Global and Stochastic Analysis
Vol. 1, No. 2, July-December 2014

www.mukpublications.com

Riemann-Hilbert Boundary Value Problem for Generalized Analytic Functions in Smirnov Classes

S. Klimentov

Southern Federal University, Southern Mathematical Institute
Milchakova 8-a, 344090, Rostov-on-Don, Russia
E-mail: sklimentov@pochta.ru

Received by the Editorial Board on August 14, 2011

Abstract

The Riemann-Hilbert boundary value problem for generalized analytic functions in Smirnov classes is under consideration. The domain is supposed simply connected with Lyapunov or Radon boundary without cusps. In the work the special representation for generalized analytic functions of Smirnov classes is built. This representation has independent interest.

Key words: Riemann-Hilbert boundary value problem, generalized analytic functions, Smirnov classes.
2010 Mathematics subject classification: 35F45

1. Introduction. Basic definitions

Riemann-Hilbert problem for complex holomorphic functions in the classic Smirnov classes was studied in the works: [1] for the Lyapunov boundaries and [2] for domains with Radon boundaries. Smirnov classes for the generalized analytic functions for the first time were introduced by K.M. Musaev [3]. Later he investigated various properties of these classes in [4]–[8]. Boundary value problems are not considered except «jump problem» [8]. Some new properties of these classes (including criteria for the solvability of boundary value problems for Hardy class of generalized analytic functions) were received in [9]–[14]. In this paper Riemann-Hilbert problem for generalized analytic functions is reduced to 2D integral equations. This approach generalizes the scheme, developed by I.N. Vekua [15, Chapter 4, § 7] for a unit circle and Hölder up-to-edge solutions. For this purpose in the proposed work the special representation «of the second type» is built for generalized analytic functions in the domains with non-smooth boundaries.

This paper generalizes the results of the works of the author [1], [2], [10]. The main difficulty is the impossibility in the case of the non-smooth border to reduce the problem by conformal mapping to one in Hardy class of generalized analytic functions.

Let G is bounded simply connected domain in complex z-plane, $z = x + iy, , i^2 = -1$, with rectifiable boundary $\Gamma = \partial G$; $\overline{G} = G \cup \Gamma$; $A(z), B(z) \in L_s(\overline{G})\, s > 2$ [1], are given complex functions. Without limiting the generality, we assume that the point $z = 0$ is located inside G.

We consider in $\overline{G}$ canonical elliptic system in the complex entry

$$\partial_{\bar{z}} w + A(z)w + B(z)\overline{w} = 0, \tag{1.1}$$

where $w = w(z) = u(z) + iv(z)$ is unknown complex function, u and v are its real and imaginary parts, $\partial_{\bar{z}} = 1/2(\partial/\partial x + i\partial/\partial y)$ is derivative in the Sobolev sense.

The solution $w(z)$ of the system (1.1) is called *generalized analytic function* [15, p. 148].

Let $\{G_n\}$ is the sequence of domains, which closures lie inside G. The boundaries Γ_n of domains G_n are assumed rectifiable and convergent to Γ in the sense, that every point of $z \in G$ belongs to all G_n starting with some number n.

Definition 1.1. We shall say the solution of the system (1.1) belongs to a class $E_p(A, B)$, $p > 0$, if for some constant $M_p < \infty$, fullfill the inequalities

$$\int\limits_{\Gamma_n} |w(z)|^p |dz| \le M_p(w), \quad n = 1, 2, \dots$$

at least for one sequence of rectifiable curves with above formulated property.

When $A = B \equiv 0$ we get the classical Smirnov class E_p [16, p. 422], [17, p. 90].

Smirnov classes $E_p(A, B)$ are defined similarly if the coefficients $A(z)$, $(B)(z)$ and $w(z)$ are defined in the exterior area of G.

Remark 1. When we get the classical Smirnov class E_p, one can reduce the investigation to the Hardy class H_p by conformal mapping $\varphi = \varphi(\zeta)$ the unit

[1] We use the notation from book [15].

disk $D : |\zeta| < 1$ onto domain G (see [16, p. 423], [17, pp. 91-92]). We cannot do it in general case $E_p(A, B)$, because the equation (1.1) transforms to equation

$$\partial_{\bar{\zeta}} w + A(\varphi(\zeta))\overline{\varphi'(\zeta)}w + B(\varphi(\zeta))\overline{\varphi'(\zeta)}\overline{w} = 0,$$

with «bad» coefficients, which are not in $L_s(\overline{D})$, $s > 2$ in general.

Let $\varphi = \varphi(\zeta)$ is one-to-one conformal mapping the unit disk $D : |\zeta| < 1$ onto G. Without loss of generality we take $\varphi(0) = 0$. We denote the boundary of the disk D as C.

Remark 2. As in the classic case (see [17, p. 91]) we can take in the definition 1.1 as the curves Γ_n only the images of the circles $C_r = \{\zeta : |\zeta| = r < 1\}$ under conformal mapping $\varphi = \varphi(\zeta)$ [11]. We denote this images as Γ_r and put $\Gamma_1 = C$.

Definition 1.2 ([17, p. 90])**.** The domain G is called V.I. Smirnov domain or the domain of class $\mathcal{C}$ if harmonic function $\ln|\varphi'(\zeta)|$ can be represented by Poisson-Lebesgue integral. Equivalently: holomorphic function $\ln\varphi'(\zeta)$ can be represented by Schwartz integral of $\ln|\varphi'(e^{i\sigma})|$ ($\zeta = re^{i\sigma}$).

If $z = z(s)$ is the parametrical equations of rectifiable curve Γ, where $s \in [0, S]$ is the length of arc on Γ (S is the length of the whole curve Γ), then almost everywhere on Γ we have $z'(s) = e^{i\theta(s)}$. This equality defines the angle $\theta(s)$ to within 2π. Geometric meaning the angle $\theta(s)$ is obvious — this is the angle of inclination the tangent to the curve Γ.

Definition 1.3 ([17, p. 19])**.** If the angle $\theta(s)$ can be selected so that the function $\theta(s)$ has bounded variation on $[0, S]$, then we call Γ Radon curve.

We always can define the function $\theta(s)$ for $\forall s \in [0, S]$ when the modules of jumps of $\theta(s)$ are less or equal π. Further we assume it's done.

Definition 1.4 ([17, p. 20])**.** We call the point of the curve Γ the cusp point if the module of jump of the function $\theta(s)$ is at this point equal π.

Definition 1.5 ([17, p. 14])**.** If the angle $\theta(s)$ can be selected so that the function $\theta(s) \in C_\alpha$, $0 < \alpha \leq 1$, in some neighborhood of arbitrary point of the curve Γ, we call Γ Lyapunov curve.

Remark 3. Radon curves without cusp points and Lyapunov curves belongs class $\mathcal{C}$ $[17, p. 90]$.

Further we assume Γ Lyapunov curve or Radon curve without cusp points.
In this work we investigate *Riemann-Hilbert (Hilbert) problem* in the next posing: to find in the domain G solution $w = w(z)$ of the equation (1.1),

$w(z) \in E_p(A, B)$, $p > 1$, whose non-tangent limiting values on Γ satisfies almost everywhere boundary condition

$$\operatorname{Re}\left\{\overline{\lambda(t)}w(t)\right\} = g(t), \tag{1.2}$$

where $t = t(s)$, $s \in [0, S]$, is the affix of the point on Γ, $\lambda = \lambda(t)$ is complex measurable function defined on Γ and satisfies conditions $0 < k_0 \le |\lambda(t)| \le k_1 < \infty$, k_0, k_1 are real constants, $g(t) = g(t(s)) \equiv g(s) \in L_p(\Gamma) \equiv L_p[0, S]$ is real function, defined on Γ. If we divide (1.2) by $|\lambda(t)|$, we get equivalent boundary condition with $|\lambda(t)| \equiv 1$:

$$\operatorname{Re}\left\{e^{-i\omega(t)}w(t)\right\} = g(t), \tag{1.3}$$

where $\omega(t) = \arg\lambda(t)$. Further we assume $|\lambda(t)| \equiv 1$.

Further we shall use the notation $f(t) \equiv f(t(s)) = f(s)$ for the function f defined on Γ. If $w(z) \in E_p(A, B)$, $w(t) = w^+(t)$, $t \in \Gamma$, mean the non-tangent limit values on Γ when $z \to t \in \Gamma$, $z \in G$. $w^-(t)$ mean the non-tangent limit values on Γ when $z \to t \in \Gamma$, $z \in E \setminus \overline{G}$, E is the complex z-plane.

Following [15, p. 179], we say that the equation

$$\partial_{\bar z}w^* - A(z)w^*(z) - \overline{B(z)w^*(z)} = 0, \quad z \in G, \tag{1.4}$$

is *adjoint* to the equation (1.1).

