C(\mathbb{R}_+; W)),\ \mathcal{P}_w)$ is used as the standard probability space $(\Omega,\ \mathcal{F},\ \mathcal{P})$.

The expectation (i.e. the integral w.r.t. the measure $\mathcal{P} = \mathcal{P}_w$) is denoted by E.

Let $\Omega_0 = [0, 1[$, $\mathcal{B}(\Omega_0)$ be the Borel σ-algebra of subsets of Ω_0, and λ be the Lebesgue measure on $\mathcal{B}(\Omega_0)$. There exists a random element on $(\Omega_0,\ \mathcal{B}(\Omega_0),\ \lambda)$ with values in $C(\mathbb{R}_+, W)$ whose probability distribution is equivalent to $\mathcal{P}_w$ (see, e.g., [Bil68, Ch. I.4]). This random element is a Wiener process in W. It is denoted by w. Direct calculations prove that

$$\mathrm{E}\,(w(s), e_i)_H (w(t), e_j)_H = \delta_{ij} \min\{s, t\}.$$

in finite-dimensional and infinite-dimensional cases.

For any process η in W with the basic probability space $(\Omega,\ \mathcal{F},\ \mathcal{P})$, the symbol $\mathcal{F}_t^\eta$ denotes the minimal σ-algebra such that $\mathcal{F}_t^\eta \subset \mathcal{F}$ and $\forall s \in [0, t]$ the map $\omega \mapsto \eta(s, \omega)$ is $(\mathcal{F}_t^\eta,\ \mathcal{B}(W))$-measurable.

For any σ-algebra $\mathcal{R} \subset \mathcal{F}$, the symbol $\overline{\mathcal{R}}$ stands for the completion of $\mathcal{R}$ by all sets of $\mathcal{P}$-measure zero.

We investigate the stochastic equation

$$\xi(t) = \xi_0 + \int\limits_0^t A(\xi(s))\, ds + \widehat{\sigma}\, w(t). \tag{1.1}$$

It describes the so-called Brownian motion with drift. We concentrate on the stationary equation with the diffusion term $\widehat{\sigma}$ that does not depend on ξ for clarity of presentation only. The general case can be investigated similarly.

In the classical cases, $\Xi = H = Y$, $A(\cdot) : H \mapsto H$, and $\widehat{\sigma} \in \mathcal{L}(W; H)$. Equation (1.1) relates the processes in one and the same space H. Successive approximations can be used to prove that a strong solution of Eq. (1.1) exists and to investigate its properties. We discuss such case in the next section.

If $A(\cdot) : \Xi \mapsto Y$ and the spaces Ξ and Y are different, then Eq. (1.1) is intricate. Even its sense has to be specified. One way is to treat it as the equality of functionals on some space Φ whose topological dual Φ^* contains Y. Then $\Xi \subset \Phi^*$ and $H \subset \Phi^*$ also, because $\Xi \subset H \subset Y$, and Eq. (1.1) means that for any $\varphi \in \Phi$

$$\langle \varphi, \xi(t) \rangle = \langle \varphi, \xi_0 \rangle + \int_0^t \langle \varphi, A(\xi(s)) \rangle \, ds + \langle \varphi, \widehat{\sigma}\, w(t) \rangle. \tag{1.2}$$

By developing this idea one arrives at Galerkin approximations. We investigate them in Section 2.

One of examples of Eq. (1.2) is the stochastic equation of Navier–Stokes type. It describes a (velocity) field $u : U \subset \mathbb{R}^n \mapsto \mathbb{R}^n$ in open bounded domain U with smooth boundary ∂U. It is assumed that $\sum_{k=1}^n \partial u_k / \partial x_k = 0$ in U and $u = 0$ on ∂U.

The stochastic Navier–Stokes equation is the system of equations

$$u_i(t,x) = u_i(0,x) + \int_0^t \big[\Delta u_i(s,x) - \sum_{j=1}^n u_j(s,x) \frac{\partial u_i(s,x)}{\partial x_j}\big]\, ds$$

$$+ \sum_{j=1}^n \sigma_{i,j}\, w_j(t) \qquad (i = 1,\ 2,\ \ldots, n). \tag{1.3}$$

In this case,

$$\Xi = \{u \in W^{2,1}(U)^{\times n} : \sum_{k=1}^n \frac{\partial}{\partial x_k} u_k = 0 \text{ and } u|_{\partial U} = 0\},$$

$$H = L^2(U)^{\times n}, \quad Y = (C_0^{(1)}(U)^{\times n})^*, \quad W = \mathbb{R}^n.$$

$$A : u \in \Xi \mapsto A(u) = \Delta u - \sum_{k=1}^n u_k \frac{\partial}{\partial x_k} u \in Y \quad \widehat{\sigma} \in \mathcal{L}(\mathbb{R}^\backslash, \mathbb{R}^\backslash).$$

Obviously, the drift term $A(\cdot)$ cannot satisfy a Lipschitz condition. However,

$$\|A(u) - A(v)\|_Y \le \|u - v\|_\Xi [1 + \|u\|_\Xi + \|v\|_\Xi].$$

This and many other examples motivate investigations of the stochastic equations with locally Lipschitzian coefficients, mapping from one space into a different one.

2 Lyapunov functions and strong solutions

If $\Xi = H = Y$ and the map $A(\cdot)$ satisfies a Lipschitz condition, then equation (1.1) can be investigated just as in the case where Ξ is a finite-dimensional space. Such investigation was done in [Bac63], [Bac64], [Cha64], and [Cha65] (see also [Yor74]).

If $\Xi = H = Y$ and the map $A(\cdot)$ satisfies a Lipschitz condition only locally, then equation (1.1) can be investigated as the similar finite-dimensional equations were investigated in [Has69]. Of course some difficulties arise because the balls are not compact in infinite-dimensional normed spaces and traces may diverge. But we can overcome them and get a full theory on existence and uniqueness of strong solutions, Markov property, properties of corresponding semigroups (including the Feller property), ergodicity of solutions and existence of invariant measures ([Kir93], [Kir94]). We represent here a basic result of this investigation.

