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On Condensing Perturbations of Closed Linear Surjective Operators

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Abstract

The paper is devoted to investigation of solvability of operator equations containing condensing perturbations of linear surjective operators. Some solvability theorems for such equations are proved. Those theorems are applied to investigation of some classes of neutral type differential equations.

Key words: Measure of non-compactness; surjective operator; condensing mapping; neutral type differential equation.

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1 Introduction

The theory of condensing mappings (k-set contractions), both single-valued and set-valued, that finds applications in various problems of contemporary mathematics, is well-known at the moment. In [3, 4, 5, 6] the perturbations of continuous mappings condensing with respect to a certain principal part, were investigated. In these works the homotopy classification of such perturbations was considered and the topological degree for new classes of vector fields was constructed. Note paper [3] (and some other papers by this author) where the perturbations were investigated, for which the principal part was a linear Fredholm operator.

On the other hand, in [7, 8, 9] the solvability of equations of the type $A(x) - f(x) = 0$, where A was a linear surjective operator and f was a completely continuous operator, was considered. Naturally the idea of studying the equations of the same type with f condensing with respect to A, arises. Note also that in this case the homotopy classification, constructed in [3, 4, 5, 6], is not applicable.

The present paper is devoted to the above-mentioned problem. We investigate the equations with condensing perturbations of closed linear surjective operators and prove existence of solution theorems. Then the obtained results are applied to studying solvability for some classes of neutral type equations.

2 (A, ψ)-condensing mappings

Let E_1 an E_2 be Banach spaces, $A : D(A) \subset E_1 \to E_2$ be a closed linear surjective operator and $\Gamma(A) \subset E_1 \times E_2$ be the graph of A. Let also $t : \Gamma(A) \to E_1$ be a projection mapping onto the domain of A, i.e. $t(x, y) = x$.

Consider some properties of the set-valued mapping $A^{-1} : E_2 \to Cv(E_1)$, where $A^{-1}(y) = \{x \in E_1 \mid A(x) = y\}$.

Definition 2.1 *The number*

$$||A^{-1}|| = \sup_{y \in E_2} \left(\frac{\inf\{||x|| \mid x \in E_1,\, A(x) = y\}}{||y||}\right) < \infty$$

is called the norm of set-valued mapping A^{-1}.

If the sub-space $Ker(A)$ is not complementable in the space E_1, a linear continuous right-inverse operator to A does not exist but the following proposition takes place.

Lemma 2.1 *For every number l, $||A^{-1}|| < l$ and every point $x_0 \in D(A)$ there exists a continuous mapping $q : E_2 \to E_1$ such that the following properties hold:*
1) $A(q(y)) = y$ for every $y \in E_2$;
2) $||x_0 - q(y)|| \le l||A(x_0) - y||$ for every $y \in E_2$.

The proof of Lemma 2.1 can be found, e.g., in [9].

Let in E_2 a monotone non-singular algebraic semi-additive real correct measure of non-compactness ψ be given (see, e.g., [1]).

Definition 2.2 *We say that a single-valued mapping $f : D(f) \subset E_1 \to E_3$ is (A, ψ)-condensing, if:*
1) for every set $\Omega \subset (D(A) \cap D(f))$, from the inequality $\psi(A(\Omega)) \le \psi(f(\Omega))$ it follows that $\psi(A(\Omega)) = 0$;
2) if $X = t^{-1}(D(f))$, the composition $f \circ t : X \to E_3$ is a continuous mapping.

Let E_1, E_2 and E_3 be Banach spaces, $A : D(A) \subset E_1 \to E_2$ be closed surjective linear operator and $B : D(B) \subset E_1 \to E_3$ be a linear operator.

Definition 2.3 *We say that the operator B is subordinated to the operator A, if:*
(1) $D(A) \subset D(B)$;
(2) for every $x \in D(A)$ the inequality $||A(x)|| \geq ||B(x)|| holds$.

Consider a mapping $g : X \subset D(A) \to E_2$.

Definition 2.4 *We say that the mapping g is completely continuous modulo the mapping A (or A-completely continuous) if it is continuous and for every bounded set $Q \subset E_2$ and every bounded set $M \subset X$ the set $\overline{g(M \cap A^{-1}(Q))}$ is compact in E_3. The empty set is considered as compact by definition.*

It is known that the set $D(A)$ can be turned into Banach space by introducing the graph norm $||x||_{D(A)} = ||x||_{E_1} + ||A(x)||_{E_2}$ into it. Let the Banach space E be the above-mentioned set $D(A)$ equipped with the graph norm. Evidently the embedding map $j : E \to E_1$ is continuous. Introduce the notation $\tilde{X} = j^{-1}(X)$ and consider the mapping $\tilde{g} : \tilde{X} \to E_3$, $\tilde{g}(x) = g(j(x))$. The following condition for A-complete continuity of a mapping g takes place.

Proposition 2.1 *A continuous mapping g is A-completely continuous if and only if the mapping $\tilde{g}$ is completely continuous.*

Proof. Necessity. Let $N \subset \tilde{X}$ be a bounded set in E. Then the set $M = j(N)$ is bounded in E_1 and the set $Q = A(j(N)) = A(M)$ is bounded in E_2. Hence the set $\tilde{g}(N) = g(j(N)) = g(M \cap A^{-1}(Q))$ is relatively compact. The Necessity follows.

Sufficiency. Let the mapping $\tilde{g}$ be completely continuous. Consider the bounded sets $Q \subset E_2$ and $M \subset X$. Let $N = j^{-1}(M \cap A^{-1}(Q)) \subset E$. It is evident that $N \subset \tilde{X}$ and it is bounded. Then $g(M \cap A^{-1}(Q)) = \tilde{g}(N)$ and it is relatively compact. This proves the Sufficiency. □

One can easily prove the following statements.