Generalizing [15, p. 301] we call the *adjoint* (homogeneous) problem to (1.2) the problem of finding in G the solution of (1.4) $w^*(z) \in E_{p'}(-A, -\overline{B})$, $1/p + 1/p' = 1$, which non-tangent limit values on Γ almost everywhere on Γ satisfy boundary condition

$$\operatorname{Re}\{\lambda(t)t'(s)w^*(t)\} = 0. \tag{1.5}$$

Following [17, p. 190] (and [1], [2]) we *assume* that we can choose at least one starting point $s = 0$ on Γ such that the function $\omega(s)$ satisfy the next condition:

$$\omega(s) = \widetilde{\omega}_0(s) + \widetilde{\omega}_1(s) + \omega_2(s), \tag{1.6}$$

where $\widetilde{\omega}_0(s)$ is continuous function on $[0, S]$ (at the ends we mean the one-side continuity); $\widetilde{\omega}_1(s)$ is the function of finite variation on $[0, S]$; $\omega_2(s)$ is measurable function on $[0, S]$ satisfying the next conditions:

$$|\omega_2(s)| \le \nu\pi, \quad 0 < \nu < \frac{1}{2p}, \quad 0 < \nu < \frac{1}{2p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{1.7}$$

Without the loss of generality we assume [17, p. 190] that $\omega(0) = \omega(S)$ and $\widetilde{\omega}_1(s)$ is right-side continuous at the point $s = 0$. After these assumptions we can rewrite (1.6) in the next form [17, p. 190]:

$$\omega(s) = \omega_0(s) + \omega_1(s) + \omega_2(s), \tag{1.8}$$

where $\omega_2(s)$ is former; $\omega_1(s)$ is the jump function of $\widetilde{\omega}_1(s)$, $\{s_k\}$ is no more than countable set of discontinuity points of $\widetilde{\omega}_1(s)$:

$$\omega_1(0) = 0, \quad \omega_1(s) = \sum_{0<s_k<s} h_k + [\widetilde{\omega}_1(s) - \widetilde{\omega}_1(s-0)], \quad 0 < s \le S,$$

$h_k = \widetilde{\omega}_1(s_k + 0) - \widetilde{\omega}_1(s_k - 0)$. The continuous on $[0, S]$ function $\omega_0(s)$ equals to the sum $\widetilde{\omega}_0(s) + [\widetilde{\omega}_1(s) - \omega_1(s)]$.

No more than countable set of jump points of the function $\omega(s)$ we denote $\Xi = \{s_k\}$.

If Γ is Radon curve, i.e. $\theta(s)$ is the function of finite variation, than for $\theta(s)$ there is the expansion analogous (1.8) (with $\omega_2(s) \equiv 0$):

$$\theta(s) = \theta_0(s) + \theta_1(s). \tag{1.9}$$

Here the function $\theta_0(s)$ is continuous on $[0, S]$, and $\theta_1(s)$ is the jump function [17, p. 11–13]. The jumps of the function $\theta(s)$ we denote f_n, and no more than countable set of jump points of the function $\theta(s)$ we denote $\Theta = \{s_n\}$.

If Γ is Lyapunov curve then in (1.9) we get $\theta_1(s) \equiv 0$, $\theta(s) = \theta_0(s) \in C_\alpha$.

It's obvious we can assume that the point $s = 0$ is not in the set $\Xi \cup \Theta$.

Definition 1.6. We say that for the boundary value problem (1.2) ((1.3)) the condition **D** is hold if:

1) when Γ is Lyapunov curve, in (1.8) or $\omega_1(s) \equiv 0$, or $\omega_2(s) \equiv 0$;

2) when Γ is Radon curve without cusp points, in (1.8) $\omega_2(s) \equiv 0$.

2. Supporting information

The problem under consideration for the equation (1.1) is reduced to the corresponding problem for holomorphic functions. The holomorphic problem in turn is reduced to the problem in the unit disk [1], [2]. The index of the problem under consideration is defined through the index of the last problem in the unit disk. We need some constructions from [1], [2] to define the index of the problem.

We build on the disk $D : |\zeta| < 1$ in the complex ζ-plain the function

$$\Psi(\zeta) = \Phi(\varphi(\zeta)) \, [\varphi'(\zeta)]^{1/p}, \tag{2.1}$$

where $\varphi = \varphi(\zeta)$ is one-to-one conformal mapping of the unit disk D on the domain G, and $\Phi(z)$ is holomorphic function on G. It's known [17, p. 91], that $\Phi(z) \in E_p$ in G if and only if $\Psi(\zeta) \in H_p$ — Hardy class (class E_p in the disk D).

As when $|\zeta| < 1$ $\varphi'(\zeta) \neq 0$, and the curve Γ is in the class $\mathcal{C}$, for $|\zeta| < 1$ one can define univalent harmonic function $\arg \varphi'(\zeta)$ and this function everywhere on the circle $\zeta = e^{i\sigma}$ has non-tangent limit values. At every smooth point of Γ the next relation takes place [17, p. 88, 272]:

$$\arg \varphi'(e^{i\sigma}) = \theta(s(\sigma)) - \sigma - \frac{\pi}{2}. \qquad (2.2)$$

Owing to (2.1) the boundary value problem (1.3) for holomorphic function $\Phi(z)$ is equivalent to the boundary value problem

$$\operatorname{Re}\left\{e^{-i\nu(\sigma)}\Psi(\zeta)\right\} = g(t(\zeta))|\varphi'(e^{i\sigma})|^{1/p},\ \zeta = e^{i\sigma}, \qquad (2.3)$$

where, with taking into account (2.2)

$$\nu(\sigma) = \omega(s(\sigma)) + \frac{1}{p}\left(\theta(s(\sigma)) - \sigma - \frac{\pi}{2}\right) \qquad (2.4)$$

and the right side of (2.3) is in $L_p(C)$.

Index of the boundary condition. If in (1.8) $\omega_1(s) \equiv 0$, Γ is Lyapunov curve and (1.7) takes place, we call as index the boundary condition (1.3) (also (1.2)) the number

$$\varkappa = \frac{1}{2\pi}(\omega_0(S) - \omega_0(0)). \qquad (2.5)$$

Following [17, p. 215] we assume it integer. We use notation $\mathrm{ind}_\Gamma \lambda(t) = \varkappa$, where $\lambda(t)$ is the coefficient of the boundary condition (1.2).

Let now in (1.8) $\omega_2(s) \equiv 0$ and Γ is Radon curve without cusp points. Because the function $s(\sigma)$ and inverse one are absolutely continuous [17, p. 87], the function $\nu(\sigma)$ has finite variation and the next expansion, analogous (1.8), (1.9):

$$\nu(\sigma) = \nu_0(\sigma) + \nu_1(\sigma). \qquad (2.6)$$

Here the jump function $\nu_1(\sigma)$ looks like:

$$\nu_1(\sigma) = \omega_1(s(\sigma)) + \frac{1}{p}\,\theta_1(s(\sigma)). \qquad (2.7)$$

It's obvious, no more than countable set of jump points of the function $\nu(\sigma)$ is the image of the set $\Xi \cup \Theta$ under mapping $\sigma(s) : [0, S] \to [0, 2\pi]$. We enumerate by some way this set and denote one $\{\sigma_k\}$. It's obvious, that jumps of the function $\nu(\sigma)$ at the point σ_k is equal

$$n_k = h_k + \frac{1}{p}\, f_k, \quad \text{where} \quad h_k \ \text{and} \ f_k \qquad (2.8)$$

are the jumps of the functions $\omega(s)$ and $\theta(s)$ at pre-image of the point σ_k in $[0, S]$. At the point of continuity of one of this functions we assume its jump equal zero.

We denote n_k^+ and n_k^- accordingly positive jumps and modules of negative jumps from (2.8). The points of this jumps we accordingly denote σ_k^+ and σ_k^-. We arrange them in decrease sequences $\{n_k^+\}$ and $\{n_k^-\}$. We assume the conjugate numbers p and p' $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$ satisfying the next relatons:

$$\frac{n_k^-}{2\pi} \neq \frac{1}{2p'}, \quad \frac{n_k^+}{2\pi} \neq \frac{1}{2p}.$$

Following [17, p. 209] we denote $\overset{(p)}{\varkappa}_1$ the number satisfying

$$\frac{n_k^-}{2\pi} > \frac{1}{2p'}, \quad k=1,2,\ldots,\overset{(p)}{\varkappa}_1; \quad \frac{n_k^-}{2\pi} < \frac{1}{2p'}, \quad k > \overset{(p)}{\varkappa}_1; \tag{2.9}$$

and $\overset{(p)}{\varkappa}_2$ the number satisfying

$$\frac{n_k^+}{2\pi} > \frac{1}{2p}, \quad k=1,2,\ldots,\overset{(p)}{\varkappa}_2; \quad \frac{n_k^+}{2\pi} < \frac{1}{2p}, \quad k > \overset{(p)}{\varkappa}_2. \tag{2.10}$$

Further we denote $n_0 = n_0^{(1)} - n_0^{(0)}$ where $n_0^{(0)} = \nu_0(2\pi) - \nu_0(0)$, $n_0^{(1)} = \nu_1(0+0) - \nu_1(2\pi - 0)$. Following [17, p. 206] we define the integer $\varkappa_0$ from the relation

$$2n_0 = 2\pi \cdot \varkappa_0 + n_0^+, \quad 0 \le n_0^+ < 2\pi. \tag{2.11}$$

In this case as index of the boundary condition (1.3) we define the integer

$$\mathrm{ind}_\Gamma \lambda(t) = \overset{(p)}{\varkappa} = \overset{(p)}{\varkappa}_1 - \overset{(p)}{\varkappa}_2 - \varkappa_0, \tag{2.12}$$

provided that the solution of boundary value problem is searching in the class $E_p(A,B)$ (i. e. in this case index depends on the class in which solution is searching).