Let $\widehat{L}$ be the generator corresponding to (1.1), i.e.

$$\widehat{L}f(\xi) = f'(\xi)(A(\xi)) + \frac{1}{2}\mathrm{Tr}_{\mathrm{H}}\big[\widehat{\sigma}^{*}\mathrm{f}''(\xi)\widehat{\sigma}\big]. \tag{2.1}$$

Here $f: \ H \mapsto \mathbb{R}$ is any "good" function and f' and f'' are the first and the second Fréshet derivatives of f.

Theorem 2.1. ($\Xi = H = Y \subset W$) *Let these conditions be satisfied.*

1. *The drift term $A(\cdot)$ satisfies a local Lipschitz condition.*
2. *There is a non-negative function $V(\cdot) \in C^{(2)}_{\mathrm{loc}}(\Xi)$ such that*

   2.1 $\lim\limits_{r\to\infty} \inf\limits_{\|\xi\|_H > r} V(\xi) = \infty\,.$

   2.2 $\widehat{L}V(\cdot) \le c\,V(\cdot)$ *for some c.*

Then these assertions are valid.

1. *For any ξ_0 such that $\mathrm{E}V(\xi_0) < \infty$, and for $\mathcal{P}$-a.e. $\omega \in \Omega$, a global solution*
   $$t \in [0,\infty) \mapsto \xi(t,\omega,0,\xi_0) \in H$$
   of Eq. (1.1) *exists, is unique, and continuous.*
2. *For every $t \ge 0$,*
   $$\mathrm{E}V(\xi(t,\cdot,0,\xi_0)) \le e^{c\,t}\,\mathrm{E}V(\xi_0). \tag{2.2}$$
3. *The solution is $(\overline{\mathcal{F}^{w}_{t}})$-adapted,*
4. *The map $\omega \mapsto \xi(\cdot,\omega,0,\xi_0)$ is $(\overline{\mathcal{F}},\overline{\mathcal{F}})$-measurable.*

5. *The solution is a homogeneous Markov process. Its transition probability is* $t \in [0,\infty),\ h \in H,\ G \in \mathcal{B}(H) \mapsto \mathcal{P}\{\xi(t,\cdot,0,h) \in G\}$. *It is stochastically continuous, satisfies the Feller condition, and for any* $G \in \mathcal{B}(H)$ *is measurable as a function of two variables* t *and* h.

Remarks.

1. Any non-negative function $V(\cdot) \in C^{(2)}_{\rm loc}(\Xi)$ that satisfies the condition 2.1 is called a Lyapunov function.

2. If $A(\cdot)$ satisfies the global Lipschitz condition

$$\|A(\xi_1) - A(\xi_2)\|_H \le K\|\xi_1 - \xi_2\|_H,$$

then

$$\|A(\xi)\|_H \le K\|\xi\|_H + \|B(0)\|_H,$$

and conditions 2 are satisfied by $\xi \mapsto V(\xi) = \|\xi\|^2_H + 1$, because

$$\widehat{L}V(\xi) = 2(\xi, A(\xi))_H + \mathrm{Tr}\,\widehat{\sigma}^*\widehat{\sigma} \le 2\mathrm{K}\|\xi\|^2_{\mathrm{H}} + 2\|\mathrm{A}(0)\|_{\mathrm{H}}\|\xi\|_{\mathrm{H}} + \mathrm{Tr}\,\widehat{\sigma}^*\widehat{\sigma} \le$$

$$\le (2K+1)\|\xi\|^2_H + \|A(0)\|^2_H + \mathrm{Tr}\,\widehat{\sigma}^*\widehat{\sigma} \le \mathrm{c}\,\mathrm{V}(\xi)\,,$$

where $c = \max\{2K+1,\ \|A(0)\|^2_H + \mathrm{Tr}\,\widehat{\sigma}^*\widehat{\sigma}\}$.

3. The dissipativity condition

$$2(A(\xi),\xi)_H + \mathrm{Tr}_H\sigma\sigma^* \le c\,\|\xi\|^2_H + D$$

is often used to prove the existence of solutions to stochastic partial differential equations. It looks like condition 2.2 but it is stronger and more restrictive, because it implies condition 2.2 with $V(\xi) = \|\xi\|^2_H + D/c$ and because there are non-dissipative systems that satisfy the conditions 2. For example, the system

$$\begin{cases} d\xi_1 &= \ \ \xi_2^3\, dt + dw_1 \\ d\xi_2 &= -\xi_1\, dt + dw_2 \end{cases} \tag{2.3}$$

does not satisfy the dissipativity condition, because

$$2(A(\xi),\xi)_H + \mathrm{Tr}_H\sigma^*\sigma = 2\xi_1\xi_2(\xi_2^2 - 1) + 2 = 2\|\xi\|^4_H - 2\|\xi\|^2_H + 2$$

if $\xi_1 = \xi_2$.

On the other hand, the system satisfies conditions 2 with $V(\xi) = \xi_1^2 + \xi_2^4/2 + 2$ and $c = 2$, because

$$\widehat{L}V(\xi) = 3\xi_2^2 + 1 \le 2\xi_1^2 + \xi_2^4 + 4 = 2V(\xi).$$

4. There are systems that have global solutions because their drift terms $A(\cdot)$ satisfy the coerciveness condition

$$\lim_{r\to\infty}\ \inf\{(\xi, A(\xi))_H :\ \xi \in \Xi,\ \|\xi\|_H > r\} = -\infty\,.$$

(Such drift terms prevent the solutions of going to infinity for a finite time.)

The coerciveness condition is often used to prove the existence of solutions to stochstic partial differential equations. It is stronger and more restrictive than conditions 2, because it implies them with $V(\xi) = \|\xi\|_H^2 + \dots$ ($\widehat{L}V(\xi)$ becomes negative for big $\|\xi\|_H$) and because there are systems with non-coercive drifts that satisfy the conditions 2, e.g., the system (2.3).