Corollary 2.1 *Let E_1, E_2 and E_3 be Banach spaces. If a mapping $g : X \subset E_1 \to E_2$ is A-completely continuous and $f : E_2 \to E_3$ is continuous, the mapping $f \circ g : X \subset E_1 \to E_3$ is A-completely continuous.*

Corollary 2.2 *If a mapping g is A-completely continuous, then for every bounded set $Y \subset X$ such that the set $A(Y)$ is bounded in E_2, the set $g(Y)$ is relatively compact.*

Consider some examples of (A, ψ)-condensing mappings.

Example 1. Assume that the operator B is subordinated to A and a bounded set X in $D(A)$ is such that the set $A(X)$ is bounded in E_2. Let $\varphi : X \times E_3 \to E_2$ be a continuous mapping satisfying the following conditions:
1) there exists a number $k \in (0, 1)$ such that for every point $x \in X$ and every $y_1, y_2 \in E_3$ the inequality $||\varphi(x, y_1) - \varphi(x, y_2)|| \leq k||y_1 - y_2||$ holds;
2) for every $y \in E_3$ the mapping $\varphi(\cdot, y) : X \to E_2$ is A-completely continuous.

Consider the mapping $f : X \to E_3$, $f(x) = \varphi(x, B(x))$. Let in the space E_2 the Hausdorff measure of non-compactness χ be given.

Proposition 2.2 *Under the above hypotheses the mapping f is (A, χ)-condensing.*

Proof. Let us prove first that for every set $Q \subset X$ the inequality

$$\chi(f(Q)) \leq k\,\chi(A(Q))$$

takes place, from which and from Definition 2.2 condition (1) follows.

Let $S = \{s_1, ..., s_n\}$ be a finite $(\chi(A(Q)) + \varepsilon)$-net of $A(Q)$. Since $S \subset A(Q)$, there exist points $\{x_1, ..., x_n\} \subset Q$ such that $s_i = A(x_i)$, $i = 1, 2, ..., n$.

Introduce the notation $B(x_i) = z_i$, $i = 1, 2, ..., n$. Then the points $\{z_i\}$ also form a $(\chi(A(Q)) + \varepsilon)$-net in $B(x)$. Indeed, if $z \in B(Q)$, there exists a point $x \in Q$ such that $B(x) = z$. Then for some i_0 we obtain

$$||z - z_{i_0}|| = ||B(x) - B(x_{i_0})|| \leq ||A(x) - A(x_{i_0})|| \leq \chi(A(Q)) + \varepsilon.$$

Let $S_1 = \{z_1, z_2, ..., z_n\}$. Consider the set $\varphi(Q, z_i)$, where $i = 1, ..., n$. It is relatively compact since the mapping φ is A-completely continuous in the first argument.

Introduce the notation $L = \varphi(Q \times S_1) = \bigcup\limits_{i=1}^{n} \varphi(Q, z_i)$ and show that this set is a completely continuous $k(\chi(A(Q)) + \varepsilon)$-net in $f(Q)$.

Let z be an arbitrary point in $f(Q)$. Then there exists a point $x \in Q$ such that $z = \varphi(x, B(x))$. Let the point $z_i \in S_1$ be such that $z_i = B(x_i)$ and

$$||A(x) - A(x_i)|| \leq \chi(A(Q)) + \varepsilon.$$

Then $\varphi(x, z_i) \in \varphi(Q, z_i) \subset L$. We have

$$||z - \varphi(x, z_i)|| = ||\varphi(x, B(x)) - \varphi(x, B(x_i))|| \leq$$

$$\leq k||B(x) - B(x_i)|| \leq k||A(x) - A(x_i)|| < k(\chi(A(Q)) + \varepsilon).$$

Hence, $\chi(f(Q)) \leq k(\chi(A(x)) + \varepsilon)$ for every $\varepsilon > 0$. Then $\chi(f(Q)) \leq k(\chi(A(x)))$ and Condition (1) from Definition 2.2 is proved.

Now let us prove that the composition $f \circ t : t^{-1}(X) \to E_3$ is continuous. For this it is enough to prove that the composition $B \circ t : \Gamma(A) \to E_3$ is continuous. Let $(x_0, A(x_0))$ be an arbitrary point in $\Gamma(A)$, and let the sequence $\{(x_n, A(x_n))\}$ tend to $(x_0, A(x_0))$, i.e.

$$||x_0 - x_n|| + ||A(x_0) - A(x_n)|| \to 0.$$

From the last property it follows that $x_0 \in D(A) \subset D(B)$ and $||B(x_0) - B(x_n)|| \to 0$ since $||B(x_0) - B(x_n)|| \leq ||A(x_0) - A(x_n)||$. The continuity is proved. □

Example 2. Let (as above) $A : D(A) \subset E_1 \to E_2$ be a closed surjective linear operator and let $B : D(B) \subset E_1 \to E_3$ be subordinated to A. Suppose that the Kuratowski measure of non-compactness α is given in E_2. Let a set X be a bounded sub-set in $D(A)$ such that $A(X)$ is bounded in E_2 as well. Suppose that a continuous mapping $f_1 : X \to E_1$, satisfies the following condition:

there exists a number $k \in (0, 1)$ such that for every points $x_1, x_2 \in X$ the inequality

$$||f_1(x_1) - f_1(x_2)|| \leq k||B(x_1) - B(x_2)||$$

holds, i.e., f_1 is a B-contracting mapping.