If in (1.8) $\omega_2(s) \equiv 0$ and Γ is Lyapunov curve, we define the index analogously with the simplification because in (2.7) $\theta_1(s(\sigma)) \equiv 0$ and in (2.8) $f_k = 0$ for every k, i.e. the set Θ is empty.

Further everywhere if $\omega_1(s) \equiv 0$, we assume that Γ is Lyapunov curve, and if $\omega_2(s) \equiv 0$, we assume that Γ is or Radon curve without cusp points, or Lyapunov curve.

Index of the adjoint boundary condition. We shall mark by asterisk all objects and quantities related the adjoint boundary condition. Now we shall express the index of the adjoint boundary condition through the index of the boundary condition (1.5) ($\varkappa$ or $\overset{(p)}{\varkappa}$).

If Γ is Lyapunov curve and $\omega_1(s) \equiv 0$, it's obvious

$$\varkappa^* = -\varkappa - 1. \tag{2.13}$$

Let $\omega_2(s) \equiv 0$ and Γ is or Lyapunov curve or Radon curve without cusp points.

Similar (2.3) we transform the boundary condition (1.5) to the view

$$\operatorname{Re}\left\{e^{-i\nu^*(\sigma)}\Psi^*(\zeta)\right\} = 0,\ \zeta = e^{i\sigma}, \tag{2.14}$$

where with allowance for (2.2) and searching the solution $\Psi^*(\zeta)$ in $E_{p'}$, $1/p + 1/p' = 1$,

$$\nu^*(\sigma) = -\omega(s(\sigma)) + \frac{1}{p'}\left(\theta(s(\sigma)) - \sigma - \frac{\pi}{2}\right) - \theta(s(\sigma)).$$

It's easy to see

$$\nu^*(\sigma) = -\left[\omega(s(\sigma)) + \frac{1}{p}(\theta(s(\sigma)) - \sigma) + \sigma\right] - \frac{\pi}{2p'}. \tag{2.15}$$

With allowance for (2.15), (2.9) and (2.10) we receive $\overset{(p)}{\varkappa^*}_1 = \overset{(p)}{\varkappa}_2$; $\overset{(p)}{\varkappa^*}_2 = \overset{(p)}{\varkappa}_1$.

If in (2.11) $n_0^+ = 0$, then from (2.15) the relation $\varkappa_0^* = \varkappa_0 - 2$ is obvious and we get

$$\overset{(p)}{\varkappa^*} = -\overset{(p)}{\varkappa} - 2. \tag{2.16}$$

In general case (2.16) follow from the coincidence the number of the conditions of solvability non-homogeneous problem (2.3), which is equal to the number of linear independent solutions of the homogeneous problem (2.14) $(\overset{(p)}{\varkappa^*} + 1)$, with $-\overset{(p)}{\varkappa} - 1$ [2]. As for $n_0^+ \neq 0$ formula (2.16) is not true, we can conclude $n_0^+ = 0$ always.

3. The formulation of the main results

Theorem 3.1. *If in* (1.8) $\omega_1(s) \equiv 0$ *and* Γ *is Lyapunov curve, then at* $\varkappa \geq 0$, *where* $\varkappa$ *is defined in* (2.5), *homogenous problem* (1.1), (1.3) *(at* $g(t) \equiv 0$*) has exactly* $2\varkappa + 1$ *linear independent in the real sense solutions in the class* $E_p(A, B)$, $p > 1$. *Non-homogenous problem is solvable in* $E_p(A, B)$ *at arbitrary right side* $g(t) \in L_p(\Gamma)$ *of the boundary condition.*

If $\varkappa < 0$ *then the homogenous problem* (1.1), (1.3) *has not in* $E_p(A, B)$ *non-zero solution and non-homogenous problem has unique solution in* $E_p(A, B)$ *if and only if* $-2\varkappa - 1$ *(real) conditions on the right side* $g(t)$ *of the boundary condition* (1.3) *are held:*

$$\int_\Gamma g(s)e^{i\omega(s)}w_k^*(t)t'(s)ds = 0. \tag{3.1}$$

Here $w_k^*(t) \in E_{p'+\varepsilon}(-A, -\overline{B})$, $E_{p+\varepsilon}(-A, -\overline{B})$, $k = 1, \ldots, -2\varkappa - 1$, *is the full system linear independent in the real sense solutions of the adjoint to* (1.1), (1.3) *boundary value problem* (1.4), (1.5) *with index* $\varkappa^* = -\varkappa - 1 \geq 0$, $\varepsilon > 0$ *is little.*

Theorem 3.2. *If in* (1.8) $\omega_2(s) \equiv 0$, *and* Γ *is Lyapunov curve or Radon curve without cusp points, then at* $\overset{(p)}{\varkappa} \geq 0$, *where* $\overset{(p)}{\varkappa}$ *is defined in* (2.12), *homogenous problem* (1.1), (1.3) *(at* $g(t) \equiv 0$*) has exactly* $\overset{(p)}{\varkappa} + 1$ *linear independent in the real sense solutions in the class* $E_p(A, B)$, $p > 1$. *Non-homogenous problem is solvable in* $E_p(A, B)$ *at arbitrary right side* $g(t) \in L_p(\Gamma)$ *of the boundary condition.*

If $\overset{(p)}{\varkappa} < 0$, *homogenous problem* (1.1), (1.3) *has not non-zero solution in* $E_p(A, B)$, $p > 1$, *and non-homogenous problem has unique solution if and only if* $-\overset{(p)}{\varkappa} - 1$ *(real) conditions on the right side* $g(t)$ *of the boundary condition* (1.3) *are held:*

$$\int_\Gamma e^{i\omega(s)} w_k^*(t(s)) t'(s) g(s) ds = 0, \quad k = 1, 2, \ldots, -\overset{(p)}{\varkappa} - 1. \tag{3.2}$$

Here $\{w_k^*(z)\} \in E_{p'}(-A, -\overline{B})$ *is the full system linear independent in the real sense solutions of the adjoint to* (1.1), (1.3) *boundary value problem* (1.4), (1.5).

It should be noted then if $\overset{(p)}{\varkappa} = -1$, *we get* $k = 0$. *It means uniquely unconditionally solvability of the non-homogenous problem.*

4. Supporting constructions

Lemma 4.1. *We denote, as above, the complex plane by* E. *At* $0 < \beta < 1$

$$\sup_{t \in E, 1/2 < r < 1} \int_{\Gamma_r} \frac{|dz|}{|t - z|^\beta} < \infty. \tag{4.1}$$

Proof. The integral $\int_{\Gamma_r} \frac{|dz|}{|t - z|^\beta}$ monotonically increases by r with arbitrary position of the point t [17, p. 77]. Hence, it's enough to estimate this integral at $r = 1$.

For $\forall t \in \Gamma$ there is $\varepsilon > 0$ (independent on t) such, as the disk $U_t^\varepsilon = \{z : |t - z| < \varepsilon\}$ has connected intersection with Γ [17, p. 21] (if Γ is Lyapunov curve, it's obvious).

On the other hand, if σ and s are the arc abscises on Γ, corresponding the points t and z, then there are such constants $k_3 > 0$ and $\varepsilon > 0$, that

$$|\sigma - s| \geq |t - z| \geq k_3 |\sigma - s|, \tag{4.2}$$

as soon as $|\sigma - s| \leq \varepsilon$ [17, p. 20].

Fix $k_3 > 0$ and $\varepsilon > 0$ so, that intersection $U_t^\varepsilon \cap \Gamma$ is connected arc $\forall t \in \Gamma$ and (4.2) is hold.

Let now the distance between t and Γ be $\geq \varepsilon$. Then

$$\int\limits_{\Gamma} \frac{|dz|}{|t - z|^\beta} \leq |\Gamma| \varepsilon^{-\beta}, \tag{4.3}$$

where $|\Gamma|$ is the length of the curve Γ.