5. The condition that the map $A(\cdot)$ is monotone, i.e.

$$(A(\xi_1) - A(\xi_2), \xi_1 - \xi_2)_H \leq -c\|\xi_1 - \xi_2\|_H^2, \tag{2.4}$$

with some $c > 0$, is often used to prove the existence of solutions to stochastic partial differential equations. It is stronger and more restrictive than conditions 2, because it implies them with $V(\xi) = \|\xi\|_H^2$ and because there are systems with non-monotone $A(\cdot)$'s that satisfy the conditions 2. For example, such is the system (2.3).

Inequality (2.4) with $\xi_2 = 0$ is

$$(A(\xi_1), \xi_1)_H \leq (A(0), \xi_1)_H - c\|\xi_1\|_H^2.$$

This implies the coerciveness condition.

Examples of applications of Theorem 2.1.

1. *Van-der-Paul generator with white noise* is described by the system

$$\begin{cases} dx_t &= v_t\, dt\,, \\ dv_t &= [-x_t + \varepsilon(1 - x_t^2)v_t]dt + \sigma dw_t, \end{cases} \tag{2.5}$$

were $\varepsilon \geq 0$.

This system is fundamental in statistical radiotechnics. Theorem 2.1 applies to it with the Lyapunov function

$$V(x, v) = \left[v + \varepsilon\frac{x^3}{3} - \varepsilon x\right]^2 + x^2 + \frac{\sigma^2}{2\varepsilon}\,,$$

because $\widehat{L}V(x, v) = 2\varepsilon x^2 - 2\varepsilon x^4/3 + \sigma^2 \leq 2\varepsilon V(x, v)$.

2. *Hamiltonian systems with the stochastic forces* have the following general form:

$$\begin{cases} d\vec{x}_t &= \vec{v}_t\, dt\,, \\ d\vec{v}_t &= -U'(\vec{x}_t)\, dt + \widehat{\sigma}(\vec{x}_t, \vec{v}_t)d\vec{w}_t\,. \end{cases} \tag{2.6}$$

Such systems with $\vec{x}_t \in \mathbb{R}$, $\vec{v}_t \in \mathbb{R}$, and $\widehat{\sigma} = 1$, were investigated in [McK69], and [MW88]. The systems where $\vec{x}_t \in \mathbb{R}^d$, $\vec{v}_t \in \mathbb{R}^d$ ($d \geq 1$), and $\widehat{\sigma}$ is identity operator were investigated in [AH89], [AK92], and [AHZ92] under the conditions $U(x) = \varphi(\|x\|^2)$, $\lim_{r\to\infty} \varphi(r) = +\infty$.

Theorem 2.1 applies to the system (2.6) with

$$V(\vec{x}, \vec{v}) = \frac{\vec{v}^{\,2}}{2} + U(\vec{x})$$

provided

$$\lim_{r\to\infty} \inf\{U(\vec{x}) : \vec{x} \in \mathbb{R}^d, \|\vec{x}\|_{\mathbb{R}^d} > r\} = \infty$$

and

$$\mathrm{Tr}\,\{\widehat{\sigma}(\tilde{\mathrm{x}}_\mathrm{t}, \tilde{\mathrm{v}}_\mathrm{t})^* \widehat{\sigma}(\tilde{\mathrm{x}}_\mathrm{t}, \tilde{\mathrm{v}}_\mathrm{t})\} \le \mathrm{c}\,\mathrm{V}(\tilde{\mathrm{x}}, \tilde{\mathrm{v}})$$

for some $c > 0$. The Lyapunov function V is the energy of the system in this case.

3. *Ornstein–Uhlenbeck process* is a solution of

$$\xi(t) = \xi_u - \int_u^t \xi(s)\,ds + \widehat{\sigma}[w(t) - w(u)].$$

We assume that $\xi(\cdot)$ belongs to a Hilbert space Ξ. Theorem 2.1 applies to this equation if $V(\cdot) = \|\cdot\|_\Xi^2 + \mathrm{Tr}_\Xi \widehat{\sigma}\widehat{\sigma}^*/2$ and $c = 2$.

Proof of Theorem 2.1

We use almost the same reasoning as in secs. III.3–III.5 of [Has69].

First we consider the case where the probability distribution of ξ_0 is concentrated in a bounded domain. Let $r > 0$ be such that the ball $B_r = \{\xi \in H : \|\xi\|_H < r\}$ contains the support of the probability distribution of ξ_0. For any $m \in \mathbb{N}$, let $A_m(\cdot)$ be a Lipschitz map $H \mapsto H$ such that $A_m(\xi) = A(\xi)$ if $\|\xi\|_H \le r + m$. The Kirszbraun–Valentine theorem states that such approximations exist provided H is a Hilbert space and $A(\cdot)$ satisfies a local Lipschitz condition [Kir34], [Val45].

Eq. (1.1) with $A(\cdot)$ replaced by $A_m(\cdot)$ satisfies conditions of the classical theorem, proved in [Bac63], [Bac64], [Cha64], and [Cha65]. We refer to the proof presented in section VII.2.1 of [DF83]. Conditions 2 were not used at this stage.

Let ξ_m denote the solution of Eq. (1.1) with $A(\cdot)$ replaced by $A_m(\cdot)$. Let $\tau_{m,n}$ be the first (random) moment t when $\|\xi_m(t)\|_H = r + n$. Here $r > 0$ is such that the ball $\{\xi \in H : \|\xi\|_H < r\}$ contains the support of the probability distribution of the initial state ξ_0.

Following [DB89] (p.139) we prove that if $n \le m_1 \le m_2$, then, for almost every ω, we have $\tau_{m_1,n}(\omega) = \tau_{m_2,n}(\omega)$ and $\forall t \in [0, \tau_{m_2,m_1}(\omega)]$ $\xi_{m_1}(t,\omega) = \xi_{m_2}(t,\omega)$.

Thus, if $m \ge n$, then $\tau_{m,m} \ge \tau_{n,n}$ a.s. and there is the unique process ξ such that $\xi(t) = \xi_m(t)$ for every $t < \tau_{m,m}$ a.s. Its (random) living time is $\tau = \sup_m \tau_{m,m}$ and $\forall t \in [0, \tau)$ the process ξ satisfies Eq. (1.1). Conditions 2 were not used at this stage.