As an example of such mapping on may consider the mapping $f_1 = q \circ B$, where $q : E_3 \to E_1$ is a contracting mapping.

Let $f_2 : X \to E_2$ be A-completely continuous mapping. Consider the mapping $f(x) = f_1(x) + f_2(x)$.

Proposition 2.3 *Under the above conditions the mapping f is (A, α)-condensing.*

Proof. Let $Q \subset X$ and

$$\alpha(f(Q)) \geq \alpha(A(Q)). \tag{2.1}$$

Let ε be an arbitrary number greater than $\alpha(A(Q))$. Then there exists a finite number of sets $\{N_i\}_{i=1}^n,\ N_i \subset A(Q)$ such that $\bigcup\limits_{i=1}^{n} N_i = A(Q)\ \ diam(N_i) < \varepsilon$ for every $i = 1, 2, ..., n$. Let $Q_i = A^{-1}(N_i) \bigcap Q$. Introduce the notation $M_i = f_1(Q_i)$. Calculate the diameter of M_i.:

$$diam(M_i) = \sup_{u,v \in M_i} ||u - v|| = \sup_{x,y \in Q_i} ||f_1(x) - f_1(y)|| \leq$$

$$\leq \sup_{x,y \in Q_i} k\,||B(x) - B(y)|| \leq \sup_{x,y \in Q_i} k\,||A(x) - A(y)|| \leq$$

$$\leq k \sup_{a,b \in N_i} ||a - b|| = k\, diam(N_i) < k\varepsilon.$$

Thus, for every $\varepsilon > \alpha_A(Q)$ there exists a finite number of sets M_i, $i = 1, 2, ..., n$, such that $\bigcup_{i=1}^{n} M_i = f_1(Q)$ $diam(M_i) < k\varepsilon$. Hence,

$$k\,\alpha_A(Q) \geq \alpha(f_1(Q)). \tag{2.2}$$

Since the sets X and $A(X)$ are bounded and the mapping f_2 is A-completely continuous, $\alpha(f_2(\Omega)) = 0$ for every $\Omega \subset X$.

By the algebraic semi-additivity of α we have:

$$\alpha(f_1(Q)) = \alpha(f_1(Q)) + \alpha(f_2(Q)) \geq \alpha(f(Q)) \geq \alpha(A(Q)). \tag{2.3}$$

Comparing inequalities (2.2) and (2.3), we obtain that $\alpha(A(Q)) = 0$ and so Condition (1) from Definition 2.2 is proved.

The fact that the composition $f \circ t : \Gamma(A) \to E_3$ is continuous, is proved by the same way as in Example 1. □

3 Equations with (A, ψ)-condensing mappings

Let E_1, E_2 be Banach spaces, $A : D(A) \subset E_1 \to E_2$ be a closed linear surjective operator. Let in E_2 a monotone, non-singular, algebraically semi-additive, real, correct measure of non-compactness ψ be given. Let $q : E_2 \to E_1$ be a continuous mapping that is right-inverse to A (see Lemma 2.1). Consider the set $X \subset E_1$ and (A, ψ)-condensing mapping $f : X \to E_2$.

Lemma 3.1 *Let a set $V \subset E_2$ be such that $q(V) \subset X$. Then the mapping $g = f \circ q : V \to E_2$ is ψ-condensing.*

Proof. Suppose that for a certain set $Q \subset V$ the inequality $\psi(g(Q)) \geq \psi(Q)$ holds. Then

$$\psi(g(Q)) = \psi(f(q(Q))) \geq \psi(Q) = \psi(A(q(Q))).$$

Introduce the notation $q(Q) = \Omega$. Then $\psi(f(\Omega)) \geq \psi(A(\Omega))$, hence

$$\psi(A(\Omega)) = \psi(A(q(Q))) = \psi(Q) = 0.$$

Now check that the mapping g is continuous. For this purpose consider the mapping $\hat{q} : E_2 \to \Gamma(A)$ given by the condition $\hat{q}(y) = (q(y), y)$. It is evident that this mapping is continuous and that $q = t \circ \hat{q}$. Then the mapping $g = f \circ q = f \circ t \circ \hat{q}$ is continuous. Thus $g = f \circ q$ is ψ-condensing. □

Let $x_0 \in D(A)$ be a certain point, $B_R[x_0] \subset E_1$ be a closed ball of radius R centered at x_0. Introduce the set

$$P = \{x \in D(A) \bigcap B_R[x_0] \mid ||A(x) - A(x_0)|| \leq m\},$$

where m is a certain positive number. Let the mapping $f : P \to E_2$ be (A, ψ)-condensing. Consider the equation

$$A(x) = f(x). \tag{3.1}$$

Theorem 3.1 *If there exists a number $l > \max\{\, ||A^{-1}||,\ \frac{R}{m}\}$ such that*

$$||A(x_0) - f(x)|| \le \frac{R}{l},$$

equation (3.1) has a solution.

Proof. Specify a point $y_0 = A(x_0)$. Consider the ball $B_{\frac{R}{l}}[y_0] \subset E_2$. Let $q : E_2 \to E_1$ be a continuous mapping that satisfies the conditions of Lemma 2.1. Check that $q(B_{\frac{R}{l}}[y_0]) \subset P$.