Consider the case when the distance between t and Γ less ε. Denote $t_0 \in \Gamma$ the point, nearest to the point t and denote $\gamma^\varepsilon = U_{t_0}^\varepsilon \cap \Gamma$. Because all points of the arc γ^ε posed out of the disk with center at t and with radius $|t - t_0|$, for $\forall z \in \gamma^\varepsilon$ we get the estimate:

$$|t - z| \geq \frac{1}{2} |t_0 - z|. \tag{4.4}$$

Further, given the (4.2) and (4.4), we obtain:

$$\int\limits_{\gamma^\varepsilon} \frac{|dz|}{|t - z|^\beta} \leq \frac{2}{k_3} \int\limits_{\gamma^\varepsilon} \frac{ds}{|\sigma - s|^\beta} \leq \frac{2}{k_3(1 - \beta)} |\gamma^\varepsilon|^{1-\beta}, \tag{4.5}$$

where $|\gamma^\varepsilon|$ is the length of the arc γ^ε and σ is the arc abscissa of t_0.

At the same time

$$\int\limits_{\Gamma \setminus \gamma^\varepsilon} \frac{|dz|}{|t - z|^\beta} \leq |\Gamma| \varepsilon^{-\beta}. \tag{4.6}$$

Comparing (4.3), (4.5) and (4.6), we get (4.1). □

Lemma 4.2. *The set $E_p(A, B)$, $p \geq 1$, with norm*

$$\|w\|_{E_p} = \left\{ \int\limits_{\Gamma} |w(z(s))|^p |dz| \right\}^{1/p}$$

is the real Banach space.

Here $w(z(s))$ is non-tangent limit values on Γ of the function $w(z) \in E_p(A, B)$.

In the case $A(z) = B(z) \equiv 0$ we obtain usual norm in classical Smirnov space E_p.

Proof. Is a literal repetition of the proof of the theorem 5 of work [12] with the replacement $H_p(A,B)$ to $E_p(A,B)$ and the refers on [9] to refers on [11]. $\square$

Denote

$$Tf(z) = -\frac{1}{\pi}\iint\limits_G \frac{f(t)}{t-z}dxdy, \quad t = x+iy.$$

Here is some properties of the operator T.

Lemma 4.3 ([15, p. 60–65])**.** *Operator T is completely continuous in $L_q(\overline{G})$, $q \geq 1$.*

Lemma 4.4. *If $f(t) \in L_q(\overline{G})$, $1 < q < 2$, then $Tf(z) \in L_\gamma(\Gamma_r)$, $0 < r \leq 1$, where γ is arbitrary number, satisfying conditions $1 < \gamma < \dfrac{q}{2-q}$, and there are the next inequalities:*

$$\|Tf\|_{L_\gamma(\Gamma_r)} \leq M_{q,\gamma}(G)\|f\|_{L_q(\overline{G})}, \tag{4.7}$$

$$\|Tf(z+\Delta z) - Tf(z)\|_{L_\gamma(\Gamma_r)} \leq M^*_{q,\gamma}(G)\|f\|_{L_q(\overline{G})}|\Delta z|^\alpha, \quad \alpha > 0. \tag{4.8}$$

Here the constants $M_{q,\gamma}(G)$ and $M^*_{q,\gamma}(G)$ depend on γ, q, G and do no depend on r and f; in (4.8) z and $z+\Delta z$ lay on Γ_r.

Proof. Inequality (4.7) is the direct consequence of lemma 1 from [11], see also [15, p. 67–69]. We shall prove the inequality (4.8). The proof is based on the reasoning from [15, p. 68].

At first we assume $q < \gamma < \dfrac{q}{2-q}$. We get:

$$|\Delta T| \equiv |Tf(z+\Delta z) - Tf(z)| \leq \frac{|\Delta z|}{\pi}\iint\limits_G \frac{|f(t)|dxdy}{|t-z||t-z-\Delta z|} \leq$$

$$\leq \frac{|\Delta z|}{\pi}\iint\limits_G |f(t)|^{\frac{q}{\gamma}}(|t-z||t-z-\Delta z|)^{-\frac{1}{\gamma}+\alpha} \cdot |f(t)|^{q\left(\frac{1}{q}-\frac{1}{\gamma}\right)} \times$$

$$\times (|t-z||t-z-\Delta z|)^{-\frac{2}{q'}+\alpha}dxdy,$$

where $2\alpha = \dfrac{1}{\gamma} - \dfrac{2}{q} + 1 > 0$, $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. Because $\dfrac{1}{\gamma} + \dfrac{\gamma - q}{\gamma q} + \dfrac{1}{q'} = 1$, using the Hölder inequality we obtain:

$$\begin{aligned} |\Delta T| &\leq \frac{|\Delta z|}{\pi}\left(\iint\limits_G |f(t)|^q(|t-z||t-z-\Delta z|)^{-1+\gamma\alpha}dxdy\right)^{\frac{1}{\gamma}} \times \\ &\times \left(\iint\limits_G |f(t)|^q dxdy\right)^{\frac{1}{q}-\frac{1}{\gamma}} \left(\iint\limits_G (|t-z||t-z-\Delta z|)^{-2+q'\alpha}dxdy\right)^{\frac{1}{q'}}. \end{aligned} \tag{4.9}$$

For the last factor in (4.9) we have the estimate [15, p. 56]:

$$\left(\iint_G (|t-z||t-z-\Delta z|)^{-2+q'\alpha}dxdy\right)^{\frac{1}{q'}} \leq C_{q,\gamma}(G)|\Delta z|^{\frac{1}{\gamma}-1}, \tag{4.10}$$

where the constant $C_{q,\gamma}(G)$ depends only on γ, q, G.

Further, bearing in mind that $0 < \alpha\gamma < 1$, and using known inequality $(a+b)^\beta \leq a^\beta + b^\beta$, $a \geq 0$, $b \geq 0$, $0 < \beta < 1$, and lemma 4.1, we get ($z = z(s)$):

$$\int_{\Gamma_r} (|t-z||t-z-\Delta z|)^{-1+\alpha\gamma}ds =$$

$$= |\Delta z|^{-1+\alpha\gamma} \int_{\Gamma_r} \left|\frac{1}{t-z} - \frac{1}{t-z-\Delta z}\right|^{1-\alpha\gamma} ds \leq$$

$$\leq |\Delta z|^{-1+\alpha\gamma} \left\{\int_{\Gamma_r} \frac{ds}{|t-z|^{1-\alpha\gamma}} + \int_{\Gamma_r} \frac{ds}{|t-z-\Delta z|^{1-\alpha\gamma}}\right\} \leq C^*_{q,\gamma}(G)|\Delta z|^{-1+\alpha\gamma}, \tag{4.11}$$

where the constant $C^*_{q,\gamma}(G)$ depends only on γ, q, G.

Comparing (4.9), (4.10) and (4.11) we obtain (4.8). It's obvious, now we can remove restriction $\gamma > q$. □

Corollary 1. The map $T : L_q(\overline{G}) \to L_\gamma(\Gamma_r)$, $1/2 < r \leq 1$, is completely continuous.

Proof. It is directly derived from (4.2), (4.7), (4.8) and Arzelà-Ascoli theorem. □

Lemma 4.5. *Let the condition* $\boldsymbol{D}$ *is held and or index* $\varkappa$ *of the boundary value problem* (1.2) *is non-negative, or index* $\overset{(p)}{\varkappa} \geq -1$*. There exists such function* $M(t,z)$*, that*

$$\text{Re}\{\overline{\lambda(s)}M(t,z(s))\} = \text{Re}\left\{\overline{\lambda(s)} \cdot \frac{1}{t-z(s)}\right\}, \quad \forall z(s) \in \Gamma, \tag{4.12}$$

where t *is arbitrary point in* G*, with properties:*

1) for every fixed $t \in G$ $M(t,z) \in E_p$, $p > 1$*;*

2) for $\forall q : 1 < q < 2$, $p < \dfrac{q}{2-q}$, $f(t) \in L_q(\overline{G})$

$$T_M f(z) = -\frac{1}{\pi}\iint_G f(t)M(t,z)dxdy \in E_\gamma, \tag{4.13}$$

$\forall\gamma : 1 < \gamma < \dfrac{q}{2-q}$, $\gamma \le p$, *and there is the estimation*

$$\|T_M f\|_{E_\gamma} \le \mathrm{const}\|f\|_{L_q(\overline{G})}, \tag{4.14}$$

where the constant is independent on f;

3) the operator $T_M f : L_q(\overline{G}) \to E_\gamma$ is completely continuous.

Proof. Consider in G Riemann-Hilbert boundary value problem (4.12) for the holomorphic function $M(t,z)$.

