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Derivative Securities in Markets with Bid-Ask Spreads

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Abstract

Recent developments and results concerned with pricing and hedging derivative securities in markets with proportional transaction costs represented as bid-ask spreads are reported. The focus is on a constructive approach to representing and computing the prices and hedging portfolios, leading to efficient numerical algorithms. A wide range of derivative securities, from plain vanilla European options to basket options, American and Bermudan type derivatives to game options are covered. New results are presented for hedging and pricing derivatives under deferred solvency conditions as well as game options.

Key words: Derivative securities, transaction costs, stochastic optimisation, convex duality

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1. Introduction

The study of market models and derivative securities in the presence of transaction costs has a long history. The problem was first considered by Merton [35], [36], followed by Dermody and Rockafellar [13], Boyle and Vorst [4], Bensaid, Lesne, Pagès and Scheinkman [2], Edirisinghe, Naik and Uppal [14], Jouini and Kallal [18], Kusuoka [29], Koehl, Pham and Touzi [26], [27], [28], Stettner [53], [54], Perrakis and Lefoll [41], [42], Rutkowski [50], Touzi [58], Jouini [17], Palmer [40], [39], Kociński [22], [24], [23], [25], Chen, Palmer and Sheu [6], [7], and others.

Several negative results have been established in continuous-time models by Soner, Shreve and Cvitanić [52], Levental and Skorohod [31], Cvitanić, Pham and Touzi [11] and Jakubenas, Levental and Ryznar [16]. In essence, these negative results show for various kinds of derivative securities, including European and American options, that optimal (least expensive) superhedging in the Black-Scholes model with transaction costs involves the trivial strategy of setting up holding a constant hedging portfolio up to the expiry time of the option. This motivates a study of market models in discrete time, as pursued by many authors and followed in this work.

In this article we adopt the general model of proportional transaction costs with discrete time steps put forward by Kabanov and Stricker [21] and developed by Kabanov, Rásonyi and Stricker [19], [20] and Schachermayer [51], involving several securities, with transaction costs implemented as bid-ask spreads via a matrix of exchange rates between the securities. Our aim is to present recent developments and new results concerning the pricing and hedging derivative securities of various kinds, including European options, basket options, American type derivatives and also game options in this general model.

Apart from the papers cited above, recent work in this direction, typically in more restrictive settings which can be included as special cases within Kabanov and Stricker's [21] model, includes papers by Chalasani and Jha [5], Tokarz [56], Bouchard and Temam [3], Tokarz and Zastawniak [57], Roux [45], Roux, Tokarz and Zastawniak [46], Roux and Zastawniak [48], [49], Löhne and Rudloff [33], Zhang, Roux and Zastawniak [59], Tien [55], and others.

Another group of papers, using preference-based or risk minimisation approaches (not pursued in here) under proportional transaction costs, includes Hodges and Neuberger [15], Davis and Zariphopoulou [12], Mercurio and Vorst [34], Lamberton, Pham and Schweizer [30], Constantinides and Zariphopoulou [10], and Constantinides and Perrakis [9], Monoyios [37], [38].

The study of derivative securities under transaction costs reveals some apparent paradoxes and curious effects. Here are a few examples. Overreplicating an option payoff may turn out to be less expensive than strict replication, an effect pointed out in [13] and [2], see also [47]. The worst exercise strategy that the writer of an American option needs to hedge against may be different than the best strategy for the holder, see [48]. These and other curious effects call for a careful treatment and a precise study of derivative securities under transaction costs. In this paper, we indicate another somewhat unexpected effect: under transaction costs a portfolio that is insolvent at a time instant t may sometimes be rescued by means of rebalancing to become solvent with probability 1 at a later time. We shall refer to this effect as deferred solvency.

It turns out that it is intimately liked with American options that allow for gradual exercise, see Section .

This article is organised as follows. First we describe the model with transaction costs in Section 4. This is followed by Section 2 on European options, Section 3 on American (and Bermudan) options, which also includes some numerical examples as well as ramifications involving deferred solvency and options with gradual exercise policies, and Section 5 devoted to game options. Section 6 concludes with some open problems and questions. There is an Appendix covering the notion of mixed (randomised) stopping times and associated notation.

2. Market model

We consider a model with d assets and discrete trading dates $t = 0, \dots, T$ on a probability space $(\Omega, \mathcal{F}, Q)$ equipped with a filtration $(\mathcal{F}_t)_{t=0}^T$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. The probability measure Q is the physical market probability. The exchange rates between the assets are represented by an adapted $d \times d$-matrix-valued process

$$\pi_t = (\pi_t^{ij})_{i,j=1}^d$$

for $t = 0, \dots, T$, where $\pi_t^{ij} > 0$ is the number of units (shares) of asset i that need to be exchanged to receive one unit of asset j at time t. Bid-ask spreads (proportional transaction costs) are present whenever $\pi_t^{ij}\pi_t^{ji} > 1$ for a pair of assets i, j, which means that a round-trip exchange of asset i into j and immediately back into i involves a loss. The bid-ask spread is then represented by the interval $[1/\pi_t^{ji}, \pi_t^{ij}]$. We call π the *bid-ask process*.

This model, introduced by Kabanov and Stricker [21] and further developed by Kabanov, Rásonyi and Stricker [19], [20] and Schachermayer [51], is often referred to as a currency market model under transaction costs, even though the assets do not necessarily have to be currencies. Note that, in general, it is unnecessary to specify a numéraire among the assets.

We write $\mathcal{L}_t$ for the family of $\mathcal{F}_t$-measurable $\mathbb{R}^d$-valued random variables for $t = 0, \dots, T$ and put $\mathcal{L}_{-1} = \mathcal{L}_0$ for convenience. Such random variables $x = \left(x^1, \dots, x^d\right) \in \mathcal{L}_t$ represent portfolios consisting of the assets in our model, with x^i the position in asset $i = 1, \dots, d$. We also denote by $\mathcal{L}_t^+$ the set of portfolios $y \in \mathcal{L}_t$ such that $y^i \geq 0$ for each $i = 1, \dots, d$.

A portfolio $x \in \mathcal{L}_t$ can be exchanged into a portfolio $y \in \mathcal{L}_t$ at time t whenever there are $\mathcal{F}_t$-measurable random variables $\beta^{ij} > 0$ for $i, j = 1, \dots, d$ such that

$$y^j = x^j + \sum_{i=1}^d \beta^{ij} - \sum_{i=1}^d \beta^{ji}\pi_t^{ji} \text{ for all } j = 1, \dots, d.$$

Here β^{ij} represents the number of units of asset j received as a result of exchanging asset i.

We say that portfolio x is *solvent* at time t if it can be exchanged into a portfolio $y \in \mathcal{L}_t^+$. The set of solvent portfolios at time t is a convex cone. It will be denoted by $\mathcal{K}_t$ and called the *solvency cone*.

A *self-financing strategy* $(z_t)_{t=0}^T$ is an $\mathbb{R}^d$-valued process such that $z_t \in \mathcal{L}_{t-1}$ for all $t = 0, \ldots, T$ (i.e. z is an $\mathcal{F}_t$-predictable process) and

$$z_t - z_{t+1} \in \mathcal{K}_t \text{ for all } t = 0, \ldots, T-1.$$

Note that this self-financing condition allows for rebalancing and the withdrawal of assets from the strategy (i.e. consumption) but not for the injection of any assets. We denote the set of all self-financing strategies by Φ.

Definition 2.1. Consider a model of d assets with bid-ask process π.

(i) (Definition 1.6 (i) in [51]) The model satisfies the *no-arbitrage property* if for every self-financing strategy $z \in \Phi$ starting with zero initial endowment $z_0 = 0$ there is no portfolio $x \in \mathcal{L}_T^+ \setminus \{0\}$ such that z_T can be exchanged into x, that is, such that $z_T - x \in \mathcal{K}_T$.

(ii) (Definition 1.9 in [51]) The model satisfies the *robust no-arbitrage condition* if there is an bid-ask process $\tilde{\pi}$ with smaller bid-ask spreads than π such that the model with exchange rates $\tilde{\pi}$ satisfies the no-arbitrage property.

We say that an bid-ask process $\tilde{\pi}$ has *smaller bid-ask spreads* than π if the bid-ask spread interval $[1/\tilde{\pi}_t^{ji}, \tilde{\pi}_t^{ij}]$ is contained in the relative interior of $[1/\pi_t^{ji}, \pi_t^{ij}]$ for each $i, j = 1, \ldots, d$ and $t = 0, \ldots, T$.

Property (i) was introduced by Kabanov and Stricker [21] under the name of weak no-arbitrage property. We follow Schachermayer [51] in calling it simply the no-arbitrage property. The robust no-arbitrage condition (ii) is due to Schachermayer [51].

The following version of the Fundamental Theorem of Asset Pricing for a finite probability space Ω (important for numerical work) was established by Kabanov and Stricker [21].

Theorem 2.2. Assume that Ω is finite. A model with bid-ask process π satisfies the no-arbitrage property if and only if there exist a probability measure P equivalent to Q and an $\mathbb{R}^d$-valued P-martingale $S = (S_t)_{t=0}^T$ such that S_t belongs to $\mathcal{K}_t^* \setminus \{0\}$ almost surely for all $t = 0, \ldots, T$, where $\mathcal{K}_t^*$ is the polar (see below) of $-\mathcal{K}_t$.

For any set $A \subset \mathbb{R}^d$ we denote by A^* the *polar* of $-A$, defined as (see Rockafellar [44])

$$A^* = \{y \in \mathbb{R}^d : y \cdot x \geq 0 \text{ for all } x \in A\},$$

where $\cdot$ is the scalar product.

The following ramification of this theorem applicable in the general case of an arbitrary probability space Ω is due to Schachermayer [51].

Theorem 2.3. A model with bid-ask process π satisfies the robust no-arbitrage condition if and only if there exist a probability measure P equivalent to Q and an $\mathbb{R}^d$-valued P-martingale $S = (S_t)_{t=0}^T$ such that S_t belongs to the relative interior of $\mathcal{K}_t^*$ almost surely for all $t = 0, \ldots, T$, where $\mathcal{K}_t^*$ is the polar of $-\mathcal{K}_t$.