Indeed, for every $y \in B_{\frac{R}{l}}[y_0]$ we have

$$||x_0 - q(y)|| \le l||A(x_0) - y|| = l||y_0 - y|| \le R.$$

On the other hand,

$$||A(x_0) - A(q(y))|| = ||A(x_0) - y|| \le \frac{R}{l} < m.$$

Then on the ball $B_{\frac{R}{l}}[y_0]$ the mapping $g = f \circ q : B_{\frac{R}{l}}[y_0] \to E_2$. is well-defined. Check that for every $y \in B_{\frac{R}{l}}[y_0]$ the inclusion $g(y) \in B_{\frac{R}{l}}[y_0]$ holds. Indeed,

$$||y_0 - g(y)|| = ||y_0 - f(q(y))|| \le \frac{R}{l},$$

since $q(y) \in B_R[x_0]$.

Then the ψ-condensing mapping g sends the ball $B_{\frac{R}{l}}[y_0]$ into itself, hence it has a fixed point. Suppose that $y_* = g(y_*)$, i.e., $y_* = f(q(y_*))$. If $x_* = q(y_*)$,

$$A(x_*) = A(q(y_*)) = y_* = f(q(y_*)) = f(x_*).$$

The point x_* is a solution of equation (3.1). □

Consider some consequences of Theorem 3.1. Let E_1, E_2 and E_3 be Banach spaces, $A : D(A) \subset E_1 \to E_2$ be a closed surjective linear operator and $B : D(B) \subset E_1 \to E_3$ be a linear operator subordinated to A. Let in addition $x_0 \in D(A)$, $B_R[x_0] \subset E_1$ be a closed ball with radius $R > 0$. Consider the set

$$P = \{x \in D(A) \bigcap B_R[x_0] \mid ||A(x) - A(x_0)|| \le m\},$$

where m is a certain positive number.

Let $\varphi : P \times E_3 \to E_2$ be a continuous mapping that satisfies the following conditions:

1) there exists a number $k \in (0, 1)$ such that for every point $x \in P$ and every $y_1, y_2 \in E_3$ the inequality $||\varphi(x, y_1) - \varphi(x, y_2)|| \le k||y_1 - y_2||$ holds;
2) for every $y \in E_3$ the mapping $\varphi(\cdot, y) : P \to E_2$ is A-completely continuous.

Corollary 3.1 *If for every point $x \in P$ there exists a number l greater than $\max\{\,||A^{-1}||,\ \frac{R}{m}\}$ and such that $||A(x_0) - \varphi(x, B(x))|| \le \frac{R}{l}$, equation $A(x) = \varphi(x, B(x))$ has a solution in the set P.*

Proof. Let in the space E_2 the Hausdorff measure of non-compactness χ be given. Consider the mapping $f(x) = \varphi(x, B(x))$. By Proposition 2.2 the mapping f is (A, χ)-condensing on the set P. Hence, by Theorem 3.1 equation $A(x) = \varphi(x, B(x))$ has a solution. □

Another corollary to Theorem 3.1 is as follows. Let E_1, E_2 and E_3 be Banach spaces, $A : D(A) \subset E_1 \to E_2$ be a closed surjective linear operator and $B : D(B) \subset E_1 \to E_3$ be a linear operator subordinated to A. Let $\varphi : E_1 \times E_3 \to E_2$ be a continuous mapping, satisfying the following conditions:
1) there exists a number $k \in (0, 1)$ such that for every point $x \in E_1$ and every $y_1, y_2 \in E_3$ the inequality $||\varphi(x, y_1) - \varphi(x, y_2)|| \le k||y_1 - y_2||$ holds;
2) for every $y \in E_3$ the mapping $\varphi(\cdot, y) : E_1 \to E_2$ is A-completely continuous.

Consider the mapping $f : D(A) \to E_2,\ f(x) = \varphi(x, B(x))$. Let in the space E_2 the Hausdorff measure of non-compactness χ be given. We investigate the solvability of the following equation:

$$A(x) = \varphi(x, B(x)). \tag{3.2}$$

Theorem 3.2 *If there exist constants $\gamma \ge 0$ and $\beta \ge 0$ such that $\gamma(||A^{-1}|| + 1) < 1$ and for every point $x \in E$ the inequality $||\varphi(x, y)|| \le \gamma(||x|| + ||y||) + \beta$ holds, equation (3.2) has a solution.*

Proof. From the hypothesis of Theorem it follows that

$$||A^{-1}|| < \frac{1 - \gamma}{\gamma}.$$

Specify a number l such that $||A^{-1}|| < l < \frac{1-\gamma}{\gamma}$. Let $q : E_2 \to E_1$ be a continuous mapping, the right-inverse to A and such that $q(0) = 0$ and $||q(y)|| \le l\,||y||$. Such a mapping does exist by Lemma 2.1. Consider the composition $f \circ q : E_2 \to E_2$. Let R be an arbitrary positive number. Consider the ball $B_R[0] \subset E_2$. Introduce the set

$$P = \{y \in D(A) \bigcap B_R[0] \mid ||A(y) - A(y_0)|| \le m\},$$

where m is a certain positive number. Then for every $y \in P$ the inequality

$$||f(q(y))|| = ||\varphi(q(y), B(q(y)))|| \le \gamma(||q(y)|| + ||B(q(y))||) + \beta \le$$

$$\le \gamma(l||y|| + ||A(q(y))||) + \beta \le \gamma(l||y|| + ||y||) + \beta \le \gamma(l + 1)R + \beta$$

holds. If R satisfies the inequality

$$R \geq \frac{\beta}{1 - \gamma(l+1)},$$

$||f(q(y))|| \leq R$. By the above assumptions the mapping f is (A, χ)-condensing on P (see Proposition 2.2). Hence, the set P is invariant with respect to χ-condensing mapping $f \circ q$. By Sadovskii's theorem (see [1]) this mapping has a fixed point that determines a solution of equation (3.2). □

4 On neutral type differential equations

4.1 On a certain neutral type differential equation

Consider an interval $[0, \tau]$ of real line.

