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Probabilistic Approaches to Nonlinear Parabolic Equations in Jet-Bundles

Ya. Belopolskaya
St.Petersburg State University for Architecture and Civil Engineering
2-ja Krasnoarmejskaja 4, 190005, St.Petersburg, Russia
e-mail: yanabeus@yahoo.com

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Abstract

We develop two probabilistic approaches that allows to construct a solution of the Cauchy problem for a class of fully nonlinear second order PDEs. Within the first one we reduce an original PDE to a quasilinear PDE in the second order jet-bundle and construct a probabilistic counterpart in terms of forward stochastic equations for the resulting Cauchy problem. The second is based on a reduction to a semilinear PDE that allows to reduce the original problem to a fully coupled forward-backward BSDE.

Key words: fully nonlinear parabolic equations, Cauchy problem, fully coupled forward-backward parabolic equations, jet-bundles.

1 Introduction

A deep connection that there exists between stochastic differential equation (SDE) theory and partial differential equation (PDE) theory was well established at the first steps of the SDE theory development. This connection can be described as follows.

Assume that $g(s,x) \in R^d$ is a smooth bounded function and $u(s,x) \in R^d, s \in [0,T], x \in R^d$ is a classic of the Cauchy problem

$$u_s + \mathcal{A}u + g = 0, \quad u(T,x) = u_0(x), \tag{1.1}$$

$$\mathcal{A}u(x) = \frac{1}{2}\sum_{i,j,k=1}^{d} A_{ik}(x)\nabla_i\nabla_j u A_{jk}(x) + \sum_{j=1}^{d} a_j(x)\nabla_i u. \tag{1.2}$$

The function u is a classic solution to (1.1) if $u \in C^{1,2}$ and satisfies (1.1) for all $s \in [0,T], x \in R^d$.

We denote by C^k the set of functions $f : R^d \to R^1$ with k continuous bounded derivatives and by $C^{1,2}$ the set of functions $f : [0,T] \times R^d \to R^1$ such that $f(s,x)$ has a bounded continuous derivative in s and $f(s) \in C^2$.

If $u \in C^{1,2}$ solves (1.1) then there exists probabilistic representation of the form

$$u(s,x) = E\left[u_0(\xi_{s,x}(T)) + \int_s^T g(\tau, \xi_{s,x}(\tau))d\tau\right], \tag{1.3}$$

where $\xi_{s,x}(t)$ satisfies a stochastic differential equation

$$d\xi(t) = a(\xi(t))dt + A(\xi(t))dw(t), \quad \xi(s) = x. \tag{1.4}$$

Here $w(t) \in R^d$ is a Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$.

An inverse statement is true as well. Namely, let there exists a solution of SDE (1.4) and the function u given by (1.3) belongs to $C^{1,2}$, then $u(s,x)$ is a unique classical solution of (1.1) [1].

One can extend these statements tothe case of a quasilinear PDE [2]-[4].

Let $a(x,u), A(x,u), x, u \in R^d$ be twice differentiable in x functions with sublinear growth in x polynomial growth in u. Consider the Cauchy problem (1.1) with

$$\mathcal{A}_u f(x) = \frac{1}{2}\sum_{i,j,k=1}^d A_{ik}(x,u)\nabla_i\nabla_j f A_{jk}(x,u) + \sum_{j=1}^d a_j(x,u)\nabla_j f, \tag{1.5}$$

Then the function $u(s,x)$ admits a probabilistic representation of the form

$$u(s,x) = E\left[u_0(\xi_{s,x}(T)) + \int_s^T g(\xi_{s,x}(\tau), u(\tau, \xi_{s,x}(\tau))d\tau\right], \tag{1.6}$$

where the process $\xi_{s,x}(t)$ satisfies a stochastic differential equation

$$d\xi(t) = a(\xi(t), u(t,\xi(t)))dt + A(\xi(t), u(t,\xi(t))dw(t), \quad \xi(s) = x. \tag{1.7}$$

Note that this time the equations (1.6),(1.7) make a closed system. Provided we can prove the existence and uniqueness of a solution $(\xi(t), u(s,x)$ to this system with $u(s) \in C^2$ we can check that this function satisfies the Cauchy problem

$$u_s(s,x) + \mathcal{A}_u u(s,x) + g(s,x,u(s,x)) = 0, \quad u(T,x) = u_0(x), \tag{1.8}$$

with $\mathcal{A}_u$ given by (1.5). At the other hand any C^2-smooth bounded solution $u(s)$ of (1.8) can be represented in the form (1.6) with $\xi_{s,x}(t)$ satisfying (1.7).