We use conditions 2 to prove that $\tau = \infty$ a.s. For that purpose we use $\tau_m(t) = \min\{\tau_{m,m}, t\}$, $\zeta_m(t) = \xi_m(\tau_m(t))$, and prove that

$$d\zeta_m(t) = \theta(\tau_{m,m} - t)[A_m(\zeta_m(t))dt + \widehat{\sigma}dw(t)],$$

where $\theta(x) = 0$ if $x < 0$ and $\theta(x) = 1$ if $x > 0$.

Itô formula and conditions 2 imply the inequality

$$\mathrm{E}V(\zeta_m(t)) \leq e^{ct}\mathrm{E}V(\xi_0). \tag{2.7}$$

Consequently

$$\mathcal{P}\{\tau_{m,m} \leq t\} \leq \frac{e^{ct}\mathrm{E}V(\xi_0)}{\inf_{\|\xi\|_H \geq m} V(\xi)} \tag{2.8}$$

Therefore, $\lim_{m\to\infty} \mathcal{P}\{\tau_{m,m} \leq t\} = 0$.

We let $m \to \infty$ in (2.7) and obtain (2.2).

Let $\xi^{(h)}(\cdot,\cdot)$ denote the solution $\xi(\cdot,\cdot,0,\xi_0)$ that a.s. starts at a point $h \in H$, i.e., the solution of Eq. (1.1) with initial state ξ_0 whose probability distribution is the Dirac measure concentrated at h. We prove that a set $\Omega_p \subset \Omega$ exists such that $\mathcal{P}(\Omega_p) = 1$ and $\forall \omega \in \Omega_p$ and $\forall h \in H$ the sample path $t \mapsto \xi^{(h)}(t,\omega)$ is in $C([0,\infty), H)$.

Thus, we have constructed a measurable map $H \times \Omega_p \mapsto C([0,\infty), H)$ such that, for every $(h,\omega) \in H \times \Omega_p$ the function $t \mapsto \xi^{(h)}(t,\omega)$ is the unique solution of Eq. (1.1). Therefore, for any probability distribution ν of ξ_0 such that $\mathrm{E}V(\xi_0) < \infty$, Eq. (1.1) $\nu \times \mathcal{P}$-a.s. has a unique solution (cf. sec. VIII.2 in [GS77]).

The successive approximations ξ_n are properly measurable, adapted, and have the Markov property because they are images one of another under the map with (globally) Lipschitzian coefficients. Therefore, the limit process ξ is also properly measurable, adapted, and has the Markov property.

Let $P(t,h,G) = \mathcal{P}\{\xi^{(h)}(t) \in G\}$ for any $t \geq 0$, $h \in H$, and $G \in \mathcal{B}(H)$. We prove that these quantities form a family of Markov transition probabilities, and that these probabilities are continuous in t and h for any $G \in \mathcal{B}(H)$. Then we verify that if $f \in C_b(H)$, then the function $h \mapsto \mathrm{E}f(\xi^{(h)}(t))$ is also in $C_b(H)$ for any $t \geq 0$ (the Feller property). This is used in proof that the functions $(t,h) \mapsto P(t,h,G)$ are $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(H)$-measurable for any $G \in \mathcal{B}(H)$. This completes the proof of Theorem 2.1.

The Feller property inspired us to investigate the semigroup of linear operators $f \in C_b(H) \mapsto \mathrm{E}f(\xi^{(h)}(t)) \in C_b(H)$. Such investigation clarifies the ergodic properties of the semigroup and the conditions for existence of invariant measure. This measure is defined as

$$P(G) = \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \int_H P(t,h,G)dv(h)dt\,,$$

where T_1, T_2, …is an increasing sequence of positive numbers, $\lim_{n\to\infty} T_n = \infty$. Such invariant measure exists if an invertible operator D exists such that the sets $\{h \in H : \|D^{-1}h\|_H < r\}$ are compact in H, the conditions of Theorem 2.1 are satisfied w.r.t. a norm $\|D^{-1} \cdot \|_H$, except that we have

$$\lim_{r\to\infty} \sup_{\|D^{-1}h\|_H>r} \widehat{L}V(h) = -\infty$$

in place of condition 2.2.

3 Reducible Lyapunov functions and generalized solutions

If the domain Ξ and the range Y of the map $A(\cdot)$ in (1.1) are different, then $\Xi \subset H \subset Y$. Equation 1.1 is understood in the generalized sense (1.2). We assume that $\Phi \subset \Xi$ and a Banach space $X \subset \Phi$ exists such that the natural embedding $X \subset H$ is dense and the natusral embedding $X \subset Y$ is compact. The case were $X = \Xi$ is allowed. We assume that the vectors $e_1, e_2, \dots$ are in X.

We use the Galerkin approximations in our investigation of Eq. (1.1). Their properties are derived from the conditions expressed in terms of Lyapunov functions of a special kind. We call them reducible w.r.t. the basis e_1, e_2, ... That means that they satisfy the following two conditions.

1) $\forall k,\ n \in \mathbb{N} \quad k > n \Rightarrow \partial_k V(\widehat{\pi}_n \,\cdot) = 0$,

2) $\forall j, k, n \in \mathbb{N} \quad j > n$ or $k > n \Rightarrow \partial^2_{jk} V(\widehat{\pi}_n \,\cdot) = q_k(\widehat{\pi}_n \,\cdot)\delta_{jk} \geq 0$.

Here ∂_k denotes the Gateau differentiation in the direction e_k, $\partial^2_{jk} = \partial_j \partial_k$. and

$$h \mapsto \widehat{\pi}_n(h) = \sum_{k=1}^{n} (e_k, h)_H \, e_k$$

is the projection $H \mapsto \text{Span}\{e_1, e_2, \dots, e_n\}$.

It is easy to verify that if $V(\cdot)$ is reducible w.r.t. e_1, e_2, ..., then the function $f(V(\cdot))$ is also reducible w.r.t. e_1, e_2, ..., provided the function $f(\cdot)$ has the second derivative and $f'(\cdot) \geq 0$. Moreover, if the functions $V_1(\cdot)$, $V_2(\cdot)$, ... are reducible w.r.t. e_1, e_2, ... and a_1, a_2, ... are nonnegative numbers, then the function $\sum_{i=1}^{\infty} a_i \, V_i(\cdot)$ is also reducible w.r.t. e_1, e_2, ..., provided the coefficients a_1, a_2, ... are such that the first and the second derivatives of the series are equal to the sums of the first and the second derivatives of its terms.