Definition 4.1 *A function $\tilde{\alpha} : [0, \tau] \to [0, \tau]$ is called admissible if:*
1) $\tilde{\alpha}$ is continuous;
2) for every $t \in [0, \tau]$ we have $\tilde{\alpha}(t) \leq t$.

Let $g : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a mapping that satisfies the following conditions:
(g1) g is continuous jointly in all variables;
(g2) there exists a number $k \in (0, 1)$ such that for every $t \in [a, b]$ and $x, y_1, y_2 \in \mathbb{R}^n$ the inequality

$$||g(t, x, y_1) - g(t, x, y_2)|| \leq k||y_1 - y_2||$$

holds.

Let $\tilde{\alpha}$ and $\tilde{\beta}$ be admissible functions. Consider the following differential equation:

$$x'(t) = g(t, x(\tilde{\alpha}(t)), x'(\tilde{\beta}(t))). \tag{4.1}$$

Let $h \in (0, \tau]$. We define a solution of equation (4.1) on the interval $(0, h]$ as a continuously differentiable function x_*, given on that interval, such that for every $t \in [0, h]$ it satisfies equation (4.1).

Consider the following problem:

$$x'(t) = g(t, x(\tilde{\alpha}(t)), x'(\tilde{\beta}(t))), \tag{4.2}$$

$$x(0) = 0. \tag{4.3}$$

Problem (4.2), (4.3) has the following operator interpretation. Suppose that $0 < h \leq \tau$ and denote by $D(A)$ the set of continuously differentiable functions $x : [0, h] \to \mathbb{R}^n$ such that $x(0) = 0$. Let $A : D(A) \subset C_{[0,h]} \to C_{[0,h]}$ be a differentiation operator.

Evidently A is continuously convertible and $A^{-1}(y)(t) = \int\limits_0^t y(s)\,ds$. It is also easy to see that $||A^{-1}|| = h$.

Let $\phi : C_{[0,h]} \times C_{[0,h]} \to C_{[0,h]}$ be a superposition operator generated by the mapping g, i.e., $\phi(x,y)(t) = g(t, x(t), y(t))$.

Lemma 4.1 *Under the above assumptions ψ is a continuous mapping and satisfies the following conditions:*
(1) for every point $x \in C_{[0,h]}$ and every $y_1, y_2 \in C_{[0,h]}$ the inequality $||\phi(x,y_1) - \phi(x,y_2)|| \leq k||y_1 - y_2||$ holds;
(2) for every $y \in C_{[0,h]}$ the mapping $\phi(\cdot, y) : C_{[0,h]} \to C_{[0,h]}$ is A-completely continuous.

The proof of Lemma 4.1 is obvious.

Consider the operator $K : C_{[0,h]} \to C_{[0,h]}$ determined by the condition $K(x)(t) = x(\tilde{\alpha}(t))$. Evidently it is a linear continuous operator and $||K|| \leq 1$. Consider also the operator $B : D(A) \to C_{[0,h]}$ determined by the condition $B(x)(t) = x'(\tilde{\beta}(t))$.

Lemma 4.2 *Operator B is subordinated to A.*

Proof. It is sufficient to prove that
(1) $D(A) \subset D(B)$;
(2) for every $x \in D(A)$ the inequality $||A(x)|| \geq ||B(x)||$ holds.

Property (1) is evident since the domains of $D(A)$ and of $D(B)$ coincide. Property (2) is a consequence of the following arguments:

$$||B(x)|| = \max_{0 \leq t \leq h} ||x'(\tilde{\beta}(t))|| \leq \max_{0 \leq s \leq h} ||x'(s)|| = ||A(x)||.$$

□

Consider the mapping $\varphi(x,y) : C_{[0,h]} \times C_{[0,h]} \to C_{[0,h]}$, determined by the condition : $\varphi(x,y) = \phi(K(x), y)$.

Lemma 4.3 *Under the above assumptions the mapping $\varphi_y = \varphi(\cdot, y) : C_{[0,h]} \to C_{[0,h]}$ is A-completely continuous for a specified y.*

Proof. Let in the set $D(A)$ the graph norm $||x||_{D(A)} = ||x||_C + ||x'||_C$ be given. Denote by $C^1_{[0,h]}$ the space $(D(A), ||\cdot||_{D(A)})$. Consider a bounded set $N \subset C^1_{[0,h]}$. For $x \in N$ we have $x(\tilde{\alpha}(\cdot)) \in K(N)$. It is evident that $K(N)$ is bounded in $C_{[0,h]}$.

Introduce the notation $M = \phi_y(K(N))$ and prove that M is relatively compact. For this we use the Arzelá-Ascoli theorem.

First of all M is uniformly bounded. Indeed, let $x \in N$, then there exists $R > 0$ such that $||x||_C \leq R \;\; ||x'||_C \leq R$. Since $y(t)$ is a specified function on the interval $[0,\tau]$, there exists a number $P>0$ such that $||y(t)|| \le P$ for every $t\in[0,\tau]$. Then $x(\tilde{\alpha}(t)) \in B_R[0]$, $y(t)\in B_P[0]$. Since $g:[0,\tau]\times B_R[0]\times B_P[0] \to \mathbb{R}^n$ is jointly continuous in all variables and the set $[0,\tau]\times B_R[0]\times B_P[0]$ is compact, the set $g([0,\tau]\times B_R[0]\times B_P[0])$ is compact in $\mathbb{R}^n$. Hence it is closed and bounded. Thus $\phi_y(x)(t) \subset g([0,\tau]\times B_R[0]\times B_P[0])$ for every function $x\in N$ and every $t\in[0,h]$. Hence, the set M is uniformly bounded.