A P-martingale S satisfying the conditions in Theorem 2.2 (in Theorem 2.3) is called a *consistent* (respectively, *strictly consistent*) *price system*, such an equivalent probability measure P is called a *risk-neutral probability*, and (P, S) a *consistent* (*respectively, strictly consistent*) *pricing pair*.

Kabanov and Stricker [21] also formulated an alternative notion of no-arbitrage, called the *strict no-arbitrage property* (Definition 1.6 (ii) in [51]) and established the corresponding version of the Fundamental Theorem of Asset Pricing, subject to an additional assumption of 'efficient friction.' This approach will not be pursued in the present paper.

An earlier version of the model with bid-ask spreads (proportional transaction costs) can be traced back to Jouini and Kallal [18], who studied a market model with $d-1$ risky assets and a risk-free asset (numéraire) in which the bid and ask prices $0 < S_t^{bi} \leq S_t^{ai}$ of the risky assets $i = 1, \ldots, d-1$ are expressed in terms of the units of the numéraire asset. Jouini and Kallal's model is a special case of the model with a matrix-valued bid-ask process π due to Kabanov and Stricker [21] such that exchanges between the risky assets can only be performed via the numéraire asset, that is, with

$$\begin{array}{ll} \pi_t^{di} = S_t^{ai} & \text{for all } i = 1, \ldots, d-1, \\ \pi_t^{id} = 1/S_t^{bi} & \text{for all } i = 1, \ldots, d-1, \\ \pi_t^{ij} = S_t^{aj}/S_t^{bi} & \text{for all } i = 1, \ldots, d-1 \text{ such that } i \neq j, \\ \pi_t^{ii} = 1 & \text{for all } i = 1, \ldots, d. \end{array} \tag{2.1}$$

A similar version of the Fundamental Theorem of Asset Pricing was established by Jouini and Kallal [18] in this context. Note that in the setting of Jouini and Kallal's work [18], the condition that S_t belongs to $\mathcal{K}_t^* \setminus \{0\}$ (to the relative interior of $\mathcal{K}_t^*$) is equivalent to S_t^i belonging to the bid-ask interval $[S_t^{bi}, S_t^{ai}]$ (respectively, to the relative interior of $[S_t^{bi}, S_t^{ai}]$) for each $i = 1, \ldots, d-1$.

3. European options

The focus in this section is on computing option prices, hedging portfolios and hedging strategies from the point of view of the writer (also referred to as the seller) and the holder (or buyer) of the option. We assume the robust no-arbitrage condition, or, when Ω is finite, just the no-arbitrage property (see Definition 2.1) within the d-asset model with matrix-valued bid-ask process π described in Section 2.

The payoff of an option with physical settlement can be expressed as the portfolio $\xi = (\xi^1, \ldots, \xi^d)$ of assets that is passed from the option writer to the holder when the option is exercised (negative positions in the portfolio correspond to assets passed in the opposite direction). In the presence of transaction costs (bid-ask spreads) options with physical settlement need to be considered for full generality because normally there is no equivalent option with cash settlement.

For example, in a market consisting of just two assets, a stock and a cash account, the payoff of a put option on the stock with physical delivery can be represented by the portfolio $(-1, K)$, where K is the strike price: when exercising the option the holder transfers one share of stock to the writer and receives the strike price K in cash. Meanwhile, a put with cash settlement would be represented by the portfolio $(0, (K-S)^+)$: no stock changes hands and the option holder receives the cash amount $(K-S)^+$, where $S \in [S^b, S^a]$ is a notional stock price within the bid-ask spread (for instance, the mid-price $S = (S^a + S^b)/2$). These two puts are not the same when $S^b < S^a$.

Let $\xi \in \mathcal{L}_T$. A European option with payoff ξ and exercise time T gives the option holder the right to receive the portfolio ξ and obliges the writer to deliver this portfolio on demand at time T.

Definition 3.1. (i) An initial endowment $y \in \mathbb{R}^d$ is said to *hedge a short* (*writer's*) *position in a European option* ξ if there is a self-financing strategy $z \in \Phi$ with $z_0 = y$ such that almost surely

$$z_T - \xi \in \mathcal{K}_T. \tag{3.1}$$

Such a $z \in \Phi$ is called a *hedging strategy for a short* (*writer's*) *position in the European option* ξ.

(ii) An initial endowment $y \in \mathbb{R}^d$ is said to *hedge a long* (*holder's*) *position in a European option* ξ if there is a self-financing strategy $z \in \Phi$ with $z_0 = y$ such that almost surely

$$z_T + \xi \in \mathcal{K}_T. \tag{3.2}$$

Such a $z \in \Phi$ is called a *hedging strategy for a long* (*holder's*) *position in the European option* ξ.

Condition (3.1) means that an option writer who follows strategy z will be solvent a.s. after delivering payoff ξ to the holder when the option is exercised at time T. According to condition (3.2), an option holder who follows strategy z will be solvent a.s. after receiving payoff ξ from the writer when the option is exercised at T.

Definition 3.2. For $i = 1, \dots, d$ let e_i denote the vector in $\mathbb{R}^d$ with components $e_i^j = 0$ for $i \neq j$ and $e_i^i = 1$.

(i) The *ask price* (or *writer's price*) at time 0 of a European option ξ expressed in units of asset i is defined as

$$p_i^a(\xi) = \inf\{s \in \mathbb{R} : se_i \text{ hedges a short position in } \xi\}.$$

(ii) The *bid price* (or *holder's price*) at time 0 of a European option ξ expressed in units of asset i is defined as

$$p_i^b(\xi) = \sup\{-s \in \mathbb{R} : se_i \text{ hedges a long position in } \xi\}.$$

The meaning of the option ask price is that an endowment of at least $p_i^a(\xi)$ units of asset i at time 0 would enable the writer to settle the option at time T and remain a.s. in a solvent position if a suitable hedging strategy is followed. On the other hand, the bid price $p_i^b(\xi)$ is the largest number of units of asset i that can be shorted at time 0 by the option holder that would leave him a.s. in a solvent position after exercising the option at time T, provided that a suitable hedging strategy is followed.

Another interpretation of $p_i^a(\xi)$ and $p_i^b(\xi)$ is that they represent liquidity prices for the option: it should be possible to buy the option on demand at or above the ask price $p_i^a(\xi)$ or sell it on demand for or below the bid price $p_i^b(\xi)$. Option prices outside the bid-ask interval $[p_i^b(\xi), p_i^a(\xi)]$ create arbitrage opportunities. Option prices inside $(p_i^b(\xi), p_i^a(\xi))$ are free of arbitrage. Depending on the model, option prices at the end-points of this interval may or may not involve arbitrage.

It is an immediate consequence of the definitions above that $z \in \Phi$ hedges a short position in the European option with payoff ξ if and only if it hedges a long position in the European option with payoff $-\xi$, and that

$$p_i^a(\xi) = -p_i^b(-\xi).$$

As a consequence, we only need to focus on the ask price and hedging a short position in the option (the writer's case).

Theorem 3.3. Consider a European option with exercise time T and payoff ξ. Let the following sequences of sets be constructed by backward induction:

$$\mathcal{Z}_T = \xi + \mathcal{K}_T, \tag{3.3}$$

and for $t = T-1, \ldots, 1, 0$

$$\begin{aligned} \mathcal{W}_t &= \mathcal{Z}_{t+1} \cap \mathcal{L}_t, \\ \mathcal{Z}_t &= \mathcal{W}_t + \mathcal{K}_t. \end{aligned} \tag{3.4}$$

Then:

(i) $\mathcal{Z}_0$ is the collection of initial endowments $y \in \mathbb{R}^d$ that hedge a short position in the European option with payoff ξ (equivalently, hedge the long position in the option with payoff $-\xi$).

(ii) The ask price of the European option can be expressed as

$$p_i^a(\xi) = \min\{s \in \mathbb{R} : se_i \in \mathcal{Z}_0\}.$$

(iii) There exists a strategy $z \in \Phi$ with initial endowment $z_0 = p_i^a(\xi)e_i$ hedging a short position in the European option ξ. The strategy z can be constructed algorithmically once the sequences $(\mathcal{Z}_t)$, $(\mathcal{W}_t)$ are known.

The proof of this theorem is straightforward and can be extracted from Roux and Zastawniak [49]. An earlier version in a simpler setting involving a single underlying asset and a cash account goes back to Roux, Tokarz and Zastawniak [46].

Another approach to constructing the collection of initial endowments hedging a short (or long) position in a European option in a d-asset model subject to the stronger robust no-arbitrage condition, based on the theory of linear vector optimisation (see, for example, Löhne [32]), can be found in a recent paper by Löhne and Rudloff [33].

The construction in Theorem 3.3 lends itself well to numerical computation. In the case of a finite Ω the $\mathcal{Z}_t$ are polyhedral convex subsets in a finite-dimensional Euclidean space, and the construction involves simple operations on such sets. These can readily be implemented on a computer, which was done in Tokarz, Roux and Zastawniak [46] and in the more general context of American type options in Roux and Zastawniak [48], [49] and Zhang, Roux and Zastawniak [59]; see also the numerical examples in Section 4.3 in the present paper. Another possibility for numerical implementation are the linear vector optimisation algorithms applied in Löhne and Rudloff [33].

Seeing that the $\mathcal{Z}_t$ are convex sets, as is the collection of strategies hedging a short position in the option, it is hardly surprising that convex duality methods can be exploited to characterise the hedging strategies and option prices. In Schachermayer [51] (which is a generalisation of earlier similar results by Kabanov and Stricker [21] and Kabanov, Rásonyi, Stricker [19], [20]) the following dual characterization of hedging strategies was established.

