Global and Stochastic Analysis
Vol. 1, No. 1, January-June, 2014
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Integrable Systems and Affine Quantities

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Received by the Editorial Board on April 12, 2011

Abstract

In this paper we use the description of differential invariant algebras for affine geometrical quantities ([1]) to propose a new method of integration in quadratures of ordinary differential equations. The method is applied for differential equations given by affine differential invariants.

Key words: affine differential invariants, quadratures, Tresse derivatives.

1 Introduction

In this paper we propose a new method of integration in quadratures of ordinary differential equations. We use the description of differential invariant algebras for affine geometrical quantities given in ([1]) and consider the classes of ordinary differential equations defined by affine differential invariants. We write such equations in terms of basic differential invariants and invariant differentiations (Tresse derivatives). Solutions of such equations can be viewed as ordinary differential equations of the second order. The last equations satisfy the conditions of Lie-Bianchi theorem ([3]) and therefore are integrable in quadratures.

Recall that the affine geometry of a straight line is defined by the group of affine transformations, so-called "$ax + b$" - group.
Affine transformations of a straight line: $x \mapsto ax + b$, where $a, b \in \mathbb{R}$ and $a \neq 0$, constitute the 2-dimensional solvable Lie group. The corresponding Lie algebra $\mathfrak{a}$ is generated by vector fields $X = \partial_x$, $Y = x\partial_x$:

$$\mathfrak{a} = \langle \partial_x, x\partial_x \rangle.$$

The application of affine geometry to ordinary differential equations is based on the following observations:

1) If an ordinary differential equation of the second order possesses of 2-dimensional solvable Lie algebra $\mathfrak{a}$, which is a lift of an action on a straight line, then there is an affine structure on the line such that solutions of the differential equations are represented by affine quantities.

2) If the differential operator, which defines the differential equation, is an invariant of the Lie algebra $\mathfrak{a}$ action, then the differential equation can be written in terms of affine differential invariants.

3) Affine algebra $\mathfrak{a}$ is a solvable Lie algebra of dimension 2. Therefore, due to Lie-Bianchi theorem ([3]), the given 2-nd order differential equation can be integrated in quadratures.

Let us consider the following example.

Differential equations of the form

$$y'' = y' + f(y),$$

having two dimensional symmetry algebra, and by that integrable in quadratures, can be divided on two classes (see [3], [2]):

(I) $\quad f = a(y+b)^c - \frac{2c+2}{(c+3)^2}(y+b),$
where $a, b, c \in \mathbb{R}, \quad c \neq -3$, and

(II) $\quad f = ae^{by} - \frac{2}{b}$, where $a, b \in \mathbb{R}, \quad b \neq 0$.

At first we analyze class (I).
Assume that

$$f = ay^c - \frac{2c+2}{(c+3)^2}y.$$

Then the equation

$$y'' = y' + ay^c - \frac{2c+2}{(c+3)^2}y \tag{1.1}$$

has symmetry algebra with generators ([2]):

$$A = \partial_x, \qquad B = e^{kx}\partial_x + \frac{k+1}{2}e^{kx}u\partial_u,$$

where $k = \frac{1-c}{3+c}$.

The vector fields A and B satisfy the commutation relation: $[A, B] = kB$.

Therefore, if we put

$$X = e^{kx}\partial_x, \qquad Y = -\frac{1}{k}\partial_x, \tag{1.2}$$

we get that vector fields

$$\overline{X} = B, \qquad \overline{Y} = -\frac{1}{k}A,$$

as well as vector fields X and Y, satisfy the following commutation relations:

$$[X, Y] = X, \qquad [\overline{X}, \overline{Y}] = \overline{X}.$$

Thus, the symmetry algebra $\langle A, B\rangle = \langle \overline{X}, \overline{Y}\rangle$ can be viewed as a lift of Lie algebra $\langle X, Y\rangle$.

The Lie algebra $\langle X, Y\rangle$ defines an affine structure on the line. To see this, let us introduce an affine parameter t such that: $X(t) = 1$. In other words, let $t = -\frac{1}{k}e^{-kx}$.

As a second coordinate v on the plane $\mathbb{R}^2$, we consider the first integral of vector field $\overline{X} = B$, for example $v = e^{-x}u^{\frac{1}{\alpha}}$.