The fact that M is equicontinuous is proved by the following arguments.

Since $g:[0,\tau]\times B_R[0]\times B_P[0]\to\mathbb{R}^n$ is jointly continuous in all variables, and the set $[0,\tau]\times B_R[0]\times B_P[0]$ is compact, the mapping g is uniformly continuous on this set. Hence, for every $\varepsilon>0$, every $x_1,x_2\in B_R[0]$, every $y_1,y_2\in B_P[0]$ and every $t_1,t_2\in[0,\tau]$ there exists $\delta_1>0$ depending on ε, such that if $|t_1-t_2|<\delta_1$, $||x_1-x_2||<\delta_1$, $||y_1-y_2||<\delta_1$ then

$$||g(t_1,x_1,y_1)-g(t_2,x_2,y_2)||<\varepsilon.$$

Since the function $y(t)$ is uniformly continuous on $[0,\tau]$, there exists $\delta_2>0$ depending on δ_1, such that for every $t_1,t_2\in[0,\tau]$ from the fact that $|t_1-t_2|<\delta_2$ it follows that $||y(t_1)-y(t_2)||<\delta_1$.

By the mean value theorem $||x(t_1)-x(t_2)|| = ||x'(c)||\cdot|t_1-t_2| \le R|t_1-t_2|$. Then $||x(\tilde{\alpha}(t_1))-x(\tilde{\alpha}(t_2))|| = ||x'(c)||\cdot|\tilde{\alpha}(t_1)-\tilde{\alpha}(t_2)|$ where $c\in[\tilde{\alpha}(t_1),\tilde{\alpha}(t_2)]$.

Since α is continuous on $[0,\tau]$, it is uniformly continuous on this interval. Hence there exists $\delta_3>0$ depending on δ_1, such that for every $t_1,t_2\in[0,\tau]$ from the fact that $|t_1-t_2|<\delta_3$ it follows that $||\tilde{\alpha}(t_1)-\tilde{\alpha}(t_2)||<\frac{\delta_1}{R}$.

Introduce $\delta=\min\{\delta_1,\delta_2,\delta_3\}$. Then from the fact that $|t_1-t_2|<\delta$ it follows that $||\tilde{\alpha}(t_1)-\tilde{\alpha}(t_2)||<\delta_1$, $||y(t_1)-y(t_2)||<\delta_1$ $\;|t_1-t_2|<\delta_1$. Then

$$||\varphi_y(x)(t_1)-\varphi_y(x)(t_2)|| = ||\phi(K(x),y)(t_1)-\phi(K(x),y)(t_2)|| =$$

$$= ||g(t_1,x(\tilde{\alpha}(t_1)),y(t_1))-g(t_2,x(\tilde{\alpha}(t_2)),y(t_2))||<\varepsilon.$$

Thus, by the the Arzelá-Ascoli theorem the set M is relatively compact. Hence the mapping φ_y is A-completely continuous. □

It is evident that problem (4.2), (4.3) is equivalent to the following operator equation:

$$A(x)=\varphi(x,B(x)). \tag{4.4}$$

Theorem 4.1 *Let a mapping g satisfy conditions (g1) and (g2). Then there exists a number $h_0\in(0,\tau]$, such that problem (4.2), (4.3) has a solution on the interval $[0,h_0]$.*

Proof. Consider the mapping $g_{(0,0)}:\mathbb{R}^n\to\mathbb{R}^n$, $g_{(0,0)}(y)=g(0,0,y)$. By condition $(g2)$ this mapping is a contraction. Hence it has a unique fixed point y_0, i.e., $y_0=g(0,0,y_0)$. Consider the function $\hat{y}_0(t)=ty_0\in D(A)$, where $t\in[0,h]$.

Introduce the notation

$$T = \{x \in D(A) \subset C_{[0,h]} \,|\, ||x - \hat{y}_0|| \le r,\ ||x' - y_0||| \le m\},$$

where r and m are some positive numbers.

Let x be an arbitrary function from T. Estimate $||A(\hat{y}_0) - \varphi(x, B(x))||$. We have

$$||A(\hat{y}_0) - \varphi(x, B(x))|| = \max_{t \in [0,h]} ||y_0 - g(t, x(\tilde{\alpha}(t)), x'(\tilde{\beta}(t)))|| =$$

$$= \max_{t \in [0,h]} ||g(0, 0, y_0) - g(t, x(\tilde{\alpha}(t)), x'(\tilde{\beta}(t)))||.$$

If $t \in [0, h]$,

$$||g(0, 0, y_0) - g(t, x(\tilde{\alpha}(t)), x'(\tilde{\beta}(t)))|| \le ||g(0, 0, y_0) - g(0, 0, x'(\tilde{\beta}(t)))|| +$$

$$+ ||g(0, 0, x'(\tilde{\beta}(t))) - g(t, x(\tilde{\alpha}(t)), x'(\tilde{\beta}(t)))|| \le k\, ||y_0 - x'(\tilde{\beta}(t))|| +$$

$$+ ||g(0, 0, x'(\tilde{\beta}(t))) - g(t, x(\tilde{\alpha}(t)), x'(\tilde{\beta}(t)))||.$$

Since $x \in T$, $||y_0 - x'(\tilde{\beta}(t))|| \le m$.