Example 1. The function $V(\cdot) = (e_i, \cdot)_H$ is not reducible w.r.t. the orthonormal basis e_1, e_2, ... in H.

Indeed, $\partial_k V(\cdot) = \delta_{ki}$. Therefore, if $n < i$ and $k = i$, then $\partial_k V(\cdot)$ is equal to 1 rather than 0.

Example 2. The function $V(\cdot) = (e_i, \cdot)^2_H$ is reducible w.r.t. the orthonormal basis $e_1, e_2, \ldots$ in H.

Indeed, $\partial_k V(\cdot) = 2(e_i, \cdot)_H \delta_{ki}$. If $k > n$, then $\partial_k V(\widehat{\pi}_n \cdot) = 2\delta_{in}\delta_{ki} = 0$. Moreover, $\partial^2_{jk} V(\cdot) = 2\delta_{ki}\delta_{jk}$.

Example 3. The function $V(\cdot) = \|\cdot\|^2_H$ is reducible w.r.t. any orthonormal basis $e_1, e_2, \ldots$ in H, because $\|\cdot\|^2_H = \sum_{i=1}^{\infty}(e_i, \cdot)^2_H$ and every function $(e_i, \cdot)^2_H$ is reducible w.r.t. $e_1, e_2, \ldots$ in H.

This example implies that if $f(\cdot)$ has two derivatives and $f'(\cdot) > 0$, then the function $V(\cdot) = f(\|\cdot\|^2_H)$ is reducible w.r.t. any orthonormal basis $e_1, e_2, \ldots$ in H,

One reducible Lyapunov function is a source of infinitely many Lyapunov functions used for investigating the Galerkin approximations. It is possible to use a family $\{L_n \in C^{(2)}_l oc(\mathbb{R}^n),\ n \in \mathbb{N}\}$ of finite-dimensional Lyapunov functions instead of using one infinite-dimensional reducible Lyapunov function.

Our goal is to prove

Theorem 3.1. ($X \subset \Xi \subset H^{\subset W}_{\subset Y}$) *Suppose that these conditions are satisfied.*

1. *The drift term $A(\cdot)$ satisfies a local Lipschitz condition.*
2. *There is a reducible Lyapunov function $V(\cdot) \in C^{(2)}_{loc}(\Xi)$ such that*
   - 2.1 $\lim\limits_{r\to\infty} \inf\limits_{\|\xi\|_X > r} V(\xi) = \infty$.
   - 2.2 $\widehat{L}V(\cdot) \le cV(\cdot)$ *for some c.*
   - 2.3 $V''(\cdot)$ *is positive outside a ball $Q \subset \Xi$. for some c_A.*

Then, for any initial state ξ_0 whose the probability distribution has a bounded support in X, and any $T > 0$, there is a sequence of Galerkin approximations to (1.2) *whose probability distributions weakly converge to a probability distribution on $C([0, T], Y)$.*

Remarks.

1. Condition 2.3 enables to use the function $V(\cdot)$ for constructing the finite-dimensional Lyapunov functions for the Galerkin approximations. The formula

$$V(x) = V(0) + V'(0)(x) + \int_0^1 \int_0^1 V''(t_1 t_2 x)(x, x) t_2 dt_1 dt_2$$

suggests an idea to strengthen condition 2.3 and derive condition 2.1.

2. Recall that the space $X \subset \Xi$ is densely embedded in Y and this embedding is compact.

The space X must be in the domain of the drift term $A(\cdot)$. Therefore, we can shrink this space.

The space Y must contain the range of $A(\cdot)$. Therefore, we can extend this space.

Using these two procedures, we can arrive at the dense and compact embedding of X in Y.

Proof of Theorem 3.1

We divide the proof into 4 steps.

I. Construction of Galerkin approximations.

We use Eq. (1.2). Because every vector e_k is in Φ, we have

$$\langle e_k, \xi(t)\rangle = \langle e_k, \xi_0\rangle + \int_0^t \langle e_k, A(\xi(s))\rangle\, ds + \langle e_k, \widehat{\sigma}\, w(t)\rangle. \tag{3.1}$$

Because vectors e_k form an orthonormal basis in H, the series

$$\sum_{i=1}^{\infty} (e_i, w(t)) e_i$$

converges in W, and $\widehat{\sigma} \in \mathcal{L}(W, H)$, we have

$$\langle e_k, \xi(t)\rangle = \langle e_k, \xi_0\rangle + \int_0^t \langle e_k, A(\xi(s))\rangle\, ds + \sum_{j=1}^{\infty} \langle e_k, \widehat{\sigma} e_j\rangle (e_j\,, w(t))_H. \tag{3.2}$$

Thus, for any $m \in \mathbb{N}$, the Galerkin approximation to Eq. (1.1) of order m is the following system of m equations:

$$\xi_k^{(m)}(t) = (e_k, \xi_0)_H + \int_0^t \left(e_k, A(\xi^{(m)}(s)\,)\right)_H ds +$$

$$+ \sum_{i=1}^{m} (e_k, \widehat{\sigma} e_i)_H\, w_i(t), \quad k = 1, \ldots, m\,, \tag{3.3}$$

where $\xi^{(m)}(s) = \xi_1^{(m)}(s) e_1 + \xi_2^{(m)}(s) e_2 + \cdots + \xi_m^{(m)}(s) e_m$.

II. Proof that solutions to the Galerkin systems (3.3) exist.

Because the systems (3.3) are finite-dimensional, it suffices to verify that Theorem III.4.1 in [Has69] applies to (3.3). Equivalently, we prove that Theorem 2.1 applies to (3.3).