Below we consider both the Cauchy problem for a semilinear parabolic equation

$$u_s(s,x) + \mathcal{A}_u^1 u(s,x) + g(s,x,u,\nabla u) = 0, \quad u(T,x) = u_0(x), \tag{1.9}$$

where for $\phi \in C^2$

$$\mathcal{A}_u^1\phi(x) = \frac{1}{2}\sum_{i,j,k=1}^{d} A_{ik}(x,u,\nabla u)\nabla_i\nabla_j\phi A_{jk}(x,u,\nabla u) + \sum_{j=1}^{d} a_j(x,u,\nabla u)\nabla_j\phi \tag{1.10}$$

and the Cauchy problem for a fully nonlinear parabolic equation

$$u_s^k + f^k(s,x,u,\nabla u,\nabla^2 u^k) = 0, \quad k = 1,\dots,d_1, \quad u(T,x) = u_0(x) \tag{1.11}$$

under some assumptions on the functions g and f.

We present here two probabilistic approaches to fully nonlinear parabolic equations. First of them based on ideas by McKean and Freidlin [1]- [3] was developed in papers [4]-[7] and the second is based on ideas of the BSDE theory developed by Pardoux and Peng [8]-[9]. To explain the underlying ideas we need a differential geometry point of view for a nonlinear PDE.

Let X, Y be a couple of Euclidian spaces (or X is a smooth finite dimensional manifold modeled in a Euclidian space) and $u(s,x) \in Y$ be a time depending vector field In the main part of this paper we assume $X = Y = R^d$. Given a linear spaces X and Y let $X \otimes Y$ denote their tensor product and $X \times Y$ denote their direct product. Below we use notations $u_s, \nabla_i u = u_{x_i}, i = 1,\dots,d$ for partial derivatives of a vector field $u \in Y$ in s and x_i respectively and note that $\nabla u \in X \otimes Y$.

Let X be a smooth manifold, $\pi : \mathcal{E} \to X$ be a vector or a fibre bundle over X and given an integer k denote by $J_k(\mathcal{E})$ the corresponding k-th jet bundle. Given integers k and r we denote by $\pi_k^{k+r} : J_{k+r}(\mathcal{E}) \to J_k(\mathcal{E})$ a mapping that maps a $(k+r)$-th jet into a corresponding k-th jet, that is $\pi_k^{k+r}(f(x)) = j_k(f)(x)$. Let ∇u be a covariant derivative in sections of the corresponding bundle. In this framework it is natural to treat the Cauchy problem for a fully nonlinear parabolic equation of the second order

$$u_s + f(x,u,\nabla u,\nabla^2 u) = 0, u(T,x) = u_0(x) \tag{1.12}$$

as the Cauchy problem for an equation on the jet bundle $J_2(\mathcal{E})$ and to construct its differential prolongations of different orders. In particular we will need a differential prolongation of (1.12) to an equation on the fibre bundle $\pi_3^5 : J_2(J_3(\mathcal{E})) \to J_3(\mathcal{E})$ or to $\pi_1^3 : J_2(J_1(\mathcal{E})) \to J_1(\mathcal{E})$. Here $J_0(\mathcal{E}) = \mathcal{E}$. It will allow us to deal with a system consisting of (1.12) and its corresponding differential prolongations as a quasilinear equation in sections of π_3^5 or as a semilinear equation in sections of π_1^3 respectively. From probabilistic point of view this gives the possibility to apply the technique developed in [4]- [7]. As a result this allows to state conditions that ensure the existence and uniqueness of a (local in time) classical solution to (1.12).