Theorem 3.4. Consider a European option with exercise time T and payoff $\xi \in \mathcal{L}_T$. Let $z \in \Phi$ be a self-financing strategy. Then the following conditions are equivalent:

(i) z is hedges a short position in the option, that is, almost surely

$$z_T - \xi \in \mathcal{K}_T.$$

(ii) For each $(P, S) \in \mathcal{P}$

$$\mathbb{E}_P[\xi \cdot S_T] \le z_0 \cdot S_0,$$

where $\mathcal{P}$ denotes the collection of consistent pricing pairs (P, S) such that the negative part $(\xi \cdot S_T)^-$ of $\xi \cdot S_T$ is integrable under P.

Theorem 3.4 implies the following characterisation of initial endowments hedging the option:

$$\mathcal{Z}_0 = \{p \in \mathbb{R}^d : \mathbb{E}_P[\xi \cdot S_T] \le p \cdot S_0 \text{ for every } (P, S) \in \mathcal{P}\}.$$

According to (2.1), this, in turn, implies the following representations for the ask/bid option prices:

$$\begin{aligned} p_i^a(\xi) &= \inf\{s \in \mathbb{R} : se_i \in \mathcal{Z}_0\} \\ &= \inf\{s \in \mathbb{R} : \mathbb{E}_P[\xi \cdot S_T] \le se_i \cdot S_0 \text{ for every } (P, S) \in \mathcal{P}\} \\ &= \inf\{s \in \mathbb{R} : \mathbb{E}_P[\xi \cdot S_T] \le s \text{ for every } (P, S) \in \mathcal{P}_i\} \\ &= \sup_{(P,S)\in\mathcal{P}_i} \mathbb{E}_P[\xi \cdot S_T], \qquad (3.5) \\ p_i^b(\xi) &= -p_i^a(-\xi) = -\sup_{(P,S)\in\mathcal{P}_i} \mathbb{E}_P[-\xi \cdot S_T] = \inf_{(P,S)\in\mathcal{P}_i} \mathbb{E}_P[\xi \cdot S_T], \qquad (3.6) \end{aligned}$$

where

$$\mathcal{P}_i = \{(P, S) \in \mathcal{P} : e_i \cdot S_t = 1 \text{ for all } t = 0, \dots, T\}. \qquad (3.7)$$

These representations of ask/bid prices for a European option under transaction costs go back to Jouini and Kallal [18], who proved them in the special case (2.1) and just for $i = d$ being the numéraire asset.

The results presented above leave open the question of how to compute a consistent pricing pair (P, S) realising the supremum or infimum representing the ask/bid option prices in (3.5), (3.6). This was resolved by Roux, Tokarz, Zastawniak [46] and Roux, Zastawniak [49] in the case when Ω is finite. The solution is based on a convex duality counterpart to the construction in Theorem 3.3.

4. American and Bermudan options

We continue to work within the d-asset model with matrix-valued bid-ask process π, and assume the robust no-arbitrage condition, or, when Ω is finite, just the no-arbitrage property (see Definition 2.1).

Let $\xi = (\xi_t)_{t=0}^T$ be an $\mathbb{R}^d$-valued adapted process, that is, $\xi_t \in \mathcal{L}_t$ for all $t = 0, \dots, T$. An American option with payoff process ξ and expiry time T gives the option holder the right to receive the portfolio ξ_τ and obliges the writer to deliver this portfolio on demand at any stopping time $\tau \in \mathcal{T}$ selected by the holder. Here $\mathcal{T}$ denotes the collection of (ordinary) stopping times with values in $\{0, \dots, T\}$. (Later on we shall also need the collection $\mathcal{X}$ of so-called mixed stopping times; see Appendix .)

The results in this section extend readily to Bermudan options by replacing the set $\{0, \dots, T\}$ of possible exercise times by a subset $\{t_1, \dots, t_K\} \subset \{0, \dots, T\}$.

Definition 4.1. (i) An initial endowment $y \in \mathbb{R}^d$ is said to *hedge a short* (*writer's*) *position in an American option* ξ if there is a self-financing strategy $z \in \Phi$ with $z_0 = y$ such that for all $\tau \in \mathcal{T}$

$$z_\tau - \xi_\tau \in \mathcal{K}_\tau$$

almost surely. Such a strategy $z \in \Phi$ is called a *hedging strategy for a short* (*writer's*) *position in the American option* ξ.

(ii) An initial endowment $y \in \mathbb{R}^d$ is said to *hedge a long* (*holder's*) *position in an American option* ξ if there exist a stopping time $\tau \in \mathcal{T}$ and a self-financing strategy $z \in \Phi$ with $z_0 = y$ such that

$$z_\tau + \xi_\tau \in \mathcal{K}_\tau$$

almost surely. Such a pair $(\tau, z) \in \mathcal{T} \times \Phi$ is called a *hedging strategy for a long* (*holder's*) *position in the American option* ξ.

Condition (i) means that the writer of an American option ξ who follows a strategy $z \in \Phi$ hedging a short position in the option will be solvent a.s. after delivering the payoff ξ_τ at any exercise time $\tau \in \mathcal{T}$ chosen by the holder. On the other hand, according to (ii), the holder of an American option ξ who follows a strategy $(\tau, z) \in \Phi$ hedging a long position in the option will be solvent a.s. after receiving the payoff ξ_τ when he/she exercises the option at time τ chosen by the holder as part of the strategy.

Definition 4.2. (i) The *ask price* (or *writer's price*) at time 0 of an American option ξ expressed in units of asset i is defined as

$$p_i^a(\xi) = \inf\{s \in \mathbb{R} : se_i \text{ hedges a short position in } \xi\}.$$

(ii) The *bid price* (or *holder's price*) at time 0 of an American option ξ expressed in units of asset i is defined as

$$p_i^b(\xi) = \sup\{-s \in \mathbb{R} : se_i \text{ hedges a long position in } \xi\}.$$

In contrast to European options, there is no symmetry between the writer's and the seller's hedging strategies for an American option, and therefore no simple relationship between the ask and bid option prices $p_i^a(\xi)$ and $p_i^b(\xi)$. The writer's and seller's cases need to be treated separately. In particular, the infimum defining the ask price $p_i^a(\xi)$ can be seen as the solution to convex optimisation problem, but this is not so for the supremum defining the bid price $p_i^a(\xi)$.

4.1 Writer's case

The following result by Roux and Zastawniak [48], [49] provides a construction of a strategy hedging a short position in an American option and a representation of the ask price of such an option.

Theorem 4.3. Given an American option ξ, let

$$\mathcal{X}_t := \xi_t + \mathcal{K}_t,$$

and let the following sequences of sets be constructed by backward induction:

$$\mathcal{Z}_T = \mathcal{X}_T,$$

and for $t = T-1, \ldots, 1, 0$

$$\begin{aligned} \mathcal{W}_t &= \mathcal{Z}_{t+1} \cap \mathcal{L}_t, \\ \mathcal{V}_t &= \mathcal{W}_t + \mathcal{K}_t, \\ \mathcal{Z}_t &= \mathcal{V}_t \cap \mathcal{X}_t. \end{aligned}$$

Then:

(i) $\mathcal{Z}_0$ is the collection of initial endowments $y \in \mathbb{R}^d$ that hedge a short position in the American option ξ.

(ii) The ask price of the American option can be expressed as

$$p_i^a(\xi) = \min\{s \in \mathbb{R} : se_i \in \mathcal{Z}_0\}.$$

(iii) There exists a strategy $z \in \Phi$ with initial endowment $z_0 = p_i^a(\xi)e_i$ hedging a short position in the American option ξ. The strategy z can be constructed algorithmically once the sequences $(\mathcal{Z}_t)$, $(\mathcal{V}_t)$, $(\mathcal{W}_t)$ are known.

The writer's case for American options admits a convex dual approach. Chalasani and Jha [5] were the first to obtain relevant results in a more restricted model with transaction costs compared to that considered here, discovering the crucial if somewhat surprising role played in this context by so-called mixed (or randomised) stopping times. A convex dual characterisation of writer's hedging strategies for American options in the general case of a d-asset market with matrix-valued bid-ask process π was established by Bouchard and Temam [3]. It can be formulated as follows. The terminology and notation involving mixed stopping times are explained in Section 7.

Theorem 4.4. Consider an American option with payoff process ξ. Let $z \in \Phi$ be a self-financing strategy. Then the following conditions are equivalent:

(i) z hedges a short position in the option, that is, for all $\tau \in \mathcal{T}$

$$z_\tau - \xi_\tau \in \mathcal{K}_\tau.$$

(ii) For each mixed stopping time $\chi \in \mathcal{X}$ and for each $(P, S) \in \mathcal{P}(\chi)$

$$\mathbb{E}_P[(\xi \cdot S)_\chi] \leq z_0 \cdot S_0,$$

where $\mathcal{P}(\chi)$ is defined below.

Definition 4.5. Let $\chi \in \mathcal{X}$ be a mixed stopping time. We denote by $\mathcal{P}(\chi)$ the collection of pairs (P, S), where P is a probability measure equivalent to Q and S is an $\mathbb{R}^d$-valued adapted process such that for all $t = 0, \ldots, T$

$$S_t \in \mathcal{K}_t^* \setminus \{0\}, \quad \mathbb{E}_P(S_{t+1}^{\chi^*} | \mathcal{F}_t) \in \mathcal{K}_t^*$$

and the process $(\xi \cdot S_T)^-$ is integrable under P. We call such (P, S) a χ-*approximate consistent pricing pair*.

For any $i = 1, \ldots, d$ we also denote by $\mathcal{P}_i(\chi)$ the collection of χ-approximate consistent pricing pairs $(P, S) \in \mathcal{P}(\chi)$ such that $e_i \cdot S_t = 1$ for all $t = 0, \ldots, T$.

In a similar manner as for European options, Theorem 4.4 implies that the ask price of an American contingent claim can be represented as

$$p_i^a(\xi) = \sup_{\chi \in \mathcal{X}} \sup_{(P,S) \in \mathcal{P}_i(\chi)} \mathbb{E}_P[(\xi \cdot S)_\chi]. \tag{4.1}$$

This is interesting as it shows that the writer of the option has to be prepared for the worst-case scenario of the potion being exercised gradually at a mixed stopping time, even if the option holder is only allowed to exercise in one go, that is, at some ordinary stopping time.