The right part of (4.12) we denote $F(t,z(s)) \equiv F(t,s) \in L_\infty(\Gamma)$. Accordingly to [1], [2], because the condition **D** is hold and or index $\varkappa$ is non-negative, or index $\overset{(p)}{\varkappa} \ge -1$, the problem (4.12) is unconditionally solvable in E_p and its particular solution can be taken in the form

$$\begin{aligned} M(t,z) &= \frac{1}{2}\left\{\Psi(\zeta(z)) + \Psi_*(\zeta(z))\right\}\left[\zeta_z'(z)\right]^{1/p}, \\ \Psi(\zeta) &= \frac{Z(\zeta)}{2\pi i}\int\limits_C \frac{F_0(t,\tau)}{Z^+(\tau)}\cdot\frac{d\tau}{\tau-\zeta}, \quad \Psi_*(\zeta) = \overline{\Psi\left(\frac{1}{\overline{\zeta}}\right)}, \\ \Psi_*(\zeta) &= \frac{\zeta^{\kappa+1}Z(\zeta)}{2\pi i}\int\limits_C \frac{\overline{F_0(t,\tau)}}{\overline{Z^+(\tau)}}\cdot\frac{d\bar\tau}{1-\bar\tau\zeta}. \end{aligned} \tag{4.15}$$

Here κ equal or $2\varkappa$, or $\overset{(p)}{\varkappa}$, $Z(\zeta)$ is completely defined function from $H_{p+\varepsilon}$, $\varepsilon > 0$, in D and in outside D, and [2]

$$Z^{-1}(\zeta) \in H_{p+\varepsilon}, \ \ Z(\zeta),\ Z^{-1}(\zeta) \in H_{p'+\varepsilon}, \ \ 1/p + 1/p' = 1;$$

$$Z^+(t) = -e^{2i\nu(\sigma)}Z^-(t), \ \ t = e^{i\sigma} \in C,$$

$Z^\pm(t)$ are non-tangent limiting values on C of the function $Z(\zeta)$ accordingly inside and outside the disk D; $\zeta = \zeta(z)$ is conformal mapping G onto D, inverse to the mapping $\varphi = \varphi(\zeta)$,

$$F_0(t,\tau) = 2F(t,\varphi(\tau))\left[\varphi_\zeta'(\tau)\right]^{1/p} e^{i\omega(\varphi(\tau))};$$

functions ω and ν are the same as in (2.4).

The statement 1) is proved.

Because $E_p \subset E_\gamma$, $p \ge \gamma > 1$, and inclusion is continuous, obviously enough to prove the remaining statements of lemma at $\gamma = p$.

Consider the expression

$$F_1(z) = 2\iint\limits_G F(t,z)f(t)\,dxdy, \ \ t = x+iy, \ \ z \in \Gamma. \tag{4.16}$$

Given (4.12) and lemma 4.4, we get

$$\|F_1\|_{L_p(\Gamma)} \le \mathrm{const}\left\| \iint_G |F(t,z)| \cdot |f(t)| dxdy \right\|_{L_p(\Gamma)} \le \mathrm{const}\|f\|_{L_q(\overline{G})},$$

where the constant depends only on p and q and is independent on f. Thus, the mapping $F_1 : L_q(\overline{G}) \to L_p(\Gamma)$, setting by (4.16), is continuous.

Denote $s + \Delta s$ the arc abscissa of the point $z + \Delta z \in \Gamma$ ($z = z(s)$). Then from (4.12), (4.8) and (4.2) we get

$$\|F_1(s+\Delta s) - F_1(s)\|_{L_p(\Gamma)} \le \mathrm{const}\|f\|_{L_q(\overline{G})}|\Delta s|^\alpha,\ \alpha > 0,$$

where the constant depends only on p and q and is independent on f. It follows, that the mapping $F_1 : L_q(\overline{G}) \to L_p(\Gamma)$ is completely continuous.

Further, by obvious way we get that the mapping $F_2 : L_q(\overline{G}) \to L_p(C)$, defined by formula

$$F_2(\tau) = F_1(z(\tau)) \cdot \left[\varphi'_\zeta(\tau)\right]^{1/p} e^{i\omega(\varphi(\tau))} \in L_p(C)$$

is completely continuous and we can use Fubini's theorem to calculate $T_M f(z)$.

Thus,

$$\begin{aligned} T_M f(z) &= -\frac{1}{\pi}\left[\zeta'_z(z)\right]^{1/p}\left\{\Psi^f(\zeta(z)) + \Psi^f_*(\zeta(z))\right\}, \\ \Psi^f(\zeta) &= \frac{Z(\zeta)}{4\pi i}\int_C \frac{F_2(\tau)}{Z^+(\tau)} \cdot \frac{d\tau}{\tau - \zeta}, \\ \Psi^f_*(\zeta) &= \frac{\zeta^{\kappa+1}Z(\zeta)}{4\pi i}\int_C \frac{\overline{F_2(\tau)}}{\overline{Z^+(\tau)}} \cdot \frac{d\bar\tau}{1 - \bar\tau\zeta}. \end{aligned} \tag{4.17}$$

Because the right sides of (4.17) map $F_2 \in L_p(C) \to H_p$ continuously [17, p. 218], the maps Ψ, $\Psi_* : f \in L_q(\overline{G}) \to H_p$, defined in (4.15), are completely continuous, and we get (4.14) and the statement 3) of our lemma. □

Lemma 4.6. *Let the condition* $\boldsymbol{D}$ *is held and or index* $\varkappa$ *of the boundary value problem* (1.2) *is non-negative, or index* $\overset{(p)}{\varkappa} \ge -1$. *There exists such function* $\hat M(t,z)$, *that*

$$\mathrm{Re}\{\overline{\lambda(s)}\hat M(t,z(s))\} = -\mathrm{Im}\left\{\overline{\lambda(s)} \cdot \frac{1}{t - z(s)}\right\},\quad \forall z(s) \in \Gamma,$$

where t *is arbitrary point in* G, *with properties:*

1) for every fixed $t \in G$ $\hat M(t,z) \in E_p$, $p > 1$;

2) for $\forall q : 1 < q < 2,\ p < \dfrac{q}{2-q}$, $f(t) \in L_q(\overline{G})$

$$T_{\hat{M}} f(z) = -\frac{1}{\pi} \iint\limits_{G} f(t) \hat{M}(t,z)\, dxdy \in E_\gamma,$$

$\forall \gamma : 1 < \gamma < \dfrac{q}{2-q}$, $\gamma \le p$, *and there is the estimation*

$$\|T_{\hat{M}} f\|_{E_\gamma} \le \mathrm{const} \|f\|_{L_q(\overline{G})},$$

where the constant is independent on f*;*

3) the operator $T_{\hat{M}} f : L_q(\overline{G}) \to E_\gamma$ *is completely continuous.*

Proof is similar to the proof of lemma 4.5.

In assumptions of lemma 4.5 we introduce the operator

$$Pf = Tf - T_M(\operatorname{Re} f) - T_{\hat{M}}(\operatorname{Im} f). \tag{4.18}$$

Lemma 4.7. *1) For* $\forall q : 1 < q < 2,\ p < \dfrac{q}{2-q}$, $f(t) \in L_q(\overline{G})$ $Pf(z) \in L_\gamma(\Gamma)$ *at* $\forall \gamma : 1 < \gamma < \dfrac{q}{2-q}$, $\gamma \le p$, *and we get the estimation*

$$\|Pf\|_{L_\gamma(\Gamma)} \le \mathrm{const} \|f\|_{L_q(\overline{G})},$$

where the constant is independent on f*.*

The map $P : L_q(\overline{G}) \to L_\gamma(\Gamma)$ *is completely continuous.*

2) Operator P *is completely continuous in* $L_q(\overline{G})$, $1 < q < 2$.

3) For almost all $z \in \Gamma$ *u* $\forall q : 1 < q < 2,\ p < \dfrac{q}{2-q}$, $f(t) \in L_q(\overline{G})$

$$\operatorname{Re}\{\overline{\lambda(z)} Pf(z)\} = 0.$$

Proof. The property 1) directly follow from the lemmas 4.4, 4.5, 4.6 and corollary 1.

The maps $T : L_q(\overline{G}) \to L_q(\overline{G})$, T_M, $T_{\hat{M}} : L_q(\overline{G}) \to E_p$ are completely continuous (lemmas 4.3, 4.5, 4.6) and $1 < q < 2 < 2p$, therefore E_p is continuously included in $L_q(\overline{G})$ (lemma 4.10, see below). Hence, we get the property 2). The property 3) is direct corollary of (4.18), lemmas 4.5, 4.6 and Lebesgue's dominated convergence theorem. □

Remark 4. If G is the unit disk, $\lambda(t) = t^n$, $n \ge 0$ — non-negative integer, then the operator P coincides with the operator P_n, constructed by I.N. Vekua in [15, p. 293].