In these coordinates the Lie algebra $\langle X, Y\rangle$ has the canonical form $X = \partial_t$, $Y = t\partial_t$, and Lie algebra $\langle \overline{X}, \overline{Y}\rangle$ is generated by vector fields

$$\overline{X} = \partial_t, \qquad \overline{Y} = t\partial_t + \frac{1}{k}v\partial_v.$$

Thus, solutions of differential equation (1.1) can be viewed as linear affine quantities.
It is easy to check that they are tensors of the form

$$h(t)\big(\partial_t\big)^{\otimes\frac{1}{k}},$$

if $\frac{1}{k} \in \mathbb{Z}$.

For differential equations of class (II):

$$y'' = y' + ae^{by} - \frac{2}{b} \tag{1.3}$$

the symmetry algebra is generated by vector fields ([2]):

$$A = \partial_x, \qquad B = e^{-x}\partial_x + \frac{2}{b}e^{-x}u\partial_u,$$

satisfying to the following commutation relation: $[A, B] = -B$.

Let $X = e^{-x}\partial_x$, $Y = \partial_x$. Then, as above, $A = \overline{Y}$, $B = \overline{X}$ and $[X, Y] = X$.

We take function $t = e^x$, as an affine parameter and let $v = e^{-x}u^{\frac{b}{2}}$.

In these coordinates vector fields $\overline{X}$, $\overline{Y}$ have the following form

$$\overline{X} = \partial_t, \qquad \overline{Y} = t\partial_t - v\partial_v.$$

Therefore solutions of differential equation (1.3) can be viewed as differential forms on the affine line.

2 Classification of 1-dimensional affine quantities

We define 1-dimensional affine quantities as sections of homogeneous bundles:

$$\pi : \mathbb{R}^2 \to \mathbb{R},$$

$$\pi : (u, x,) \mapsto x.$$

In other words, affine quantities are sections of 1-dimensional bundles equipped an action of the affine algebra.

Let

$$\overline{X} = \partial_x + A(x,u)\partial_u, \qquad \overline{Y} = x\partial_x + B(x,u)\partial_u$$

be lifts of vector fields X and Y correspondingly.

The following theorem ([1]) gives the complete classification of such actions:

Theorem 1. One-dimensional affine quantities can be divided on two classes which correspondent to the following representations of the affine Lie algebra:

Class 1.

$$\overline{X} = \partial_x - \frac{\varphi_x}{\varphi_u}\partial_u, \qquad \overline{Y} = x\overline{X}.$$

Class 2.

$$\overline{X} = \partial_x - \frac{\varphi_x}{\varphi_u}\partial_u, \qquad \overline{Y} = x\overline{X} + \frac{1}{\varphi_u}\partial_u,$$

where $\varphi(x,u)$ - is a such smooth function, that $\varphi_u \neq 0$.

3 Differential equations for affine quantities of class 1

In this section we consider one-dimensional affine quantities of class 1 and ordinary differential equations associated with them. To this end we need a description of affine differential invariants.

In paper ([1]) it has been shown that the algebra of affine differential invariants for such quantities has the following structure.

Theorem 2. Differential invariants for affine quantities of class 1 have two basic invariants:

- invariant of the zero order

$$I = \varphi(x,u),$$

and

- invariant of the second order

$$J = \frac{\left(\frac{d^2\varphi}{dx^2}\right)}{\left(\frac{d\varphi}{dx}\right)^2}.$$

All other differential invariants are generated by the Tresse derivatives ([4])

$$\frac{D^k J}{DI^k}.$$

Here we denoted by $\frac{d}{dx}$ the total derivation in x.

Thus, due to this theorem, any affine differential invariant of class 1 can be written in the form:

$$F\big(I, J, \frac{DJ}{DI}, \dots, \frac{D^k J}{DI^k}\big). \tag{3.1}$$

Differential invariants of form (3.1) define the ordinary differential equations

$$F\big(I, J, \frac{DJ}{DI}, \dots, \frac{D^k J}{DI^k}\big) = 0, \tag{3.2}$$

which have order $k+2$ and possess 2-dimensional symmetry Lie algebra with generators:

$$\overline{X} = \partial_x - \frac{\varphi_x}{\varphi_u}\partial_u, \quad \overline{Y} = x\overline{X}.$$

This Lie algebra is solvable. Therefore, due to Lie-Bianchi theorem, any equation of form (3.2), having second order, can be integrated in quadratures ([3], [2]).