Theorem 4.4 does not provide a method to construct the mixed stopping time $\hat{\chi}$ that realises the maximum in (4.1) or the $\hat{\chi}$-approximate consistent pricing pair $(\hat{P}, \hat{S})$ realising the supremum in (4.1). This question was solved by Roux and Zastawniak [48], [49].

Theorem 4.6. If Ω is finite and the no-arbitrage property holds, then the ask price of the American option with payoff process ξ can be represented as

$$\begin{aligned} p_i^a(\xi) &= \max_{\chi \in \mathcal{X}} \sup_{(P,S) \in \mathcal{P}_i(\chi)} \mathbb{E}_P[(\xi \cdot S)_\chi] \\ &= \mathbb{E}_{\hat{P}}[(\xi \cdot \hat{S})_{\hat{\chi}}], \end{aligned}$$

where $\hat{\chi} \in \mathcal{X}$ and $(\hat{P}, \hat{S}) \in \overline{\mathcal{P}_i(\hat{\chi})}$ can be constructed algorithmically. (By $\overline{A}$ we denote the closure of A.)

Numerical examples based on a computer implementation of the algorithmic constructions alluded to in Theorems 4.3 and 4.6 are presented in Section 4.3. For a parallel computing implementation, see [59]. More numerical examples and a graphical illustration of the algorithms can be found in [48], [49].

4.2 Holder's case

The next theorem provides a construction of a strategy hedging a long position in an American option. The result is due to Roux and Zastawniak [48], [49].

Theorem 4.7. Given an American option ξ, let

$$\mathcal{X}_t := -\xi_t + \mathcal{K}_t,$$

and let the following sequences of sets be constructed by backward induction:

$$\mathcal{Z}_T = \mathcal{X}_T,$$

and for $t = T-1, \ldots, 1, 0$

$$\begin{aligned} \mathcal{W}_t &= \mathcal{Z}_{t+1} \cap \mathcal{L}_t, \\ \mathcal{V}_t &= \mathcal{W}_t + \mathcal{K}_t, \\ \mathcal{Z}_t &= \mathcal{V}_t \cup \mathcal{X}_t. \end{aligned}$$

Then:

(i) $\mathcal{Z}_0$ is the collection of initial endowments $y \in \mathbb{R}^d$ that hedge a long position in the American option ξ.

(ii) The bid price of the American option can be expressed as

$$p_i^b(\xi) = \max\{-s \in \mathbb{R} : se_i \in \mathcal{Z}_0\}.$$

(iii) There exists a self-financing strategy $z \in \Phi$ with initial endowment $z_0 = -p_i^b(\xi)e_i$ such that (τ, z) with

$$\tau = \min\{t = 0, \dots, T : z_t \in \mathcal{X}_t\}$$

is a strategy hedging a long position in the American option ξ. The strategy can be constructed algorithmically once the sequences $(\mathcal{Z}_t)$, $(\mathcal{V}_t)$, $(\mathcal{W}_t)$ are known.

Although formally similar to the construction in the writer's case, a crucial difference in the above theorem for the holder is the union of sets in place of the intersection in the formula for $\mathcal{Z}_t$. Because of this, the sets $\mathcal{Z}_t, \mathcal{V}_t, \mathcal{W}_t$ are not convex, in general. Because of this, it is no longer possible to apply convex duality methods. Nonetheless, the following representation of the bid price for an American option can be established, see Roux and Zastawniak [48], [49].

Theorem 4.8. If Ω is finite and the no-arbitrage property holds, then the bid price of the American option with payoff process ξ can be represented as

$$\begin{aligned} p_i^b(\xi) &= \max_{\tau \in \mathcal{T}} \inf_{(P,S) \in \mathcal{P}_i(\tau)} \mathbb{E}_P[(\xi \cdot S)_\tau] \\ &= \mathbb{E}_{\check{P}}[(\xi \cdot \check{S})_{\check{\tau}}], \end{aligned}$$

where $\check{\tau} \in \mathcal{T}$ is the same stopping time as in Theorem 4.7 (iii), and where $(\check{P}, \check{S}) \in \overline{\mathcal{P}_i(\check{\tau})}$ realising the infimum can be constructed algorithmically.

Observe that this bid price representation involves ordinary stopping times only, in contrast to the writer's case where it was necessary to admit mixed stopping times. In particular, it is remarkable that in general the worst stopping time $\hat{\chi} \in \mathcal{X}$ that the writer needs to be prepared for differs from the best stopping time $\check{\tau} \in \mathcal{T}$ for the option holder. Although the holder is likely to behave rationally and exercise the option at time $\check{\tau}$, this is not guaranteed, and to be fully hedged the writer needs to be prepared for other possibilities.

Numerical examples based on a computer implementation of the algorithmic constructions alluded to in Theorems 4.7 and 4.8 are presented in Section 4.3. For a parallel computing implementation, see [59]. More numerical examples and a graphical illustration of the algorithms can be found in [48], [49].

4.3 Numerical examples

Previously reported numerical examples [46], [48] demonstrate that the algorithmic constructions reported in Sections 3, 4.1, 4.2 apply to options with arbitrary payoffs, cover the full range of transaction costs, and are by no means restricted to the binomial model. The efficiency of the pricing algorithms is due to the fact that, when pricing options with path-independent payoffs, the number of computations grows only polynomially with the number of time steps. In this section we present two realistic examples in some detail.

In friction-free models, where assets are freely exchangeable, it is self-evident that exercising an American option is of benefit to its owner whenever the option payoff can be converted into a non-negative number of units of one of the assets. In the presence of transaction costs, the situation is no longer as clear-cut, as the desirability of the payoff (and hence the exercise decision) also depends on the current position in all the assets of the holder at the time that the payoff becomes available. The holder may therefore choose not to exercise the option at all. Motivated by the work of Perrakis and Lefoll [43], we allow for this by formally adding an extra time step $T+1$ and setting the option payoff at that time to be zero.

Example 4.9. Consider a binomial tree model with two risky assets. We assume a notional friction-free exchange rate $E=(E_t)$ between the two assets satisfying

$$E_{t+1}=\epsilon_t E_t$$

for $t=0,\dots,T-1$, where $E_0=10$ is given, and where (ε_t) is a sequence of independent identically distributed random variables taking the values

$$e^{\frac{\kappa}{T}+\sigma\sqrt{\frac{1}{T}}},\quad e^{\frac{\kappa}{T}-\sigma\sqrt{\frac{1}{T}}},$$

each with positive probability. Here $\sigma=0.1$ is the volatility of the exchange rate, $\kappa=0.05$ is the depreciation rate of the first asset in terms of the second, the time horizon is 1 year and $T=250$ is the number of steps in the model. We further assume that for $t=0,\dots,T$ the actual exchange rates are

$$\pi_t^{12}=(1+k)E_t,\quad \pi_t^{11}=\pi_t^{22}=1,\quad \pi_t^{21}=\frac{1}{(1-k)E_t},$$

where $k=0.5\%$ is the transaction cost rate. A portfolio $z_t=(z_t^1,z_t^2)$ is solvent at time t if and only if

$$\min\{z_t^1\pi_t^{21}+z_t^2, z_t^1+z_t^2\pi_t^{12}\}\geq 0.$$

Consider now an American put option on the second asset with strike 20 and physical delivery that offers the portfolio

$$\xi_t=(\xi_t^1,\xi_t^2)=(20,-1) \tag{4.2}$$

at any time t. This corresponds to an American put option on the second asset with physical delivery and strike 20. The constructions referred to in Sections 4.1 and 4.2 and described in detail in [48], [49] yield

$$p_1^b(\xi)=0.0159486,\quad p_1^a(\xi)=1.32909.$$

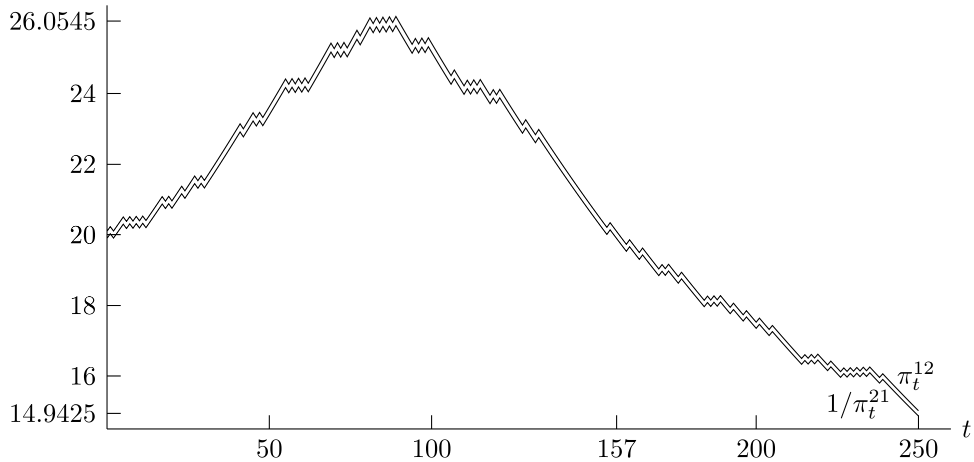

Figure 1: Exchange rate sample path in Example 4.9.

Thus an initial endowment of 1.32909 in the first asset would allow the writer to deliver this option without risk, whereas the holder would be able to borrow 0.0159486 shares in the first asset by using the payoff of the option as collateral.

Consider the optimal stopping times and hedging strategies for the sample path in Figure 1. In this scenario, the friction-free exchange rate remains above the strike price 20 until time 157, after which it remains below the strike until the expiry date of the option. On the one hand, since the exchange rate is at its lowest near the expiry of the option, and in fact reaches its minimum at time 250, one would expect that later exercise times would be more expensive for the writer to hedge against than earlier ones. On the other hand, our results lead us to expect that the holder would exercise the option as soon as he/she can do so while remaining solvent.