Let $x^{(m)} = x_1 e_i + \cdots + x_m e_m$. The function

$$(x_1, \ldots, x_m) \in \mathbb{R}^m \longmapsto v(x_1, \ldots, x_m) = V(x^{(m)})$$

is such that if we set $S_{jk}^{(m)} = (e_j, \widehat{\sigma}\widehat{\pi}_m\widehat{\sigma}^* e_k)_H$, then

$$\widehat{L}^{(m)} v(x_1, \ldots, x_m) =$$

$$= \frac{1}{2} \sum_{j,k=1}^{m} S_{jk}^{(m)} \frac{\partial^2}{\partial x_j \partial x_k} v(x_1, \ldots, x_m) + + \sum_{k=1}^{m} (e_k, A(x^{(m)})\,)_H \frac{\partial}{\partial x_k} v(x_1, \ldots, x_m) =$$

$$= \frac{1}{2} \sum_{j,k=1}^{m} S_{jk}^{(m)}\, \partial_{jk}^2 V(x^{(m)}) + \sum_{k=1}^{m} (e_k, A(x^{(m)})\,)_H\, \partial_k V(x^{(m)}).$$

Because $V(\cdot)$ is reducible, we have

$$\sum_{k=1}^{m} (e_k, A(x^{(m)})\,)_H\, \partial_k V(x^{(m)}) = \sum_{k=1}^{\infty} (e_k, A(x^{(m)})\,)_H\, \partial_k V(x^{(m)})$$

and

$$\sum_{j,k=1}^{m} S_{jk}^{(m)}\, \partial_{jk}^2 V(x^{(m)}) \leq \sum_{j,k=1}^{m} S_{jk}^{(m)}\, \partial_{jk}^2 V(x^{(m)}) + \sum_{k=m+1}^{\infty} S_{kk}^{(m)}\, q_k(x^{(m)}) =$$

$$= \sum_{j,k=1}^{\infty} S_{jk}^{(m)}\, \partial_{jk}^2 V(x^{(m)}).$$

Condition 2.3 implies that

$$\sum_{j,k=1}^{\infty} S_{jk}^{(m)}\, \partial_{jk}^2 V(x^{(m)}) \leq \sum_{j,k=1}^{\infty} S_{jk}^{(\infty)}\, \partial_{jk}^2 V(x^{(m)})$$

outside the ball Q.

Using Condition 2.2, we obtain

$$\widehat{L}^{(m)} v(x_1, \ldots, x_m) \leq \widehat{L} V(x^{(m)}) \leq c\, V(x^{(m)}) = c\, v(x_1, \ldots, x_m) \qquad (3.4)$$

outside the ball Q. Because $V(\cdot) \in C_{loc}^{(2)}(\Xi)$, $V(\cdot)$ and $V'(\cdot)$ are bounded on Q. This and Condition 2.3 imply that $\widehat{L}V(\cdot)$ is bounded on Q. Therefore, there is a constant d such that the function $\widetilde{V}(\cdot) = V(\cdot) + d$ satisfies (3.4) everywhere.

Let $\|\cdot\|_m$ be a norm in Span$(e_1,\, e_2,\, \ldots,\, e_m)$. Because all norms in a finite-dimensional space are equivalent, there are two positive numbers a_m and b_m

such that $a_m\|x\|_X \leq \|x\|_m \leq b_m\|x\|_X$ for every $x \in \mathrm{Span}(e_1, e_2, \ldots, e_m)$, Therefore, condition 2.1 implies that

$$\lim_{r\to\infty} \inf_{\|x^{(m)}\|_m > r} v(x_1, x_2, \ldots, x_m) \leq \lim_{r\to\infty} \inf_{\|x\|_X > b_m r} V(x) = \infty$$

All that and Condition 1 imply that Theorem 2.1 applies to the systems (3.3) with all $m \in \mathbb{N}$. Therefore, every such system has a unique solution $\xi^{(m)}$.

Inequality (2.2) means that $\forall t > 0$

$$\mathrm{E}\widetilde{V}(\xi^{(m)}(t)) \leq e^{ct}\mathrm{E}\widetilde{V}(\widehat{\pi}_m \xi_0). \tag{3.5}$$

III. Construction of measures on $C([0,T], Y)$ generated by the processes $\xi^{(m)}$.

The sample paths of every solution $\xi^{(m)}$ a.s. are in $C([0,\infty), Y)$. Therefore, the solutions define the measures $P^{(m)}$ such that $P^{(m)}(G) = \mathcal{P}\{\xi^{(m)}(\cdot) \in G\}$. for any $G \in \mathcal{B}(C([0,T], Y))$. The Ulam theorem states that every $P^{(m)}$ is a Radon probability measure (see, e.g., Theorem I.3.1 in [VTC85])

IV. Proof that the family $\{P^{(m)},\ m \in \mathbb{N}\}$ is relatively compact.

The Prokhorov theorem states that the family of Radon measures $\{P^{(m)}\}$ is relatively compact in the weak topology if and only if the family is tight (see, e.g., Theorem I.3.6 in [VTC85]). That means that $\forall \varepsilon > 0$ a compact subset K_ε exists in $C([0,\infty), Y)$ such that $P^{(m)}(K_\epsilon) > 1 - \varepsilon$ for every $m = 1, 2, \ldots$.

The Arzela–Ascoli theorem gives necessary and sufficient conditions for a family of functions to be compact.

Let $\mathcal{X}_T = C([0,T], X)$ and $\mathcal{Y}_T = C([0,T], Y)$.

We construct a family $\{K_n,\ n \in \mathbb{N}\}$ of compact subsets of $\mathcal{Y}_T$ that are the images under the natural embedding $\widehat{J}_T : \mathcal{X}_T \mapsto \mathcal{Y}_T$ of the sets $\tilde{K}_n \subset \mathcal{X}_T$ such that

1 $\sup_{x(\cdot)\in\tilde{K}_n} \|x(\cdot)\|_{\mathcal{X}_T} \leq r + n.$

2 $\lim_{\delta\to 0+} \sup_{x(\cdot)\in\tilde{K}_n} \sup_{|t_1-t_2|<\delta} \|x(t_1) - x(t_2)\|_Y = 0$

Here $r > 0$ is such that the ball $\{x \in X : \|x\|_X < r\}$ contains the support of the probability distribution of ξ_0.