In assumptions of lemma 4.5 now we introduce the operator

$$P_\lambda w = P(Aw + B\overline{w}).$$

Lemma 4.8. *If* $w \in L_m(\overline{G})$, $1 < m < 2p$, $A(z), B(z) \in L_s(\overline{G})$, $s > 2$, *then for suitable* m *and for* $q : \dfrac{1}{m} + \dfrac{1}{s} = \dfrac{1}{q}$, $1 < q < 2$, *we get* $p < \dfrac{q}{2-q}$; $Aw + B\overline{w} \in L_q(\overline{G})$ *and the next properties:*

1) the operator P_λ *is completely continuous in* $L_m(\overline{G})$;

2) $P_\lambda w(z) \in L_\gamma(\Gamma)$ *at* $\forall\gamma : 1 < \gamma < \dfrac{q}{2-q}$, $\gamma \le p$, *and there is the estimation*

$$\|P_\lambda w\|_{L_\gamma(\Gamma)} \le \mathrm{const}\|w\|_{L_m(\overline{G})},$$

where the constant is independent on w; *the map* $P_\lambda : L_m(\overline{G}) \to L_\gamma(\Gamma)$ *is completely continuous;*

3) for almost all $z \in \Gamma$

$$\mathrm{Re}\{\overline{\lambda(z)}P_\lambda w(z)\} = 0.$$

Proof. Obviously, at the expense of choice m we can consider $1 < q < 2$. If at the expense of choice m we can obtain q arbitrarily close to 2, then the inequality $p < \dfrac{q}{2-q}$ can be provided for any $p > 1$.

Let for every m, $1 < m < 2p$, we get $1 < q(m) < q_0 < 2$, where $q_0 = \sup\limits_{1<m<2p} q(m) = \dfrac{2ps}{2p+s}$. Then supremum on $m : 1 < m < 2p$ of the monotonically increasing on m function $\dfrac{q(m)}{2-q(m)}$ equals $\dfrac{ps}{2p+s-ps}$, and at the same time

$$p - \frac{ps}{2p+s-ps} = p^2\frac{2-s}{2p+s-ps} < 0.$$

Thus, in this case the ineqality $p < \dfrac{q}{2-q}$ can be provided for any $p > 1$ too.

If it's provided, then owing to Hölder inequality $Aw + B\overline{w} \in L_q(\overline{G})$. Other assertions of the lemma follow from lemma 4.7 and Hölder inequality. □

We shall investigate the behavior of the norm of operator $P_\lambda : L_m(\overline{G}) \to L_m(\overline{G})$, in which $A(z)$, $B(z)$ — coefficients of the equation (1.1), under homotheties

$$\widetilde{z} = \varepsilon z, \quad \varepsilon > 0, \tag{4.19}$$

of the domain G.

Lemma 4.9. *The norm of operator*

$$T_0 w = T(Aw + B\overline{w}) : L_m(\overline{G}) \to L_m(\overline{G}),$$

where m *is defined by lemma 4.8, under homotheties* (4.19) *has the asymptotics* $O(\varepsilon^\delta)$, *where* $\delta = \dfrac{s-2}{2s}(m-1) > 0$.

Proof. Under homothety (4.19) the coefficients of the equation (1.1) are transformed by formulas

$$\widetilde{A}(\widetilde{z}) = \frac{1}{\varepsilon}A(z), \quad \widetilde{B}(\widetilde{z}) = \frac{1}{\varepsilon}B(z). \tag{4.20}$$

Here and further we mark by tilde the objects, depending on variable $\widetilde{z}$. Obviously, for investigating the norm of operator T_0, without loss of generality we can assume $B(z) \equiv 0$.

For the operator $T_0 w = T(Aw)$ we get the estimation [15, p.64]:

$$\|T_0 w\|_{L_\gamma(\overline{G})} \leq \frac{1}{\pi^\gamma}\left[M(q'\alpha,\, G)\right]^{1/q'} M(\gamma\alpha,\, G)\|Aw\|_{L_q(\overline{G})}, \tag{4.21}$$

where $\alpha = \dfrac{1}{\gamma} - \dfrac{1}{q} + \dfrac{1}{2} > 0$, $\dfrac{1}{m} + \dfrac{1}{s} = \dfrac{1}{q}$, $1 < q < 2$, $1 < \gamma < \dfrac{2q}{2-q}$, $\dfrac{1}{q} + \dfrac{1}{q'} = 1$,

$$M(\lambda,\, G) = \sup_{z\in E} \iint\limits_G |\zeta - z|^{-2+\lambda} d\xi d\eta,\ \zeta = \xi + i\eta. \tag{4.22}$$

We note, that from the Hölder inequality one gets

$$\|Aw\|_{L_q(\overline{G})} \leq \|A\|_{L_s(\overline{G})} \cdot \|w\|_{L_m(\overline{G})}.$$

From here we obtain the corresponding estimation of the right side of (4.21) through $\|w\|_{L_m(\overline{G})}$.

We shall compare the factor before $\|w\|_{L_m(\overline{G})}$ in the right side (4.21) before transformation (4.19) and after one.

Taking into account (4.22), we shall get:

$$\begin{aligned} &\|\widetilde{A}\|_{L_s(\overline{\widetilde{G}})} = \varepsilon^{\frac{2-s}{s}}\|A\|_{L_s(\overline{G})};\ M(\lambda,\, \widetilde{G}) = \varepsilon^\lambda M(\lambda,\, G);\\ &\left[M(q'\alpha,\, \widetilde{G})\right]^{1/q'} = \varepsilon^\alpha\left[M(q'\alpha,\, G)\right]^{1/q'};\ M(\gamma\alpha, \widetilde{G}) = \varepsilon^{\gamma\alpha}M(\gamma\alpha,\, G). \end{aligned} \tag{4.23}$$

Denote $\widetilde{N}$ the factor before $\|w\|_{L_m}$ in the right side (4.21) after transformation (4.19). From (4.23) we obtain $\widetilde{N} = \varepsilon^\delta N$, where $\delta = \alpha(1+\gamma) + \dfrac{2-s}{s}$. If we put $\gamma = m$, what owing to lemma 4.8 is possible, then finally one gets $\delta = \dfrac{s-2}{2s}(m-1) > 0$. □

Lemma 4.10. *If $w(z) \in E_p(A,B)$, $p > 1$, then $w(z) \in L_m(\overline{G})$, $\forall m : 1 < m < 2p$, and the inclusion $E_p(A,B) \subset L_m(\overline{G})$ is continuous, i. e., there is inequality:*

$$\|w\|_{L_m(\overline{G})} \leq \text{const}\left[M\left(\frac{m}{m+\mu};\, \Gamma\right)\right]^{1/p'}\left[M\left(\frac{2\mu}{m+\mu};\, G\right)\right]^{1/p}\|w\|_{E_p}, \tag{4.24}$$

where $1/p + 1/p' = 1$, $0 < \mu < 1$,

$$M(\lambda;\,\Gamma) = \sup_{z\in E}\int\limits_{\Gamma} |z-\zeta|^{-\lambda} ds, \tag{4.25}$$

$M(\lambda;\,G)$ *is defined by* (4.22)*, the constant depends on* G *and coefficients* $A(z)$*,* $B(z)$*, and is not dependent on* w *and homothety* (4.19)*.*

Proof. The first assertion is a special case of the theorem 4 from [11].

Remark 5. For holomorphic functions lemma 4.10 (without inequality) was formulated in [18, p. 96]. Thus the proof has a mistake, which leads to the wrong inequality for the norms. In this regard, we give the detailed proof (based on the construction from [18, p. 96]).

At first let's consider the case of holomorphic functions (i. e. the case $A(z) = B(z) \equiv 0$).

Representing function $\Phi(z) \in E_p$, $p > 1$, by Cauchy-Lebesgue integral [16, p. 423-424], and using Hölder inequality with the exponents $m = (2-\mu)p$, $r = p/\alpha$ and $\lambda = p/(p-1)$, $\dfrac{1}{m} + \dfrac{1}{r} + \dfrac{1}{\lambda} = 1$, we get

$$(2\pi)^m \iint\limits_G |\Phi(z)|^m dxdy \le \iint\limits_G \left(\int\limits_\Gamma |\Phi(t)||t-z|^{-1} ds\right)^m dxdy \le$$

$$\le \iint\limits_G \left[\int\limits_\Gamma |\Phi(t)|^{1-\alpha}|t-z|^{-\beta}\left(|\Phi(t)|^{\alpha}|t-z|^{\beta-1}\right) ds\right]^m dxdy \le$$

$$\le \|\Phi\|^{\alpha m}_{L_p(\Gamma)} \iint\limits_G \left(\int\limits_\Gamma |t-z|^{-\beta\lambda} ds\right)^{m/\lambda} \left(\int\limits_\Gamma |\Phi(t)|^p |t-z|^{(\beta-1)m} ds\right) dxdy,$$

where

$$\alpha = \frac{1-\mu}{2-\mu}, \quad \beta = \frac{m-2+\mu}{m+\mu}.$$

Since $\beta\lambda = \dfrac{m}{m+\mu}$, $(1-\beta)m = \dfrac{2m}{m+\mu}$, and also $\alpha + p/m = 1$, from here we obtain (4.24) in the case of holomorphic functions.