An optimal stopping time χ and hedging strategy $z = (z^1, z^2)$ for the writer starting from the initial endowment $z_0 = (1.32909, 0)$ is presented on the left-hand side of Figure 2. This strategy changes from z_0 into the portfolio

$$z_1 = (10.6362, -0.467692) \tag{4.3}$$

at time 0. As we can see, the optimal stopping time for the writer is very heavily weighted towards exercise at or after time 221. At this time step, the writer changes from $z_{221} = (19.9998, -0.999989)$ to $z_{222} = (20, -1) = \xi_{222}$, which he/she holds unchanged until the expiry date of the option. This feature allows delivery of the option at any of the most expensive exercise times, while leaving the writer in a solvent position if the option is not exercised. Indeed, if the option remains not exercised at time 250, then the writer can change his/her portfolio into

$$z^1_{250} + z^2_{250}\pi^{12}_{250} = 20 - 1 \cdot 1.005 \cdot 14.9425 = 4.9828$$

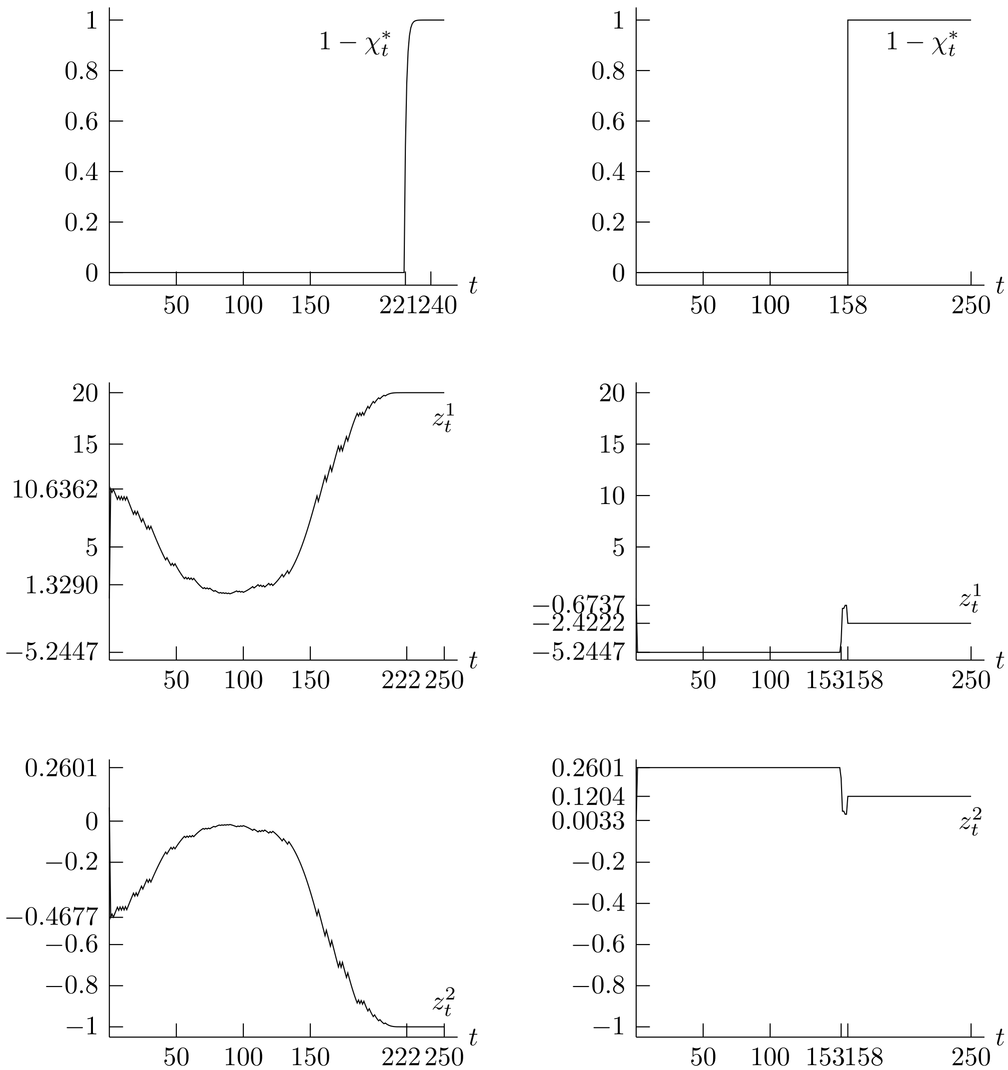

Figure 2: Cumulative optimal stopping time and hedging strategy for the option writer (left) and holder (right) in Example 4.9.

units of asset 1 or

$$z^1_{250}\pi^{21}_{250} + z^2_{250} = \frac{20}{0.995 \cdot 14.9425} - 1 = 0.3452$$

units of asset 2.

The right-hand side of Figure 2 contains an optimal stopping time $\chi = \chi^\tau$ (an ordinary stopping time) and hedging strategy $z = (z^1, z^2)$ for the holder of this option. This strategy starts with initial endowment $z_0 = (0.0159486, 0)$, from which it changes into

$$z_1 = (-5.24469, 0.260137) \tag{4.4}$$

at time 0; this position is held unchanged until time 152. The optimal stopping time for the holder is 158.

Table 1 gives the exchange rate and optimal strategy for the holder between time steps 151 and 159 in full detail. In a world free of transaction costs, the option ξ would be in the money for the first time at time 157, when the friction-free exchange rate drops below the strike price, and the exercise decision would be straightforward. However, in the present case with transaction costs, exercising the option at time 157 leaves the holder with the portfolio $z_{157} + \xi_{157}$, which not solvent at time 157 since

$$z^1_{157} + \xi^1_{157} + [z^2_{157} + \xi^2_{157}]\pi^{12}_{157} \approx -0.09764 < 0.$$

In contrast, the portfolio $z_{158} + \xi_{158}$ is solvent at time 158 since

$$\begin{aligned}[z^1_{158} + \xi^1_{158}]\pi^{21}_{158} + z^2_{158} + \xi^2_{158} &\approx 0.009329 > 0,\\ z^1_{158} + \xi^1_{158} + [z^2_{158} + \xi^2_{158}]\pi^{12}_{158} &\approx 0.0102 > 0.\end{aligned}$$

The holder would be solvent after exercising the option at time 158, which confirms that 158 is the optimal stopping time for the holder.

Revisiting (4.2), we could alternatively interpret the option ξ as a basket of 20 American call options with strike 0.05 on the first asset (with the provision that the whole basket must be exercised simultaneously). Reusing the initial set returned by the constructions referred to in Sections 4.1 and 4.2 and described in detail in [48], [49] (the set is called $\mathcal{Z}_0$ in both cases), we obtain

$$p^b_2(\xi) = 0.000793463, \quad p^a_2(\xi) = 0.0667885.$$

The optimal stopping times and hedging strategies for both the writer and the holder are the same as in Figure 2, except for the initial endowments. Indeed, at time 0, the writer is able to change from the portfolio $z_0 = (0, 0.0667885)$ to the portfolio z_1 of (4.3) in a self-financing way since $z_0 - z_1$ is solvent, i.e.

$$\begin{aligned}[z^1_0 - z^1_1]\pi^{21}_0 + z^2_0 - z^2_1 &\approx 0.0001095 > 0,\\ z^1_0 - z^1_1 + [z^2_0 - z^2_1]\pi^{12}_0 &\approx 0.1289 > 0.\end{aligned}$$

t	$1/\pi_t^{21}$	E_t	π_t^{12}	y_t^1	y_t^2
151	20.3809	20.4833	20.5857	-5.24469	0.260137
152	20.2564	20.3582	20.4600	-5.24469	0.260137
153	20.1327	20.2339	20.3351	-4.19557	0.208344
154	20.0097	20.1103	20.2109	-0.97055	0.048156
155	20.1408	20.2420	20.3432	-0.97055	0.048156
156	20.0178	20.1184	20.2190	-0.67370	0.033418
157	19.8955	19.9955	20.0955	-0.67370	0.033418
158	19.7741	19.8735	19.9729	-2.42221	0.120427
159	19.6533	19.7521	19.8509	-2.42221	0.120427

Table 1: Exchange rate and optimal trading strategy for the option holder in Example 4.9.

Likewise, the portfolio $z_0 = (0, -0.000793463)$ allows the option holder to change into the portfolio z_1 of (4.4) in a self-financing way due to the fact that

$$[z_0^1 - z_1^1]\pi_0^{21} + z_0^2 - z_1^2 \approx 0,$$
$$z_0^1 - z_1^1 + [z_0^2 - z_1^2]\pi_0^{12} \approx 0.$$

Example 4.10. Consider a trinomial tree model with two risky assets. We assume a notional friction-free exchange rate $E = (E_t)$ between the two assets which satisfies

$$E_{t+1} = \epsilon_t E_t$$

for $t = 0, \ldots, T-1$, where $E_0 = 10$ is given, and where (ε_t) is a sequence of independent identically distributed random variables taking the three values

$$e^{\sigma\sqrt{\frac{1}{T}}}, \quad 1, \quad e^{-\sigma\sqrt{\frac{1}{T}}},$$

each with positive probability. Here $\sigma = 0.1$ is the volatility of the exchange rate, the time horizon is 1 year, and $T = 250$ is the number of steps in the model. We further assume that for $t = 0, \ldots, T$ the exchange rates are

$$\pi_t^{12} = (1+k)E_t, \quad \pi_t^{11} = \pi_t^{22} = 1, \quad \pi_t^{21} = \frac{1}{(1-k)E_t},$$

where $k = 0.5\%$ is the transaction cost rate.