For general differential equation (3.2) let us assume that this equation, considered as the differential equation in Tresse derivatives, can be integrated.

Then solutions of the last equation can be considered as ordinary differential equations of the second order. These equations possess 2-dimensional solvable symmetry Lie algebra and therefore can be integrated in quadratures for arbitrary function

$$J = f(I). \tag{3.3}$$

For example, assume that equation (3.2) is a linear differential equation of the 1-order with respect to Tresse derivatives:

$$\frac{DJ}{DI} + A(I)J = B(I). \tag{3.4}$$

It is worth to note that this equation considered as an ordinary differential equation has the 3-order.

Solving equation (3.4) with respect to function J, we find 1-parameter family of solutions

$$J = F(c, I), \tag{3.5}$$

which are ordinary differential equations of the 3-rd order

$$\frac{d^2\varphi}{dx^2} = F(c,\varphi)\left(\frac{d\varphi}{dx}\right)^2,$$

where c - is a constant.

The last equations have 2-dimensional solvable Lie algebra of symmetries. Therefore, these equations can be integrated in quadratures.

As another example we consider 4-th order differential equations, corresponding to the second order linear differential equations in Tresse derivatives. In the case of constant coefficients and distinct characteristic roots their solutions have the form:

$$J = K_1 e^{\lambda_1 I} + K_2 e^{\lambda_2 I}, \tag{3.6}$$

and the last second order differential equations can be integrated in quadratures.

Thus, for the case of harmonic oscillator, we have differential equations of the form

$$\frac{D^2 J}{DI^2} + \omega^2 J = 0,$$

and general solutions

$$J = K_1 \cos(\omega I) + K_2 \sin(\omega I). \tag{3.7}$$

As we have seen relations (3.6) and (3.7) can be considered as ordinary differential equations of the second order having 2-dimensional solvable Lie algebra of symmetries. Hence, they can be integrated in quadratures.

Finally, let's consider another classes of differential equations integrated in quadratures.

These differential equations have the form:

$$u_4 - 7u_1^{-1}u_2u_3 + 8u_2^2 + W(u)u_2 = A(u)u_1^2, \tag{3.8}$$

and posses the two dimensional symmetry algebra, corresponding to $\varphi = u$. In these equations A and W are arbitrary functions.

Writing these equations in terms of differential invariants we get the following differential equations of the second order in Tresse derivatives:

$$\frac{D^2 J}{DI^2} + W(I)J = A(I).$$

The last equations are integrable in quadratures if the potential functions $W(I)$ is integrable in the sense of paper ([3]).

For example, it's true when

$W(u) = Cu^{-2}$, or $W(u) = C(u^2 + pu + q)^2$,

where C, p, q are constants.

This is also true, when $W(u)$ is a solution of the stationary Korteweg and de Vries equation, or its higher generalizations (see [3]).

Differential equations of the form:

$$u_1^2 u_4 - 7u_1 u_2 u_3 + 8u_2^2 - u_1^3 u_3 + 2u_1^2 u_2^2 = a\frac{u_2^k}{u_1^{2k-6}} - \frac{2(k+1)}{(k+3)^2} u_1^4 u_2 \qquad (3.9)$$

also have the two dimensional symmetry algebra, corresponding to $\varphi = u$. Writing these equations in terms of differential invariants we get the following differential equations of the second order in Tresse derivatives:

$$\frac{D^2 J}{DI^2} = \frac{DJ}{DI} + aJ^k - \frac{2(k+1)}{(k+3)^2} J.$$

The last equation is integrable in quadratures due to ([2], [3]), and therefore, differential equation (3.9) is integrable in quadratures too.

Examples of such equations are the following:

$k = a = 2:$ $\quad u_1^2 u_4 - 7u_1 u_2 u_3 + 8u_2^3 - u_1^3 u_3 + \frac{6}{25} u_1^4 u_2 = 0,$

$k = 3, a = 8:$ $\quad u_1 u_4 - 7u_2 u_3 - u_1^2 u_3 + 2u_1 u_2^2 + \frac{2}{9} u_1^2 u_2 = 0.$

4 Differential equations for affine quantities of class 2

In this section we consider ordinary differential equations associated with affine quantities of class 2. In paper ([1]) it has been shown that the algebra of affine differential invariants for such quantities has the following structure.