The general case (4.24) follows from proving special case and the basic representation for generalized analytic functions of class $E_p(A,B)$ [11]:

$$w(z) = \Phi(z)\exp\{-T(A + B(\overline{w}/w))(z)\}, \quad \Phi(z) \in E_p, \tag{4.26}$$

and formula (4.20). We have to take into account, that in (4.26) $\exp\{-T(A + B(\overline{w}/w))\} \in C_\beta(\overline{G})$, $\beta = (s-2)/s$, [15, p. 60] and it is not changed under homothety (4.19) (in the sense that the values of this expression coincide in the corresponding points $\widetilde{z}$ and z). □

Lemma 4.11. *The operator* $T_M(Aw + B\overline{w}) : L_m(\overline{G}) \to L_m(\overline{G})$*, defined by* (4.13)*, where* m *defined by lemma 4.8,* A*,* B *— coefficients of the equation* (1.1)*, is completely continuous. Under homotheties* (4.19) *its norm has asymptotics* $O(\varepsilon^\delta)$*, where* $\delta > 0$*.*

The similar assertion takes place for $T_{\hat{M}}(Aw + B\overline{w})$*.*

Proof. Obviously, it's enough to prove lemma for $T_M(Aw)$.

Complete continuity was proved in the proof of lemma 4.7. Now we shall establish the asymptotics of the norm of operator $T_M(Aw)$.

From (4.17) we get [17, p. 218]:

$$\begin{aligned} &\|T_M(Aw)(z)\|_{L_p(\Gamma)} \le \frac{2}{\pi}\|\Psi^{Aw}(\zeta)\|_{L_p(C)} \le \\ &\le \text{const}\|F_2(\zeta)\|_{L_p(C)} \le \text{const}\|F_1(z)\|_{L_p(\Gamma)}, \end{aligned} \tag{4.27}$$

where the constants under homothety (4.19) is not dependent on ε. Now we estimate $\|F_1(z)\|_{L_p(\Gamma)}$.

From (4.13), (4.16), [15, p. 68] and Hölder inequality we get:

$$\begin{aligned} &\|F_1(z)\|_{L_p(\Gamma)} \le 2\pi\|T(Aw)\|_{L_p(\Gamma)} \le \\ &\le \frac{2}{\pi^{p-1}}\left[M(q'\alpha;\, G)\right]^{1/q'}\left[M(1-p\alpha;\, \Gamma)\right]^{1/p}\|A\|_{L_s(\overline{G})}\|w\|_{L_m(\overline{G})}, \end{aligned} \tag{4.28}$$

where $1/q' + 1/q = 1$, $2\alpha = 1/p - 2/q + 1$; $M(\lambda;\, G)$ is defined by (4.22); $M(\lambda;\, \Gamma)$ is defined by (4.25). Denote the factor before $\|w\|_{L_m(\overline{G})}$ in the right side (4.28) by N_1 and after transformation (4.19) denote the same factor $\widetilde{N}_1$. Owing to (4.20), (4.23) and $\widetilde{M}(\lambda; \widetilde{\Gamma}) = \varepsilon^{1-\lambda}M(\lambda;\, \Gamma)$, we get $\widetilde{N}_1 = \varepsilon^{\delta_1}N_1$, where

$$\delta_1 = -\frac{1}{m} + \frac{1}{s} - p\left(\frac{1}{q} - \frac{1}{2}\right) + \frac{1}{2p}.$$

Further, substituting (4.27), (4.28) in (4.24), we obtain the inequality

$$\|T_M(Aw)\|_{L_m(\overline{G})} \le N\|w\|_{L_m(\overline{G})},$$

where the factor N under homothety (4.19) is multiplied by ε^δ,

$$\delta = \delta(\mu) = \delta_1 + \frac{\mu}{m+\mu}\cdot\frac{p+1}{p}.$$

Put $m = (2-\mu)p$, where $\mu > 0$ is little. Then, in view of the fact that $\dfrac{1}{m} + \dfrac{1}{s} = \dfrac{1}{q}$,

$$\delta(\mu) = \frac{1}{s} - p\left(\frac{1}{(2-\mu)p} + \frac{1}{s} - \frac{1}{2}\right) + \mu\left(\frac{1}{(2-\mu)p+\mu}\cdot\frac{p+1}{p} - \frac{1}{2p(2-\mu)}\right).$$

Since at $\mu = 0$ $\delta(0) = \left(\frac{1}{2} - \frac{1}{s}\right)(p-1) > 0$, at little $\mu > 0$ $\delta(\mu) > 0$. $\square$

Remark 6. Using lemma 4.8 we can assume s enough closing to 2, and for arbitrary large m, by choosing s we can provide $q < 2$.

Since the condition $p < \frac{q}{2-q}$ in lemma 4.8 is equivalent the condition $\frac{1}{m} < \frac{1}{2p} + \frac{1}{2} - \frac{1}{s}$, at m close to $2p$ (and, may be large) it's held provided appropriate s.

The next statement is the obvious corollary of lemmas 4.9 and 4.11.

Lemma 4.12. *After homothety* (4.19) *with sufficient little* $\varepsilon > 0$ *the operator* $P_\lambda : L_m(\overline{G}) \to L_m(\overline{G})$ *at some* $m : 1 < m < 2p$ *becomes contraction operator.*

Theorem 4.13. *Let the condition* $\boldsymbol{D}$ *is held and or index* $\varkappa$ *of the boundary value problem* (1.2) *is non-negative, or index* $\overset{(p)}{\varkappa} \geq -1$.

If $w(z) \in E_p(A, B)$, $p > 1$, *the next representation takes place:*

$$w(z) + P_\lambda w(z) = \Phi(z), \tag{4.29}$$

where $\Phi(z) \in E_p$ *and almost everywhere on* Γ

$$\mathrm{Re}\{\overline{\lambda}(t)w(t)\} = \mathrm{Re}\{\overline{\lambda}(t)\Phi(t)\}, \quad t \in \Gamma. \tag{4.30}$$

If $\Phi(z) \in E_p$, *then the relation* (4.29) *uniquely defines the function* $w(z) \in E_p(A, B)$, *satisfying almost everywhere on* Γ *condition* (4.30). *Formula* (4.29) *establishes (real) liniar isomorphism between Banach spaces* $E_p(A, B)$ *u* E_p, *and also the operator* $P_\lambda : E_p(A, B) \to L_p(\Gamma)$ *is completely continuous.*

Proof. If $w(z) \in E_p(A, B)$, $p > 1$. there is the relation [11]:

$$w(z) + T(Aw + B\overline{w})(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{w(t)dt}{t - z} \in E_p. \tag{4.31}$$

Imposing in [11] condition $p > \frac{s}{2(s-1)}$ is superfluous, because $s > 2$ always can be taken enough close to 2 and this condition will be held.

Subtracting from the both sides of (4.31) the holomorphic function

$$\Phi^*(z) = T_M\{\mathrm{Re}(Aw + B\overline{w})\}(z) + T_{\hat{M}}\{\mathrm{Im}(Aw + B\overline{w})\}(z),$$

which owing to lemmas 4.5 and 4.6 belongs to E_p, and denoting

$$\Phi(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{w(t)dt}{t - z} - \Phi^*(z) \in E_p,$$

we obtain (4.29). The relation (4.30) follows from the property 3 of lemma 4.8.

Let now $\Phi(z) \in E_p$ and we shall assume $P_\lambda : L_m(\overline{G}) \to L_m(\overline{G})$ is contraction operator at some $m : 1 < m < 2p$. Owing to lemma 4.12 we can do it without loss of generality.

In this case the equation (4.29) is uniquely solvable in $L_m(\overline{G})$ with arbitrary right side from the same space. Because owing to lemma 4.10 $\Phi(z) \in L_m(\overline{G})$, we get unique solution $w(z) \in L_m(\overline{G})$, which, obviously, is in G the solution of the differential equation (1.1). Owing to lemmas 4.4, 4.5 and 4.6

$$\int\limits_{\Gamma_r} |P_\lambda w(t)|^p ds < \text{const} < \infty, \quad \forall r : 0 < r < 1,$$

where the constant is not dependent on r. From here we get $w(z) \in E_p(A, B)$. The relation (4.30) follows from lemma 4.8.

Thus, the operator $I + P_\lambda$, where I is identity mapping, carries out (real) isomorphism $E_p(A, B)$ and E_p. Here the operator $P_\lambda : E_p(A, B) \to L_p(\Gamma)$, owing to lemmas 4.8 and 4.10, is completely continuous. □

5. Proof of the main results

At $\varkappa \geq 0$ or $\overset{(p)}{\varkappa} \geq -1$ the assertions of the theorems 3.1 and 3.2 directly follow from the corresponding results for holomorphic functions [1], [2] and theorem 4.13. Really, in this case the operator $I + P_\lambda$ carries out (real) isomorphism the space of solutions in E_p of boundary value problem (1.3) for holomorphic functions and the space of solutions in $E_p(A, B)$ of boundary value problem (1.1), (1.3).

Let's consider the case of negative index. Now we shall prove the necessity of conditions (3.1) and (3.2).