Consider now an American butterfly spread written on the second asset that offers the payoff $\xi_t = (\xi_t^1, \xi_t^2)$ at any time step t, where

$$\xi_t^1 = 0, \quad \xi_t^2 = [E_t - 10]^+ - 2[E_t - 11]^+ + [E_t - 12]^+.$$

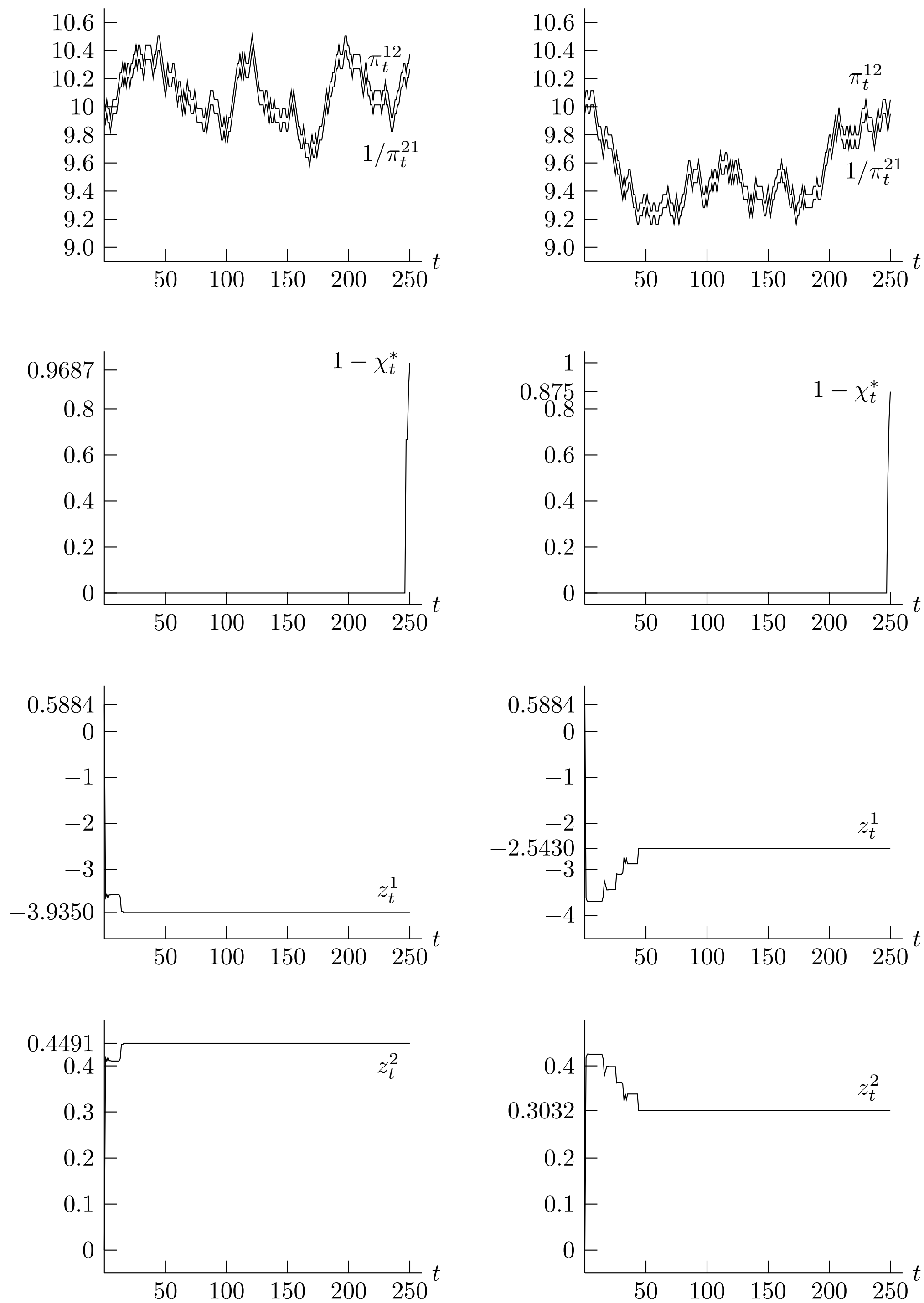

Figure 3: Exchange rates, cumulative optimal stopping time for the writer, and hedging strategy for the writer along sample paths ending in the money (left) and out of the money (right) in the trinomial model in Example 4.10.

Applying the constructions referred to in Section 4.1 and described in detail in [48], [49], we find that the ask price of the option expressed in terms of asset 1 is $p^a_1(\xi) = 0.588364$. For two randomly selected scenarios, in Figure 3 we present the sample paths of exchange rates π^{12}_t, π^{21}_t together with the cumulative optimal stopping time $1 - \chi^*_t$ (the proportion of the initial option holding exercised up to time t) and hedging strategy $z_t = (z^1_t, z^2_t)$ for the writer. Both the hedging strategies start from the same initial portfolio z_0 consisting of $p^a_1(\xi) = 0.588364$ units of the first asset and none of the second. It appears that the magnitude of transaction costs combines with the incompleteness of the model to prevent trading from taking place at every time step, as in both cases trading takes place sporadically until a certain point, and then a fixed portfolio is held until expiry. In both cases the optimal stopping time is a mixed stopping time, $\chi \in \mathcal{X}$; in the case ending out of the money the optimal stopping time contains the probability of 12.5% that the option is never exercised.

It follows from the constructions referred to in Section 4.2 and described in detail in [48], [49] that the bid price of the option is $p^b_1(\xi) = 0$; this means that this derivative security cannot be hedged without risk by the holder; in other words, no asset can be shorted without risk against holding this option. As the butterfly is not in the money at time 0, the (first) optimal stopping time for the holder is at the root node at time 0. It is immediately clear that the most expensive stopping time for the writer is different from the best stopping time for the holder.

4.4 Deferred solvency and gradual exercise

Here we briefly introduce two new ideas in the study of options under transaction costs. For simplicity, we specialise to the case of finite Ω and assume the no-arbitrage property.

First, observe that in the model with bid-ask process π it may be possible for a portfolio y that is insolvent at some time instant t in the sense that $y \notin \mathcal{K}_t$ to be rebalanced so it becomes solvent with probability 1 at a later time. This possibility, which is illustrated by Example 4.11, will be referred to as deferred solvency, and is defined in precise terms below. (In this context, we shall refer to the notion of solvency introduced in Section 2 as immediate solvency.)

Second, recall from Section 4.1 that the writer of an American option needs to allow for mixed stopping times in order to prepare for the worst-case scenario even if the holder is only allowed to exercise the option at an ordinary stopping time. The question arises, what happens if the holder is allowed to exercise the option gradually, that is, at a mixed stopping time?

It turns out that these two ideas are linked together via the self-financing and hedging conditions for short and long positions in an option with gradual exercise, see Definitions 4.13 and 4.14. These conditions involve rebalancing the strategy up to and including the time horizon T even if the option is fully exercised before that time, and that brings deferred solvency into the picture.

Example 4.11. In this toy example we consider a single-step binomial model with two assets and bid-ask process $\pi_t = [\pi_t^{ij}]$ for $t = 0, 1$ as follows:

$$\begin{bmatrix} 1 & 1/5 \\ 8 & 1 \end{bmatrix} \quad \begin{matrix} \nearrow \\ \searrow \end{matrix} \quad \begin{matrix} \begin{bmatrix} 1 & 1/7 \\ 7 & 1 \end{bmatrix} \\ \\ \begin{bmatrix} 1 & 1/6 \\ 6 & 1 \end{bmatrix} \end{matrix}$$

Portfolio $(-1, 7)$ is insolvent at time 0. However, if kept unchanged until time 1, it can be rebalanced into the solvent portfolios $(0, 0)$ at the up node or $(0, 1)$ in the down node. While the portfolio $(-1, 7)$ fails the immediate solvency condition at time 0, it does satisfy the deferred solvency condition defined below.

Definition 4.12. We say that a portfolio $x \in \mathcal{L}_t$ satisfies the *deferred solvency* condition at time t whenever there exists a sequence $(z_s)_{s=t}^T$ such that $z_s \in \mathcal{L}_s$ for $s = t, \ldots, T$ and

$$\begin{aligned} x - z_t &\in \mathcal{K}_t, \\ z_s - z_{s+1} &\in \mathcal{K}_{s+1} \text{ for } s = t, \ldots, T - 1, \\ z_T &\in \mathcal{K}_T. \end{aligned}$$

The collection of such portfolios x is a convex cone, which will be denoted by $\tilde{\mathcal{K}}_t$ and called the *deferred solvency cone*.

Let $\xi = (\xi_t)_{t=0}^T$ be an $\mathbb{R}^d$-valued adapted process. An American option with payoff process ξ and expiry time T that allows for *gradual exercise* gives the option holder the right to receive and obliges the writer to deliver the portfolio $\chi_t \xi_t$ at each time instant $t = 0, \ldots, T$ for any mixed stopping time $\chi \in \mathcal{X}$ selected by the holder. Here χ_t is the fraction of the option exercised by the holder at time t. By contrast, if the holder of the American option is only allowed to exercise at an ordinary stopping time $\tau \in \mathcal{T}$, we shall say that the option allows for *full exercise*.

Note that at time t the writer knows the fractions $\chi_0, \ldots, \chi_t$ of the option exercised up to and including time t, but not the future values $\chi_{t+1}, \ldots, \chi_T$ to be chosen by the holder. The strategy that can be adopted by the writer to hedge a short position in the option must reflect this: the strategy up to and including the portfolio created at time t (i.e. up to an including the portfolio indexed by $t + 1$) may depend on $\chi_0, \ldots, \chi_t$ but not on $\chi_{t+1}, \ldots, \chi_T$.

In addition, when rebalancing at time t the writer needs to include in the self-financing condition the portfolio $\chi_t \xi_t$ to be delivered to the holder, and can use deferred solvency rather than immediate solvency.

Accordingly, we need to modify the definition of hedging a short position in an American option (cf. Definition 4.1 (i)).