Now estimate $||g(0, 0, x'(\tilde{\beta}(t))) - g(t, x(\tilde{\alpha}(t)), x'(\tilde{\beta}(t)))||$. It is evident that there exists a bounded closed set $G \in [0, h] \times \mathbb{R}^n \times \mathbb{R}^n$ such that from $x \in T$ it follows that $(t, x(\tilde{\alpha}(t)), x'(\tilde{\beta}(t))) \in G$. Since g is continuous, it is uniformly continuous on G, i.e., for every $\delta > 0$ there exists $\eta = \eta(\delta) > 0$ such that from $|t_1 - t_2| < \eta$ and $||x_1 - x_2|| < \eta \;\; ||y_1 - y_2|| < \eta$ uit follows that

$$||g(t_1, x_1, y_1) - g(t_2, x_2, y_2)|| < \delta.$$

Note that there exists a number $S > 0$ such that $||x'(t)|| \le S$ for every $t \in [0, h]$ and every $x \in T$. Then $||0 - x(\tilde{\alpha}(t))|| = ||x(0) - x(\tilde{\alpha}(t))|| \le S\, \tilde{\alpha}(t) \le S\, t$. Consider $\delta = m$. Then there exists $h_1 > 0$ such that $h_1 < \eta \;\; Sh_1 < \eta$. Hence

$$||g(0, 0, x'(\tilde{\beta}(t))) - g(t, x(\tilde{\alpha}(t)), x'(\tilde{\beta}(t)))|| < m$$

for every $t \in [0, h_1]$.

Thus, we obtain that if $t \in [0, h_1]$, then

$$||g(0, 0, y_0) - g(t, x(\tilde{\alpha}(t)), x'(\tilde{\beta}(t))|| \le k\, m + m = m(1 + k).$$

Specify a number l so that $l\, m(1 + k) < r$. Take a positive number $h_0 < \min\{h_1, l, \tau\}$. Consider equation (4.4) in the space $C_{[0,h_0]}$. Let x be an arbitrary function from T. We have

$$||A(\hat{y}_0) - f(x)|| \le m(1 + k) < \frac{r}{l},$$

where $l > h_0 = ||A^{-1}||$. Thus the hypothesis of Theorem 3.1 is fulfilled and so a solution of problem (4.2), (4.3) does exists. □

4.2 On global solutions of neutral type equations

Let (as well as above) $[0, \tau]$ be an interval of the real line.

Consider continuous functions $\lambda : [0, \tau] \to [0, \tau]$ and $\mu : [0, \tau] \to [0, \tau]$.

Let the mapping $g : [0, \tau] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be the same as in the previous subsection. Consider the following differential equation:

$$x'(t) = g(t, x(\lambda(t)), x'(\mu(t))), \tag{4.5}$$

$$x(0) = 0. \tag{4.6}$$

A continuously differentiable function x_* given on $[0, \tau]$ and satisfying equation (4.5) and condition (4.6), is called a solution of problem (4.5), (4.6) on $[0, \tau]$.

This problem has the following operator interpretation. Denote by $D(A)$ the st of continuously differentiable functions $x : [0, \tau] \to \mathbb{R}^n$ such that $x(0) = 0$. Let $A : D(A) \subset C_{[0,\tau]} \to C_{[0,\tau]}$ be an operator of differentiation. It is evident that A is continuously invertible and $A^{-1}(y)(t) = \int\limits_0^t y(s)\, ds$. It is easy to see that $||A^{-1}|| = \tau$.

Let $\phi : C_{[0,\tau]} \times C_{[0,\tau]} \to C_{[0,\tau]}$ be the superposition operator generated by the mapping g, i.e., $\phi(x, y)(t) = g(t, x(t), y(t))$.

Lemma 4.4 *Under the above assumption ϕ is continuous and satisfies the following conditions:*
(1) for every point $x \in C_{[0,\tau]}$ and every $y_1, y_2 \in C_{[0,\tau]}$ the inequlity $||\phi(x, y_1) - \phi(x, y_2)|| \leq k||y_1 - y_2||$ holds;
(2) for every $y \in C_{[0,\tau]}$ the mapping $\phi(\cdot, y) : C_{[0,\tau]} \to C_{[0,\tau]}$ is A-completely continuous.

The proof of Lemma 4.4 is obvious.

Consider the operator $K : C_{[0,\tau]} \to C_{[0,\tau]}$ determined by the following condition: $K(x)(t) = x(\lambda(t))$. Evidently it is a continuous linear operator and $||K|| \leq 1$. Consider also the operator $B : D(A) \to C_{[0,\tau]}$ determined by the condition $B(x)(t) = x'(\mu(t))$.

Lemma 4.5 *B is subordinated to A.*

The proof of Lemma 4.5 is quite analogous to that of Lemma 4.2.

Consider the mapping $\varphi(x, y) : C_{[0,\tau]} \times C_{[0,\tau]} \to C_{[0,\tau]}$, determined by the condition : $\varphi(x, y) = \phi(K(x), y)$.

Lemma 4.6 *Under the above conditions the mapping $\varphi_y = \varphi(\cdot, y) : C_{[0,\tau]} \to C_{[0,\tau]}$ is A-completely continuous for a specified y.*

The proof of Lemma 4.6 is quite analogous to that of Lemma 4.3.

Problem (4.5) is evidently equivalent to the following operator equation

$$A(x) = \varphi(x, B(x)). \tag{4.7}$$

Theorem 4.2 *If there exist constants $p \geq 0$ and $s \geq 0$ such that for every point $x \in C_{[0,\tau]}$ the inequality $||g(t, x(t), y(t))|| \leq p(||x(t)|| + ||y(t)||) + s$ holds and $p(\tau + 1) < 1$, then equation (4.6) has a solution.*

Proof. The mapping $\varphi(x, B(x))$ satisfies the hypothesis of Theorem 3.2.