Every set $\tilde{K}_n$ is not compact if X is not finite-dimensional. Nevertheless the Arzela–Ascoli theorem implies that if $\widehat{J}$ is the compact natural embedding of X in Y, then every set

$$K_n = \{y(\cdot) \in \mathcal{Y} : y(t) = \widehat{J}x(t),\ x(\cdot) \in \tilde{K}_n,\ t \in [0,T]\} \tag{3.6}$$

is relatively compact in $\mathcal{Y}_T$.

We estimate the measures of the sets $\tilde{K}_n$.

Let B_n denote the ball $\{x(\cdot) \in \mathcal{X}_T : \|x(\cdot)\|_{\mathcal{X}_T} < r + n\}$ Using conditional measures we obtain

$$P^{(m)}(\tilde{K}_n) = P^{(m)}(B_n) P^{(m)}(\mathcal{X}_T^{(ec)} \,|B_n), \tag{3.7}$$

where $\mathcal{X}_T^{(ec)}$ denotes the subspace of equicontinuous functions in $\mathcal{X}_T$.

We estimate $P^{(m)}(B_n)$ as $\mathcal{P}\{\xi^{(m)} \in B_n\}$.

Let $\tau_n^{(m)}$ be the first (random) moment t when $\|\xi^{(m)}(t)\|_X = r + n$. The events $\{\xi^{(m)} \in B_n\}$ and $\{\tau_{m,n} > T\}$ are equivalent. Therefore, we estimate $\mathcal{P}\{\tau_n^{(m)} > T\}$.

For that purpose we use $\tau_n^{(m)}(t) = \min\{\tau_n^{(m)}, t\}$, and $\zeta^{(m)}(t) = \xi^{(m)}(\tau_n^{(m)}(t))$, and prove that

$$d\zeta_k^{(m)} = \theta(\tau_{m,m} - t)[(e_k, A(\zeta^{(m)}))_H \, dt + \sum_{i=1}^{m} (e_k, \widehat{\sigma} e_i)_H \, dw_i],$$

where $\theta(x) = 0$ if $x < 0$ and $\theta(x) = 1$ if $x > 0$.

Itô formula and conditions 2 imply the inequality

$$\mathrm{E}V(\zeta^{(m)}(t)) \leq e^{ct} \mathrm{E}V(\hat{\pi}_m \xi_0).$$

Consequently

$$\mathcal{P}\{\tau_n^{(m)} \leq T\} \leq \frac{e^{cT} \mathrm{E}V(\hat{\pi}_m \xi_0)}{\inf_{\|\xi\|_X \geq n} V(\hat{\pi}_m \xi)} \leq \frac{e^{cT} \sup_m \mathrm{E}V(\hat{\pi}_m \xi_0)}{\inf_{\|\xi\|_X \geq n} V(\xi)} \tag{3.8}$$

In other words, the probability that the process $\xi^{(m)}$ reaches the boundary of the ball B_n during the time $t < T$ tends to zero as the ball radius $R = r + n$ tends to ∞. Thus

$$P^{(m)}(B_n) > 1 - \frac{e^{cT} \sup_m \mathrm{E}V(\hat{\pi}_m \xi_0)}{\inf_{\|\xi\|_X \geq n} V(\xi)}. \tag{3.9}$$

Note that the we get the estimate that does not depend on m. It implies that $\lim_{n\to\infty} P^{(m)}(B_n) = 0$ uniformly in m.

We estimate $P^{(m)}(\mathcal{X}_T^{(ec)} \,|B_n)$.

It follows from the Galerkin system (3.3) that

$$\|\xi^{(m)}(t_2) - \xi^{(m)}(t_1)\|_Y \leq \int_{t_1}^{t_2} \|A(\xi^{(m)}(s))\|_Y ds + \|\widehat{\sigma}\hat{\pi}_m[w(t_2) - w(t_1)]\|_H \leq$$

$$\leq \int_{t_1}^{t_2} \|A(\xi^{(m)}(s))\|_Y ds$$

Because $\xi^{(m)} \in B_n$ and $A(\cdot)$ satisfies a local Lipschitz condition, A positive number C exists such that $\sup_{0\leq s\leq T} \|A(\xi^{(m)}(s))\|_Y \leq C$. Note that C does not depend on m.

Thus we obtain the inequality

$$\|\xi^{(m)}(t_2) - \xi^{(m)}(t_1)\|_Y \leq (t_2 - t_1)C + \|\widehat{\sigma}\|_{\mathcal{L}(\mathcal{W},\mathcal{H})}\|w(t_2) - w(t_1)\|_W. \quad (3.10)$$

It is valid for every ω such that $\sup_{0\leq t\leq T} \|\xi^{(m)}(t,\omega)\|_X < r + n$. Combining (3.10) with the Levi theorem on modulus of continuity and inequality (3.9), we arrive at conclusion that

$$P^{(m)}(\tilde{K}_n) > 1 - \frac{e^{cT}\sup_m \mathrm{E}V(\hat{\pi}_m\xi_0)}{\inf_{\|\xi\|_X\geq n} V(\xi)}.$$

Therefore, the family of measures $\{P^{(m)},\ m \in \mathbb{N}\}$ is tight in the space of Radon probability measures on the Borel subsets of $\mathcal{Y}_T$.

By the Prokhorov theorem, a subsequence $\{P^{(m_k)},\ k \in \mathbb{N}\}$ exists in $\{P^{(m)}, m \in \mathbb{N}\}$ that weakly converges to a probability measure $P^{(\infty)}$ on $\mathcal{B}(\mathcal{Y}_T)$. The theeorem is proved.