Definition 4.13. Consider an American option with payoff process ξ that allows for gradual exercise. A predictable process $z^\chi = (z_t^\chi)_{t=0}^T$ depending on $\chi \in \mathcal{X}$ is called a *hedging strategy for a short* (*writer's*) *position* in the option if the following conditions are satisfied:

(i) Self-financing and hedging: for all $t = 0, \ldots, T$ and for all $\chi \in \mathcal{X}$

$$z_t^\chi - \chi_t \xi_t - z_{t+1}^\chi \in \tilde{\mathcal{K}}_t,$$

where we put $z_{T+1}^\chi = 0$ for simplicity.

(ii) Dependence on mixed stopping time: for all $t = 0, \ldots, T$ and for all $\chi, \chi' \in \mathcal{X}$ such that

$$\chi_s = \chi'_s \text{ for each } s = 0, \ldots, t-1$$

we have

$$z_s^\chi = z_s^{\chi'} \text{ for each } s = 0, \ldots, t.$$

We say that an initial endowment $y \in \mathbb{R}^d$ hedges *a short* (*writer's*) *position* in the option if there is a strategy $(z^\chi)_{\chi \in \mathcal{X}}$ hedging a short position in the option such that $y = z_0^\chi$ (note that, according to (ii), z_0^χ does not in fact depend on χ).

The holder of an American option with gradual exercise chooses the fractions χ_t of the option to be exercised at each time instant t, and may know the mixed stopping time $\chi \in \mathcal{X}$ in advance. In indeed the choice of χ becomes part of the holder's hedging strategy. Condition (ii) from Definition 4.13 does not, therefore, apply in this case. The self-financing and hedging condition reflects the fact that the portfolio $\chi_t \xi_t$ is received rather than delivered by the holder at time t. Deferred solvency is used here too rather than immediate solvency.

Definition 4.14. Consider an American option with payoff process ξ that allows for gradual exercise. A pair (χ, z) consisting of a mixed stopping time $\chi \in \mathcal{X}$ and a predictable process $z = (z_t)_{t=0}^T$ is called a *hedging strategy for a long* (*holder's*) *position* in the option if the following self-financing and hedging condition is satisfied: for all $t = 0, \ldots, T$

$$z_t + \chi_t \xi_t - z_{t+1} \in \tilde{\mathcal{K}}_t,$$

where we put $z_{T+1} = 0$ for simplicity.

We say that an initial endowment $y \in \mathbb{R}^d$ hedges *a long (holder's) position* in the option if there is a strategy (χ, z) hedging a long position in the option such that $y = z_0$.

The ask and bid price for an American option with gradual exercise can now be defined in the usual way.

Definition 4.15. Consider an American option with payoff process ξ that allows for gradual exercise.

(i) The *ask price* (or *writer's price*) at time 0 of the option expressed in units of asset i is defined as

$$p_i^a(\xi) = \inf\{s \in \mathbb{R} : se_i \text{ hedges a short position in } \xi\}.$$

(ii) The *bid price* (or *holder's price*) at time 0 of the option expressed in units of asset i is defined as

$$p_i^b(\xi) = \sup\{-s \in \mathbb{R} : se_i \text{ hedges a long position in } \xi\}.$$

The following result provides a construction of a strategy hedging a short position in an American option with gradual exercise. It also gives a representation of the ask price of such an option. It reduces the problem to hedging and pricing a short position in an American option with full exercise in the model with deferred solvency cones $\tilde{\mathcal{K}}_t$ in place of the $\mathcal{K}_t$.

Theorem 4.16. We denote by $\tilde{\mathcal{Z}}_t, \tilde{\mathcal{V}}_t, \tilde{\mathcal{W}}_t$ the sequences constructed as in Theorem 4.3 for the American option with payoff ξ and full (as opposed to gradual) exercise using the deferred solvency cones $\tilde{\mathcal{K}}_t$ in place of the $\mathcal{K}_t$. Moreover, by $\tilde{p}_i^a(\xi)$ we denote the corresponding ask price of the option with full exercise, and by $\tilde{z} \in \Phi$ the corresponding strategy hedging a short position in the option with full exercise such that $\tilde{z} = \tilde{p}_i^a(\xi)e_i$.

Then, for the American option with payoff process ξ that allows for gradual exercise:

(i) $\tilde{\mathcal{Z}}_0$ is the collection of initial endowments $y \in \mathbb{R}^d$ that hedge a short position the option with gradual exercise.

(ii) The ask price of the option with gradual exercise can be expressed as

$$p_i^a(\xi) = \tilde{p}_i^a(\xi) = \min\{s \in \mathbb{R} : se_i \in \tilde{\mathcal{Z}}_0\}.$$

(iii) The strategy $(z^\chi)_{\chi\in\mathcal{X}}$ hedging a short position in the option with gradual exercise such that $z_0^\chi = p_i^a(\xi)$ is given by

$$z_t^\chi = \chi_t^* \tilde{z}_t.$$

In a nutshell, this theorem shows that in order to hedge a short position in an American option with gradual exercise the writer should always prepare a portfolio to hedge against possible full exercise of the remaining portion χ_t^* of the option using the deferred solvency cones $\tilde{\mathcal{K}}_t$.

While the writer's case for an American option with gradual exercise reduces to that for an option with full exercise, the holder's case turns out to be essentially different.

Theorem 4.17. Given an American option ξ with gradual exercise, let

$$\mathcal{X}_t := -\xi_t + \tilde{\mathcal{K}}_t,$$

and let the following sequences of sets be constructed by backward induction:

$$\mathcal{Z}_T = \mathcal{X}_T,$$

and for $t = T-1, \ldots, 1, 0$

$$\begin{aligned} \mathcal{W}_t &= \mathcal{Z}_{t+1} \cap \mathcal{L}_t, \\ \mathcal{V}_t &= \mathcal{W}_t + \tilde{\mathcal{K}}_t, \\ \mathcal{Z}_t &= \operatorname{conv}(\mathcal{V}_t \cup \mathcal{X}_t), \end{aligned}$$

where $\operatorname{conv}(\cdot)$ is the convex hull. Then:

(i) $\mathcal{Z}_0$ is the collection of initial endowments $y \in \mathbb{R}^d$ that hedge a long position in the American option ξ with gradual exercise.

(ii) The bid price of the American option with gradual exercise can be expressed as
$$p_i^b(\xi) = \max\{-s \in \mathbb{R} : se_i \in \mathcal{Z}_0\}.$$

(iii) There exists a strategy (χ, z) hedging a long position in the American option with gradual exercise such that $z_0 = -p_i^b(\xi)e_i$. Both the mixed stopping time χ and the process z can be constructed algorithmically once the sequences $(\mathcal{Z}_t)$, $(\mathcal{V}_t)$, $(\mathcal{W}_t)$ are known.

A crucial difference between the construction in this theorem for American options with gradual exercise and that in Theorem 4.7 for options with full exercise, is in the formula for $\mathcal{Z}_t$. The convex hull in this formula restores convexity. That, in turn, means that convex duality methods are applicable in the holder's case for an American option with gradual exercise (but not for an option with full exercise). The dual representations of the ask and bid process for an option with gradual exercise are given in the following theorem.

Theorem 4.18. Consider an American option with payoff ξ allowing for gradual exercise.

(i) The ask price of this option can be represented as

$$p_i^a(\xi) = \max_{\chi\in\mathcal{X}} \sup_{(P,S)\in\tilde{\mathcal{P}}_i(\chi)} \mathbb{E}_P((\xi\cdot S)_\chi) = \mathbb{E}_{\hat{P}}((\xi\cdot\hat{S})_{\hat{\chi}})$$

where $\tilde{\mathcal{P}}_i(\chi)$ is the collection of χ-approximate consistent pricing pairs obtained by applying Definition 4.5 with the deferred solvency cones $\tilde{\mathcal{K}}_t$ in place of the $\mathcal{K}_t$. The mixed stopping time $\hat{\chi}\in\mathcal{X}$ realising the maximum and the pair $(\hat{P},\hat{S})\in\overline{\tilde{\mathcal{P}}_i(\hat{\chi})}$ for which the supremum is attained can be constructed algorithmically.

(ii) The ask price of this option can be represented as

$$p_i^b(\xi) = \max_{\chi\in\mathcal{X}} \inf_{(P,S)\in\tilde{\mathcal{P}}_i} \mathbb{E}_P((\xi\cdot S)_\chi) = \mathbb{E}_{\check{P}}((\xi\cdot\check{S})_{\check{\chi}}),$$

where $\tilde{\mathcal{P}}_i$ is the family of consistent pricing pairs obtained by applying the definition of $\mathcal{P}_i$ with the deferred solvency cones $\tilde{\mathcal{K}}_t$ in place of the $\mathcal{K}_t$. The mixed stopping time $\check{\chi}\in\mathcal{X}$ realising the maximum and the pair $(\check{P},\check{S})\in\overline{\tilde{\mathcal{P}}_i}$ for which the infimum is attained can be constructed algorithmically.

One interesting and entirely new feature in this theorem is that the processes S in the dual representation for the bid price are simply martingales under P, in contrast to the ask price case, where a wider class of processes needs to be admitted.

The results in this section have been studied in [55] in a simpler model with just two assets (a stock and cash account acting as numéraire) with transaction costs in *à la* Jouini and Kallal [18].

5. Game options

In this section we present new results for game options. We continue to use the d-asset model with matrix-valued bid-ask process π. As elsewhere in this paper, $\mathcal{K}_t$ for $t=0,\dots,T$ are the solvency cones, Φ is the collection of self-financing strategies, $\mathcal{T}$ denotes the set of (ordinary) stopping times, and $\mathcal{X}$ the set of mixed stopping times. We assume for simplicity that Ω is finite and the no-arbitrage property holds.

A *game option* is a derivative security that may be exercised by the option holder at a stopping time $\tau\in\mathcal{T}$ and cancelled by the writer at a stopping time $\sigma\in\mathcal{T}$. At time $\sigma\wedge\tau$ the holder receives the payoff

$$\eta_{\sigma,\tau} = \xi_\tau \mathbf{1}_{\{\tau<\sigma\}} + \zeta_\sigma \mathbf{1}_{\{\sigma<\tau\}} + \theta_\sigma \mathbf{1}_{\{\sigma=\tau\}}.$$

Here $\mathbf{1}_A$ is the indicator function for any $A \subset \Omega$ and $\xi = (\xi_t)_{t=0}^T$, $\zeta = (\zeta_t)_{t=0}^T$, $\theta = (\theta_t)_{t=0}^T$ are adapted $\mathbb{R}^d$-valued processes such that $\zeta_t - \theta_t \in \mathcal{K}_t$ and $\theta_t - \xi_t \in \mathcal{K}_t$ for all t.