There exists another probabilistic approach to the Cauchy problem for a semilinear PDE based on the theory of backward backward stochastic differential equations (BSDE) developed in [8]– [9]. To a fully nonlinear second order parabolic equations this approach was extended in [10]. Here we consider (1.12) as an equation on the fibre bundle $\pi_1^3 : J_2(J_1(\mathcal{E})) \to J_1(\mathcal{E})$ or in other words as a semilinear equation in sections of π_1^3. This allows us to derive a new probabilistic approach to the Cauchy problem for (1.12) based on the so called fully coupled forward-backward SDEs (FBSDEs). The theory of FBSDEs was developed by Peng, Hu, Wu and others in papers [11] - [14]. AS a result we construct a viscosity solution to (1.12).

Let us recall a notion of a viscosity solution of the Cauchy problem for a nonlinear parabolic equation [15].

Given $x, y \in R^d$ let $\langle x, y\rangle = \sum_{k=1}^d x_k y_k$ and $M_+^d \in R^d \otimes R^d$ denote the set of positive definite matrices. Given a bounded $G \subset R^d$ let u be a mapping from $G_T = [0,T] \times G$ to R^1.

The second order superjet of a function u at point $(\hat{s}, \hat{x}) \in [0,T]\times G$ relative to G_T is

$$\mathcal{P}_G^{2,+}u(\hat{s},\hat{x}) =$$
$$\{(a,p,q) \in R^1 \times R^d \times M_+^d : \ u(s,x) \le u(\hat{s},\hat{x}) + a(s-\hat{s}) + \langle p, x-\hat{x}\rangle +$$
$$\frac{1}{2}\langle q(x-\hat{x}), x-\hat{x}\rangle + o(|s-\hat{s}| + \|x-\hat{x}\|^2), G_T \ni (s,x) \to (\hat{s},\hat{x}) \in G_T\},$$

The second order subjet of a function u at point $(\hat{t}, \hat{x}) \in G_T$, relative to G_T is

$$\mathcal{P}_G^{2,-}u(\hat{s},\hat{x}) =$$
$$\{(a,p,q) \in R^1 \times R^d \times M_+^d : u(s,x) \ge u(\hat{s},\hat{x}) + a(s-\hat{s}) + \langle p, x-\hat{x}\rangle + \quad (1.13)$$
$$\frac{1}{2}\langle q(x-\hat{x}), x-\hat{x}\rangle + o(|s-\hat{s}| + \|x-\hat{x}\|^2), G_T \ni (s,x) \to (\hat{s},\hat{x}) \in G_T\}.$$

Let $USC(G) = \{$the set of upper semicontinuous functions $u : G \to R^1\}$ and $LSC(G) = \{$the set of lower semicontinuous functions $u : G \to R^1\}$.

A function $u(s,x)$ is called a subsolution of (1.11) when $u \in USC(G_T)$ and

$$a + F(x, u(s,x), p, q) \le 0, \quad \text{for } (s,x) \in G_T, \quad (a,p,q) \in \mathcal{P}_G^{2,+}u, \qquad (1.14)$$

and a supersolution of (1.11), when $u \in LSC(G_T)$ and

$$a + f(x, u(s,x), p, q) \ge 0, \quad \text{for } (s,x) \in G_T, \quad (a,p,q) \in \mathcal{P}_G^{2,-}u. \qquad (1.15)$$

Though we need a background based on differential geometry treatment of a fully nonlinear PDE, throughout the whole paper but section 2 we consider the problem only in a framework of linear spaces.

Nevertheless most of the results should be think of as local results which one needs as the first step for the investigation of nonlinear parabolic equations on manifolds and bundles.

Let us just mention one more probabilistic approach based on the theory of stochastic flows due to Kunita [16], [17] which allows to construct weak (generalized or distributional) solutions of parabolic equations. This approach was extended to a quasilinear case in [18].

At the end of the introduction let us note some advantages of the approach to nonlinear PDEs based on SDEs. One of them is the absence of nondegeneracy assumptions both for the case of scalar nonlinear equations and systems of such equations. In addition under some restrictions this approach can be used to give a probabilistic background for the vanishing viscosity technique used to construct solutions of hyperbolic systems [19]. Another advantage of a probabilistic approach is its weak dependence on a dimension of a phase space X and a model space of a fibre manifold $\mathcal{E}$ that allows to extend some results to an infinite dimensional case [4],[7]. A drawback of this approach is rather high smoothness assumptions on the Cauchy problem data. The approach based on FBSDEs needs less smooth data but allows to deal with a more narrow class of systems of parabolic equations.