Let us estimate $||\varphi(x, y)||$. We have:

$$||\varphi(x, y)|| = \max_{0 \leq t \leq \tau} ||\varphi(x, y)(t)|| \leq \max_{0 \leq t \leq \tau} ||g(t, x(\lambda(t)), y(t))|| \leq$$

$$\leq \max_{0 \leq t \leq \tau} (p(||x(\lambda(t))|| + ||y(t)||) + s) \leq p(||x|| + ||y||) + s$$

By Theorem 3.2 equation (4.6) has a solution if $p(||A^{-1}|| + 1) < 1$. Since $||A^{-1}|| = \tau$, this takes place if $p(\tau + 1) < 1$. □

4.3 An abstract scheme of neutral type equations

Let E_1, E_2, $E_2^{'}$ and E_3 be Banach spaces, $A : D(A) \subset E_1 \to E_2$ be a closed surjective linear operator and $J : E_2 \to E_2^{'}$ be a continuous surjective linear operator that satisfies the following condition:

(J1) there exists a continuous linear operator $Q : E_2^{'} \to E_2$ such that for every $z \in E_2$ the equality $J(Q(z)) = z$ holds.

Let $B : D(B) \subset E_1 \to E_3$ be a closed linear operator subordinated to A.

Let $\varphi : E_1 \times E_3 \to E_2^{'}$ be a continuous mapping. We assume that φ satisfies the following conditions:

(φ1) for every $z \in E_3$ the mapping $\varphi(\cdot, z) : E_1 \to E_2$ is A-completely continuous;

(φ2) there exists a number $k \in (0, \frac{1}{||Q||})$ such that for every $x \in E_1$ and every z_1, $z_2 \in E_3$ the inequality

$$||\varphi(x, z_1) - \varphi(x, z_2)|| \leq k\, ||z_1 - z_2||$$

holds.

We investigate the solvability of the following operator equation:

$$J(A(x)) = \varphi(x, B(x)). \tag{4.8}$$

It is easy to see that every solution of equation

$$A(x) = Q(\varphi(x, B(x)) \tag{4.9}$$

is a solution of equation (4.8). Let in E_2 a measure of non-compactness χ be given. We apply the results of previous subsection to studying equation (4.9). The following theorem takes place.

Theorem 4.3 *Let all above-mentioned assumptions be satisfied and let there exist constants* $\gamma \geq 0$ *and* $\beta \geq 0$ *such that for every* $x \in E_1$ *and* $z \in E_3$ *the inequality* $||\varphi(x,z)|| \leq \gamma(||x|| + ||z||) + \beta$ *holds. If* $||Q||\,\gamma\,(||A^{-1}|| + 1) < 1$, *equation (4.8) has a solution.*

Proof. Consider the mapping $\hat{\varphi} : E_1 \times E_3 \to E_2$ determined as follows: $\hat{\varphi}(x,z) = Q(\varphi(x,z))$. By the above-mentioned assumptions this mapping satisfies the hypothesis of Theorem 3.2. Let us estimate $||\hat{\varphi}(x,z)||$. We have:

$$||\hat{\varphi}(x,z)|| = ||Q(\varphi(x,z))|| \leq ||Q||\,(\gamma\,(||x|| + ||z||) + \beta) =$$

$$= ||Q||\,\gamma\,(||x|| + ||z||) + ||Q||\beta.$$

By Theorem 3.2 equation (4.9) has a solution if $||Q||\,\gamma\,(||A^{-1}|| + 1) < 1$. This proves the theorem. □

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References

[1] R.R. Ahmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina, B.N. Sadovskii, "Measures of Noncompactness and Condensing Operators", Operator Theory: Advance and Appl. V.55, Birkhauser, 1992.

[2] M. Kamenskii, V. Obukhovskii, P. Zecca, "Condensing multivalued maps and semilinear differential inclusions in Banach spaces", Walter de Gruyter, Berlin–New York, 2001.

[3] G. Hetzer, "Some remarks on operators and the coincidence degree for Fredholm equation with noncompact nonlinear perturbation", Ann. Soc. Sci. Bruxelles, Ser.1, Vol. 89, No. 1, pp. 497-508, 1975

[4] Yu.G. Borisovich, "Modern approach to the theory of topological characteristics of nonlinear operators. I.", Lecture Notes in Mathematics, Vol 1334, pp. 199-220, 1988

[5] Yu.G. Borisovich, "Modern approach to the theory of topological characteristics of nonlinear operators. II", Lecture Notes in Mathematics, Vol 1453, pp. 21-49, 1990.

[6] V.T. Dmitrienko, V.G. Zvyagin, "Homotopy classification of a class of continuous mappings", Mathematical Notes, Vol. 31, No. 5, pp 801-812, 1982.

[7] B. Ricceri, "On the topological dimension of the solution set of a class of nonlinear equations", C. R. Acad. Sci. Paris Sér. I Math., Vol. 325, No. 1, pp. 65–70, 1997

[8] B.D. Gel'man, "On a Class of Operator Equations", Mathematical Notes, Vol. 70, No 3-4, pp. 494-501, 2001.

[9] B.D. Gel'man, "Operator equations and Cauchy problems for degenerated differential equations", Proceedings of Voronezh State University, Series Physics Mathematics, No. 2., pp. 86-91, 2007.

[10] B.D. Gel'man, "Set-valued contraction mappings and their applications", Proceedings of Voronezh State University, Series Physics Mathematics, No. 1, pp. 74–86, 2009