Theorem 3. Algebra of differential invariants for affine quantities of class 2 has basic differential invariant of the first order

$$J = e^{\varphi} \frac{d\varphi}{dx},$$

and invariant differentiation

$$\nabla = e^{\varphi} \frac{d}{dx},$$

such that any differential invariant of order $(k+1)$ can be represented in the form

$$F\Big(J, \nabla J, \ldots, \nabla^k J\Big). \qquad (4.1)$$

Each differential invariant of the above form generates an ordinary differential equation of the $(k+1)$-order

$$F\Big(J, \nabla J, \nabla^2 J, \dots, \nabla^k J\Big) = 0. \tag{4.2}$$

This equation has two dimensional symmetry algebra with generators

$$\overline{X} = \partial_x - \frac{\varphi_x}{\varphi_u}\partial_u, \qquad \overline{Y} = x\overline{X} + \frac{1}{\varphi_u}\partial_u.$$

For the case $k = 1$ such equations

$$F\Big(J, \nabla J\Big) = 0$$

have the second order, and therefore, due to Lie-Bianchi theorem, are integrable in quadratures.

In the case $k = 2$ such equations

$$F\Big(J, \nabla J, \nabla^2 J\Big) = 0 \tag{4.3}$$

have the 3-rd order and can be integrated in quadratures if the first integral of this equation is known.

Namely, assume that $H(J, \nabla J)$ is such an integral. Then function $\nabla\big(H(J, \nabla J)\big)$ is proportional to $F\Big(J, \nabla J, \nabla^2 J\Big)$ and equation (4.3) is equivalent to the family of differential equations

$$H(J, \nabla J) = const$$

of the second order.

The last differential equations have the two dimensional symmetry algebra and therefore can be integrated in quadratures.

Thus, the harmonic oscillator equation with respect to derivation ∇:

$$\nabla^2 J + J = 0, \tag{4.4}$$

which is the ordinary differential equation of the 3-rd order has the first integral

$$H = (\nabla J)^2 + J^2. \tag{4.5}$$

Therefore, differential equation (4.4) is equivalent to 1-parametrical family of the second order differential equations:

$$H = (\nabla J)^2 + J^2 = const.$$

These equations have the two dimensional symmetry algebra and therefore they are integrable in quadratures.
In general, the integration method for differential equations of type (4.3) can be formulated as follows: introduce a formal variable s, which we call *affine parameter*, in such a way, that $\nabla = \frac{d}{ds}$. Then differential equation (4.3) can be considered as an ordinary differential equation with respect to function $J = J(s)$.

Let $J = f(s)$ be a solution of this equation. Applying operator ∇ to this relation we get the following system of equations:

$$J = f(s),$$

$$\nabla J = f'(s).$$

Eliminating parameter s, we get the following relation:

$$G(J, \nabla J) = 0.$$

The last equation is an ordinary differential equation of the second order, which has the two dimensional symmetry algebra. Therefore, this equation is integrated in quadratures.

As an example of application of this method let us consider the following case:

$$\varphi = \ln|u|, \qquad \nabla = u\frac{d}{dx}, \qquad J = u_1.$$

Then differential equations of the 3-rd order

$$u^2 u_3 + u u_1 u_2 - u u_2 = a u_1^k - \frac{2(k+1)}{(k+3)^2} u_1 \tag{4.6}$$

have the two dimensional symmetry algebra, corresponding $\varphi = \ln|u|$.

In terms of differential invariants this equation can be writing as follows:

$$\nabla^2 J = \nabla J + a J^k - \frac{2(k+1)}{(k+3)^2} J.$$

The last equation has the first integral (see [2], [3]) and, therefore, can be integrated in quadratures.

References

[1] N.G. Konovenko, "Algebras of differential invariants for geometrical quantities on affine line", Bulletin of the University of Kiev. Series: Physics and Mathematics, No 2, pp. 9-16, 2008.

[2] S.V. Dushin and V.V. Lychagin, “Symmetries of distributions and quadrature of ordinary differential equctions”, Acta Appl. Math., Vol. 24, No 1, pp. 29-57, 1991.

[3] A. Kushner, V. Lychagin and V. Roubtsov, “Contact geometry and non-linear differential equations”, Cambridge University Press, 2007.

[4] A. Tresse, “Sur les invariants differentiels des groupes continus de transformations”, Acta Math., Vol. 18, pp. 1-88, 1894.