The remaining part of the paper is organized as follows. In section 2 we expose the differential geometry point of view for nonlinear second order PDEs and the notion of differential prolongation based mainly on [20]. In section 3 we reduce a quasilinear PDE to a stochastic problem and state conditions to ensure that its solution gives rise to a classical solution of the original PDE. Next we apply the idea of differential prolongation to reduce both semilinear and fully nonlinear PDEs to quasilinear PDEs and then apply the above probabilistic approach to the resulting Cauchy problem. In section 4 we consider an alternative approach to constructing a solution of the Cauchy problem for a semilinear parabolic equation based on the FBPDE theory and pay special attention to fully coupled FBSDEs. We check that the solution of a fully coupled FBSDE gives rise to a viscosity solution of the Cauchy problem for the original semilinear PDE. Finally in section 5 we apply the idea of a differential prolongation to a fully nonlinear system of PDEs to reduce it to a semilinear one and apply the fully coupled FBSDE technique to construct a viscosity solution of the original problem.

2 Prolongation of differential equations

To explain the main ideas which allows us to reduce a semilinear or a fully nonlinear equation to a quasilinear one in section 3 or to reduce a fully nonlinear equation to a semilinear one in section 4 we need some differential geometry constructions.

From differential geometry point of view a partial differential equation (PDE) is roughly a collection of relations between the dependent variables and their derivatives with respect to independent variables.

From this point of view it is natural to think about a PDE as a submanifold of a jet bundle. A prolongation of a PDE is a new PDE obtained by differentiation of the original one. Below we use a prolongation to reduce a fully nonlinear parabolic equation to a semilinear or even to a quasilinear one.

To explain how to treat a prolongation of an SDE it is natural to use the language of jet bundles. To this end we need some notions and notations which can be found in the book by Pommaret [20].

Assume first that all objects belong to C^k with k large enough or to C^∞. Let $\pi : \mathcal{E} \to X$ be a vector bundle. We can think about X as the set of independent variables and about the manifold $\mathcal{E}$ as the set of both independent and dependent variables (functions). Denote by $\Gamma^\infty(\pi)$ the set of smooth sections of $\pi : \mathcal{E} \to X$.

Let (U_α, Φ_α) be an atlas on $\mathcal{E}$ and (U_α, ϕ_α) be the atlas on X. We denote by x^i_α the local coordinates in U_α and by (x^i_α, y^k_α) local coordinates in U_α on $\mathcal{E}$. A (local) section of π is a continuous map $s : U \to \mathcal{E}$ such that $\pi s(x) = x$ and (U, ϕ) is a chart from the atlas.

Given an integer k and a couple f, g of local sections of a fibre bundle $\pi : \mathcal{E} \to X$ we say that f and g are q-equivalent at a point $x \in X$ if $f^j(x) = g^j(x)$ and $\partial_\mu g^j(x) = \partial_\mu f^j(x)$ for $|\mu| \leq k$ where μ is a multiindex. An equivalence class of section f is called a k-jet of this section and is denoted as $j_k(f)(x)$. In fact one can think about $j_k(f)(x)$ as a piece of Taylor series of the section f up to the order k terms inclusively.

We denote by $J_k(\mathcal{E})$ the bundle of $k{-}jets$ of sections of π and let $J_0(\mathcal{E}) = \mathcal{E}$. An element of $J_k(\mathcal{E})$ we typically denote by $j_k\pi(x)$. We let $\pi_k : J_k(\mathcal{E}) \to X$ and $\pi^k_l : J_k(\mathcal{E}) \to J_l(\mathcal{E}), l \leq k$ be the canonical projections.

A partial differential equation of order k is a fibred submanifold $\mathcal{R}_k \subset J_k(\mathcal{E})$ of $\pi_k : J_k(\mathcal{E}) \to X$. We denote by $\hat{\pi} : \mathcal{R}_k \to X$ the restriction of π_k to $\mathcal{R}_k$. We also write $\mathcal{R}_{k,x} = \mathcal{R}_k \cap \pi_k^{-1}(x)$.