Let $w^*(z) \in E_{p'}(-A, -\overline{B})$ is arbitrary solution of the homogeneous adjoint problem (1.4), (1.5), and $w(z) \in E_p(A, B)$ is the solution of the problem (1.1), (1.3). Note, that there is equality [11]:

$$\operatorname{Im} \int\limits_\Gamma w(t) w^*(t) dt = 0.$$

From here we get:

$$0 = \operatorname{Im} \int\limits_\Gamma e^{-i\omega(s)} w(t) e^{i\omega(s)} w^*(t) t'(s) ds = \int\limits_\Gamma g(s) e^{i\omega(s)} w^*(t(s)) t'(s) ds.$$

Owing to the results, proved for $\varkappa \geq 0$ and $\overset{(p)}{\varkappa} \geq -1$, and relations (2.13), (2.16), we obtain the necessity of conditions (3.1) and (3.2).

Further, let $w(z) \in E_p(A,B)$ is the solution of the homogeneous problem (1.1), (1.3) at $\varkappa < 0$ or $\overset{(p)}{\varkappa} < 0$. Then the function $\Phi(z) \in E_p$ in the representation $w(z)$ by formula (4.26) is the solution of the homogeneous problem

$$\operatorname{Re}\{\overline{\lambda_1(t)}\Phi(t)\} = 0, \quad t \in \Gamma, \tag{5.1}$$

where

$$\overline{\lambda_1(t)} = e^{-i\omega(t)} \cdot \exp\left\{-T(A + B\overline{w}/w)\right\}, \tag{5.2}$$

and the indexes of the problems (1.3) and (5.1) coincide. Thus, owing to [1], [2], $\Phi(z) \equiv 0$ and the assertions of theorems 3.1 and 3.2 about homogeneous problem with negative index are proved.

We pass to the analysis of the non-homogeneous problem at $\varkappa < 0$, or $\overset{(p)}{\varkappa} < 0$ — even. We make the replacement of the required function $w_0(z) = z^n w(z)$, where $n = -\varkappa$ or $n = -\dfrac{\overset{(p)}{\varkappa}}{2}$. Then the function $w_0(z)$ satisfies the equation

$$\partial_{\bar z} w_0 + A(z) w_0 + B_0(z)\overline{w_0} = 0, \tag{5.3}$$

where $B_0(z) = B(z)\dfrac{z^n}{\overline{z}^n}$, and boundary condition

$$\operatorname{Re}\left\{\overline{\lambda_0(t)} w_0(t)\right\} = g(t), \quad t \in \Gamma, \tag{5.4}$$

where $\lambda_0(t) = e^{i\omega(t)} \cdot (\overline{z})^{-n}$.

Given the domain G consists of the point $z = 0$, we get $\operatorname{ind}_\Gamma \lambda_0 = 0$.

Owing to already proved parts of the theorems 3.1 and 3.2, the problem (5.3), (5.4) has the solution $w_0(z) \in E_p(A, B_0)$ (even the set of solutions, depend on real parameter). Let $w_0(z) = \Phi_0(z)e^{\chi(z)}$ is the representation of the kind (4.26), $\chi(z) = \exp\{-T(A + B_0\overline{w}_0/w_0)\} = \exp\{-T(A + B\overline{w}/w)\}$, where $w(z) = w_0(z)z^{-n}$. Obviously, that such determined function $w(z)$ be (unique) solution of the problem (1.1), (1.3) in $E_p(A,B)$, necessary and sufficient that the function $\Phi_0(z)$ has a look $\Phi_0(z) = z^n\Phi(z)$, where $\Phi(z) \in E_p$ is the solution of the boundary value problem

$$\operatorname{Re}\left\{\overline{\lambda_2(t)}\Phi(t)\right\} = g(t), \quad t \in \Gamma, \ \ \lambda_2(t) = e^{i\omega(t)} \cdot e^{\overline{\chi(t)}}. \tag{5.5}$$

Because $\operatorname{ind}_\Gamma \lambda_2(t) = \operatorname{ind}_\Gamma \lambda(t) < 0$, for existing (unique) solution $\Phi(z) \in E_p$ of the problem (5.5) it is necessary and sufficient that the function $g(t)$ satisfies $2n - 1$ independent real conditions [1], [2]:

$$\int_\Gamma g(s) e^{i\omega(s) - \chi(t(s))} \Phi_k^*(t(s)) t'(s) ds = 0, \ k = 1, \ldots, 2n-1, \tag{5.6}$$

where $\{\Phi_k^*(z)\}$ is the full system of linear independent in real sense solutions of the boundary value problem, adjoint to the problem (5.5).

From here it is evident the sufficiency of the conditions (3.1) and (3.2) in the case under consideration.

If $\overset{(p)}{\varkappa} < -1$ is odd, we put $n = -\dfrac{\overset{(p)}{\varkappa} + 1}{2}$ and repeat all above reasoning with some differences. The differences are the next: in (5.4) $\mathrm{ind}_\Gamma \lambda_0(t) = -1$ and in (5.6) a number of conditions equal $-\overset{(p)}{\varkappa} - 1$.

References

[1] S.B. Klimentov, "Hilbert boundary value problem for holomorphic functions in the Smirnov classes", Studies on mathematical analysis, differential equations, and there applications. Review of Science. The South of Russia. Mathematical Forum. V. 4. SMI, Vladikavkaz, 2010, pp. 252–263 (in Russian).

[2] S.B. Klimentov, "Hilbert boundary value problem for holomorphic functions in the Smirnov classes in the domain with Radon boundary", Izvestia Vuzov. Sev.- Kav. Region. Natural Science, No. 3, pp. 14–18, 2011 (in Russian).

[3] K.M. Musaev, "Some classes of generalized analytic functions", Izv. Acad. Nauk Azerb. SSR, No. 2, pp. 40–46, 1971 (in Russian).

[4] K.M. Musaev, "On some extreme properties of generalized analytic functions", DAN SSSR, V. 203, No. 2, pp. 289–292, 1972 (in Russian).

[5] K.M. Musaev, "F. Riesz type theorems in the theory of generalized analytic functions", Special problems of the theory of functions, ELM-press, Baku, 1980, pp. 137–144 (in Russian).

[6] K.M. Musaev, On boundedness of the Cauchy singular integral in the class of generalized analytic functions", Izv. Acad. Nauk Azerb. SSR, Math. Phys. Tech. V. VII, No. 6, pp. 3–8, 1986 (in Russian).

[7] K.M. Musaev, T. Kh. Gasanova, "About annihilators of son classes of generalized analytic functions", Mem. IMM AS Azerb., V. VII(XVI), pp. 162–168, 1998 (in Russian).

[8] K.M. Musaev, T. Kh. Gasanova, "The boundary value problem in the class of generelized analytic functions – jump problem" Transactions of AS Azerbaijan, V. 5, No. 19, pp. 109–112, 1999.

[9] S.B. Klimentov, “Hardy classes of generalized analytic functions”, Izvestia Vuzov. Sev.- Kav. Region. Natural Science, No. 3, pp. 6–10, 2003 (in Russian).

[10] S.B. Klimentov, “Riemann-Hilbert boundary value problen in the Hardy classes of generalized analytic functions”, Izvestia Vuzov. Sev.- Kav. Region. Natural Science, No. 4, pp. 3–5, 2004 (in Russian).

[11] S.B. Klimentov, “Smirnov classes of generalized analytic functions”, Izvestia Vuzov. Sev.- Kav. Region. Natural Science, No. 1, pp. 13–17, 2005 (in Russian).

[12] S.B. Klimentov, “BMO classes of generalized analytic functions”, Vladikavkaz math. journal, V. 8, No. 1, pp. 27–39, 2006, (in Russian).

[13] S.B. Klimentov, “Duality theorem for the Hardy classes of generalized analytic functions”, Complex analysis. Operator theory. Mathematical simulation., Vladikavkaz scientific center of RAS, Vladikavkaz, pp. 63–73, 2006 (in Russian).

[14] S.B. Klimentov, The second type representations for the Hardy and BMO classes of generalized analytic functions”, Investigations on modern analysis and mathematical simulation, Vladikavkaz scientific center of RAS, Vladikavkaz, pp. 38–54, 2008 (in Russian).

[15] I.N. Vekua, “Generalized analytic functions”, Fizmatgiz, Moscow, 1959 (in Russian). English translation: I. N. Vekua, “Generalized Analytic Functions ”. XXIX + 668 S. Oxford/London/New York/Paris 1962. Pergamon Press.

[16] G.M. Goluzin, “Geometrical theory of functions of complex variable”, Nauka, Moscow, 1966 (in Russian). English translation: G.M. Goluzin, “Geometrical theory of functions of complex variable”, Providence, R.I. : American Mathematical Society, – Translations of mathematical monographs; v. 26, 1969.

[17] I.I. Danilyuk, “Non-regular boundary value problems in a plane”, Nauka, Moscow, 1975 (in Russian).

[18] V.N. Monakhov, “Boundary value problems with free boundaries for elliptic systems of equations”, Nauka, Novosibirsk, 1977 (in Russian). English translation: Boundary-value problems with free boundaries for elliptic systems of equations, by V. N. Monakhov, Translations of Mathematical Monographs, Vol. 57, Americal Mathematical Society, Providence, R.I., 1983.