4 Examples

The Wiener process in Theorem 3.1 is infinite-dimensional. It is easy to prove the theorem for finite-dimensional Wiener process. Such version applies to the stochastic equation of the Navier–Stokes type (1.3). The corresponding Lyapunov function is

$$V(\cdot) = \|\cdot\|_H^4 + a\|\cdot\|_H^2 + b,$$

where $a \geq 0$ and $b \geq 0$ are constants. Direct calculations show that

$$(V'(u), h)_H = (4\|u\|_H^2 + 2a)(u,h)_H,$$

and

$$(g, V''(u), h) = 4\|u\|_H^2(g,h)_H + 8(u,g)_H(u,h)_H + 2a(g,h)_H$$

Because $(A(u), V'(u))_H = -4\|u\|_H^2\|u'\|_H^2 - 2a\|u'\|_H^2 \leq 0$, it is easy to verify that the conditions of Theorem 3.1 are‘satisfied.

Another example is the *Wave equation with white noise* $\eta(t)$

$$u_{tt}(t,\vec{x}) = \Delta u(t,\vec{x}) + \sigma\,\eta(t), \qquad t > 0,\ \vec{x} \in \mathbb{R}^d$$

It can be represented as

$$\left\{ \begin{array}{lcl} du(t,\vec{x}) & = & v(t,\vec{x})\,dt\,, \\ dv(t,\vec{x}) & = & \Delta u(t,\vec{x})\,dt + \sigma dw(t)\,. \end{array} \right. \tag{4.1}$$

We assume that, for any $t > 0$, $u(t,\cdot) \in W^{(2,1)}(\mathbb{R}^d)$ and $v(t,\cdot) \in L^2(\mathbb{R}^d)$.

Obviously, the system (4.1) is the infinite dimensional Hamiltonian system with the stochastic force. Therefore, it is natural to take the Lyapunov function to be the energy of the system, i.e.

$$V(u,v) = \frac{1}{2}\|v\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^d)}\,.$$

It is easy to verify that $(A(u,v), V'(u,v))_H = 0$ (conservation of energy) and that the conditions of Theorem 3.1 are obviously satisfied.

Our third evample is *the* $\lambda\varphi^4$*-model of Quantum Field Thery*

$$d\varphi(t) = [\Delta\varphi - \varphi - \varphi^3]dt + \widehat{\sigma}\,dw(t) \tag{4.2}$$

In this case,

$$U \subset \mathbb{R}^n \text{ is bounded and open,} \quad \partial\mathrm{U} \text{ is smooth.}$$

$$X = W_0^{2,1}(U), \quad \Xi = L^4(U), \quad H = L^2(U), \quad Y = C_0^{(2)}(U)^*.$$
$$A(\varphi) = \Delta\varphi - \varphi - \varphi^3, \qquad \widehat{\sigma}(\varphi) = \widehat{\sigma} = \text{const}$$

The direct calculations show that

$$\|A(\varphi) - A(\psi)\|_Y \le \|\varphi - \psi\|_H\,[2 + 3\sqrt{2}\,(\|\varphi\|^2_{\Xi} + \|\psi\|^2_{\Xi}\,)\,]$$

and that the Lyapunov function is

$$V(\cdot) = (\|\cdot\|^2_H + \mathrm{Tr}\sigma\sigma^*)^2$$

It is easy to verify that the conditions of Theorem 3.1 are satisfied.

5 Discussion

The space $\mathcal{Y}_T$ is Polish. Therefore, the Skorokhod theorem applies to the subsequence $\{P^{(m_k)},\ k \in \mathbb{N}\}$ (see, e.g., Theorem I.2.7 in [IW81]). The theorem asserts that there are a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{P}})$, and $\mathcal{Y}_T$-valued random elements $\eta, \eta^{(1)}, \ldots, \eta^{(k)}, \ldots$ $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathcal{P}})$, that have the following properties:

1. $\forall k \in \mathbb{N}$ the probability distribution of $\eta^{(k)}$ is $P^{(m_k)}$.

2. The probability distributions of η is $P^{(\infty)}$.

3. $\lim_{k\to\infty} \|\eta^{(k)} - \eta\|_{\mathcal{Y}_T} = 0$ $\widetilde{\mathcal{P}}$-a.s.

It is interesting to prove that η is a martingale solution of Eq. (1.2), i.e. to prove that $\forall f \in C^{(2)}(Y)$,

$$f(\eta(\cdot)) - f(\xi_0) - \int_0^{\cdot} \widehat{L} f(\eta(s))\, ds \tag{5.1}$$

is a $\{\widetilde{\mathcal{F}}_t^{\eta}\}$-martingale, i.e., η is a weak solution of basic equation (1.2).

For this we should assume that some extra conditions are satisfied in addition to those of Therem 3.1.

For example, let us assume that the natural embedding $X \subset \Xi$ is compact. Then we can use the Lemma 1 of [Dub65] and prove that the family $\{P^{(m_k)},\ k \in \mathbb{N}\}$ is relatively compact on Borel subsets of the Banach space $\mathcal{Z}_T$ with the norm

$$\|z(\cdot)\|_{\mathcal{Z}_T} = \sup_{0\le t\le T} \|z(t)\|_Y + \left[\int_0^T \|z(t)\|_{\Xi}^p dt\right]^{1/p}.$$

The presence of $\|\cdot\|_{\Xi}$ in $\|\cdot\|_{\mathcal{Z}_T}$ is important because the drift $A(\cdot)$ is defined on Ξ. Therefore, we can use some extra continuity of $A(\cdot)$, corresponding norm estimates, and the de La Vallée Poussin theorem to prove that

$$\lim_{k\to\infty} \int_0^T \mathrm{E}\|\eta^{(k)}(t) - \eta(t)\|_{\Xi}^2\, dt = 0\,,$$

and, consequently,

$$\lim_{k\to\infty} \int_0^T \mathrm{E}\|A(\eta^{(k)}(s)\,) - A(\eta(s))\|_Y\, ds = 0$$

This imply that (5.1) is a martingale as a limit of

$$f(\eta^{(k)}(t)) - f(\xi_0) - \int_0^t \widehat{L} f(\eta^{(k)}(s))\, ds.$$

In other words, η is a weak solution of basic equation (1.2).

In closing we note that the method of Lyapunov functions for investigation of stochastic equations was created many years ago. Nevertheless, it is not used as frequently as its analogue in the stability theory. We showed that it is the most general now and we used it to prove two existence theorems in quite abstract settings. We showed also that those theorems have direct and easy applications.

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