In essence a game option (ξ, ζ, θ) is an American option ξ with the additional feature that it may be cancelled by the writer at any stopping time σ, at which time the writer is required to deliver the payoff ξ_σ to the holder, together with a penalty. This penalty is equal to $\zeta_\sigma - \xi_\sigma \in \mathcal{K}_\sigma$ on the set $\{\sigma < \tau\}$ where the writer cancels before the holder has exercised, and it is $\theta_\sigma - \xi_\sigma \in \mathcal{K}_\sigma$ on the set $\{\sigma = \tau\}$ where the cancellation and exercise times coincide.

Definition 5.1. (i) An initial endowment $y \in \mathbb{R}^d$ is said to *hedge a short* (*writer's*) *position in a game option* (ξ, ζ, θ) if there is a pair $(\sigma, z) \in \mathcal{T} \times \Phi$ with $z_0 = y$ such that for all $\tau \in \mathcal{T}$

$$z_{\sigma\wedge\tau} - \eta_{\sigma,\tau} \in \mathcal{K}_{\sigma\wedge\tau}.$$

Such a pair $(\sigma, z) \in \mathcal{T} \times \Phi$ is called a *hedging strategy for a short* (*writer's*) *position in the game option* (ξ, ζ, θ).

(ii) An initial endowment $y \in \mathbb{R}^d$ is said to *hedge a long* (*holder's*) *position in a game option* (ξ, ζ, θ) if there is a pair $(\tau, z) \in \mathcal{T} \times \Phi$ with $z_0 = y$ such that for all $\sigma \in \mathcal{T}$

$$z_{\sigma\wedge\tau} + \eta_{\sigma,\tau} \in \mathcal{K}_{\sigma\wedge\tau}.$$

Such a pair $(\tau, z) \in \mathcal{T} \times \Phi$ is called a *hedging strategy for a long* (*holder's*) *position in the game option* (ξ, ζ, θ).

It is straightforward to check that (τ, z) is a strategy hedging a long position in the game option (ξ, ζ, θ) if and only if hedges a short position in the game option $(-\zeta, -\xi, -\theta)$. Motivated by this symmetry, the remainder of this discussion focuses on the case of a short position (writer's case) in a game option.

Definition 5.2. The *ask price* (or *writer's price*) at time 0 of the game option (ξ, ζ, θ) expressed in units of asset i is defined as

$$p_i^a(\xi, \zeta, \theta) = \inf\{s \in \mathbb{R} : se_i \text{ hedges a short position in } (\xi, \zeta, \theta)\}.$$

The following result provides a construction of a hedging strategy and a representation of the ask price for a game option which lend themselves to numerical implementation.

Theorem 5.3. Given a game option (ξ, ζ, θ), let

$$\mathcal{X}_t := \xi_t + \mathcal{K}_t,$$
$$\mathcal{Y}_t := \begin{cases} \theta_T + \mathcal{K}_T & \text{if } t = T, \\ \zeta_t + \mathcal{K}_t & \text{if } t < T, \end{cases}$$

and let the following sequences of sets be constructed by backward induction:

$$\mathcal{Z}_T = \mathcal{Y}_T,$$

and for $t = T-1, \ldots, 1, 0$

$$\mathcal{W}_t = \mathcal{Z}_{t+1} \cap \mathcal{L}_t,$$
$$\mathcal{V}_t = \mathcal{W}_t + \mathcal{K}_t,$$
$$\mathcal{Z}_t = [\mathcal{V}_t \cap \mathcal{X}_t] \cup \mathcal{Y}_t.$$

Then:

(i) $\mathcal{Z}_0$ is the collection of initial endowments $y \in \mathbb{R}^d$ that hedge a short position in the game option (ξ, ζ, θ).

(ii) The ask price of the game option can be expressed as

$$p_i^a(\xi, \zeta, \theta) = \min\{s \in \mathbb{R} : se_i \in \mathcal{Z}_0\}.$$

(iii) There exists a strategy $(\hat{\sigma}, \hat{z}) \in \mathcal{T} \times \Phi$ with $\hat{z}_0 = p_i^a(\xi, \zeta, \theta)e_i$ hedging a short position in the game option (ξ, ζ, θ). The strategy $(\hat{\sigma}, \hat{z})$ can be constructed algorithmically once the sequences $(\mathcal{Z}_t)$, $(\mathcal{V}_t)$, $(\mathcal{W}_t)$ are known.

For any $\chi \in \mathcal{X}$ let $\mathcal{P}^i(\chi)$ be the set of consistent pricing pairs defined just as for American options.

For any $\chi \in \mathcal{X}$ and $\sigma \in \mathcal{T}$ define the truncated mixed stopping time $\chi \wedge \sigma \in \mathcal{X}$ by

$$(\chi \wedge \sigma)_t := \chi_t \mathbf{1}_{\{t<\sigma\}} + \chi_t^* \mathbf{1}_{\{t=\sigma\}}$$

for all $t = 0, \ldots, T$. For any $(P, S) \in \mathcal{P}^i(\chi \wedge \sigma)$ define

$$\eta_{\sigma,\chi}(S) := \sum_{t=0}^{\sigma-1} \chi_t \xi_t \cdot S_t + \chi_{\sigma+1}^* \zeta_\sigma \cdot S_\sigma + \chi_\sigma \theta_\sigma \cdot S_\sigma.$$

If χ takes the simple form $\chi_t = \mathbf{1}_{\{\tau=t\}}$ for some stopping time $\tau \in \mathcal{T}$, then it is easy to check that $(\chi \wedge \sigma)_t = \mathbf{1}_{\{\sigma\wedge\tau=t\}}$ for all t and $\eta_{\sigma,\chi}(S) = \eta_{\sigma,\tau} \cdot S_{\sigma\wedge\tau}$.

The dual representation of the ask price of a game option reads as follows.

Theorem 5.4. We have

$$\begin{aligned} p_i^a(\xi,\zeta,\theta) &= \min_{\sigma\in\mathcal{T}}\max_{\chi\in\mathcal{X}}\sup_{(P,S)\in\mathcal{P}_i(\chi\wedge\sigma)}\mathbb{E}_P[\eta_{\sigma,\chi}(S)] \\ &= \mathbb{E}_{\hat{P}}[\eta_{\hat{\sigma},\hat{\chi}}(\hat{S})], \end{aligned}$$

where $\hat{\sigma}$ is the stopping time of Theorem 5.3 (iii) and where $\hat{\chi}\in\mathcal{X}$ and $(\hat{P},\hat{S})\in\overline{\mathcal{P}_i(\hat{\chi}\wedge\hat{\sigma})}$ can be constructed algorithmically.

Here $\hat{\chi} = \hat{\chi}\wedge\hat{\sigma}$ and the construction of $(\hat{P},\hat{S})$ is similar to that in the American case, but where the payoff of the American option is $(\eta_{\hat{\sigma},t})$ and it has a random expiry date $\hat{\sigma}$.

6. Further prospects and open problems

In this article we have discussed recent developments in the theory of pricing and hedging options under transaction costs and presented some new results concerning game options, options with gradual exercise policies and deferred solvency.

There are a number of open problems inviting further work in this direction. These may include multiple exercise options (which appear, for example, in swing options within energy or emission markets) subject to transaction costs. Another promising research direction involves derivative securities in more sophisticated market models with friction, for example, including fixed transaction costs, or markets with convex (nonlinear) frictions used to capture the lack of liquidity in a market. Open questions more directly related to the results covered in the present paper, to name just a couple, would involve extending the dual representation of hedging strategies similar to Theorem 4.4 to the case of the holder of an American option (in the case of gradual exercise policies, where convexity is preserved in the holder's case) and possibly for game options.

7. Appendix: Mixed stopping times

A *mixed stopping time* (also called a *randomised stopping time* as in, for example, Chow, Robins and Siegmund [8], Baxter and Chacon [1], or Chalasani and Jha [5]) is defined to be a non-negative adapted process χ on a probability space $(\Omega,\mathcal{F},Q)$ with filtration $(\mathcal{F}_t)_{t=0}^T$ such that

$$\sum_{t=0}^{T}\chi_t = 1$$

almost surely. The collection of all mixed stopping times will be denoted by $\mathcal{X}$, and the collection of all ordinary stopping times with values in $\{0,\dots,T\}$ by $\mathcal{T}$.

Each ordinary stopping time $\tau \in \mathcal{T}$ can be represented as a mixed stopping time χ^τ such that for any $t = 0,\dots,T$

$$\chi_t^\tau = \mathbf{1}_{\{\tau=t\}},$$

so we have $\mathcal{T} \subset \mathcal{X}$.

For any mixed stopping time χ and any adapted process Z, we define the time-χ value of Z as

$$Z_\chi = \sum_{t=0}^T \chi_t Z_t.$$

If $\tau \in \mathcal{T}$, then Z_{χ^τ} is the familiar random variable

$$Z_{\chi^\tau} = \sum_{t=0}^T \chi_t^\tau Z_t = \sum_{t=0}^T \mathbf{1}_{\{\tau=t\}} Z_t = Z_\tau.$$

For any mixed stopping time $\chi \in \mathcal{X}$ and any adapted process Z we define predictable non-increasing process χ^* and Z^{χ^*} such that for each $t = 0,\dots,T$

$$\chi_t^* = \sum_{s=t}^T \chi_s, \quad Z_t^{\chi^*} = \sum_{s=t}^T \chi_s Z_s.$$

For convenience, we put $\chi_{T+1}^* = 0$ and $Z_{T+1}^{\chi^*} = 0$.

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