A local defining equation for $\mathcal{R}_k$ is a quintiple (U, Z, τ, Φ, η) such that U is a neighborhood of x, $\tau : Z \to U$ is a fibred manifold, $\Phi : \pi^{-1}(U) \to Z$ is a fibred manifold morphism of constant rank, η of $\tau : Z \to U$ is a smooth section and $\pi_k^{-1}(U) \cap \mathcal{R}_k = \ker_\eta \Phi = \{v_k \in \pi_k^{-1}(U) | \Phi(v_k) = \eta(\pi_k(v_k))\}$. A local solution of $\mathcal{R}_k$ is a local section (u, U) with the property that $j_k u(x) \in \mathcal{R}_k$ for every $x \in U$.

As an example let us consider the case $X = R^2$, $\mathcal{E} = R^2 \times R^1$ with $\pi((x, y), u) = (x, y)$ and denote coordinate of $J_2(\mathcal{E})$ by

$$(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}).$$

In this case the bundle of 2-jets is equal to the vector bundle $J_2(\mathcal{E}) = R^2 \times R^1 \times R^2 \times R^2 \otimes R^2$.

We define a second order partial differential equation by

$$\mathcal{R}_{Lap} = \{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \in J_2\pi | u_{xx} + u_{yy} = 0\}.$$

We note that this is indeed a fibred manifold. The subscript Lap stands for yhe Laplace equation and $\mathcal{R}_{Lap}$ is the geometric form for the Laplace equation.

Of course solutions are functions $(x, y) \mapsto u(x, y)$ satisfying

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

To obtain a local defining equation for the Laplace equation we set $U = R^2$, $Z_{Lap} = R^2 \times R$ with $\tau((x, y), z) = (x, y)$,

$$\Phi_{Lap}((x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = ((x, y), u_{xx} + u_{yy})$$

and $\eta_{Lap}(x, y) = ((x, y), 0)$.

A homogeneous (inhomogeneous) linear partial differential equation of order k is a vector (affine) subbundle $\mathcal{R}_k \subset J_k(\mathcal{E})$ of $\pi_k : J_k(\mathcal{E}) \to X$.

A quasilinear partial differential equation of order k is a partial differential equation $\mathcal{R}_k \subset J_k(\mathcal{E})$ such that $(\pi^k_{k-1})^{-1}(\pi^k_{k-1}(v_k) \cap \mathcal{R}_k$ is an affine subspace of $(\pi^k_{k-1})^{-1}(\pi^k_{k-1}(v_k) \cap \mathcal{R}_k$ for each $v_k \in \mathcal{R}_k$.

Let us describe the above objects in local coordinates. Let $\pi : \mathcal{U} \to X$ be a fibred manifold. We denote local coordinates for X by $x \in R^d$ and fibred coordinates for $\mathcal{E}$ as $(x, u) \in R^d \times R^d$. Thus the local representation of π is $(x, v) \to x$ and coordinates for $J_k\pi$ are

$$(x, u, v_1, \dots, v_k) \in R^d \times R^d \times L^1_{sym}(R^d) \times \cdots \times L^k_{sym}(R^d).$$

Let $\mathcal{R}_k$ be a k-th order PDE. Its solution is locally presented by a map $x \mapsto (x, u(x))$ with the property that the map $x \mapsto (x, u(x), Du(x), \dots D^k u(x))$ takes its value in $\mathcal{R}_k$.

Let (U, Z, τ, Φ, η) be a local defining equation and suppose that U is a coordinate chart for X with coordinates $(x, u) \in R^d \times R^d$ and $(x, z) \in R^d \times R^d$ for Z. Then Φ has the local representation

$$(x, u, v_1, \dots, v_k) \mapsto (x, \Phi(x, u, v_1, \dots, v_k))$$

defining the map Φ. If the local representation of η is $x \to (x, \eta(x))$, then $\mathcal{R}_k \cap \pi_k^{-1}(\mathcal{U})$ is given by $\{(x, u, v_1, \dots v_k) | \Phi(x, u, v_1, \dots, v_k) = \eta(x)\}$.

Prolongation of a PDE is the formal differentiation of the PDE arriving at a PDE of higher order. To be more precise let $\pi : \mathcal{E} \to X$ be a fibred manifold and let $\mathcal{R}_k \subset J_k(\mathcal{E})$ be a PDE of order k. For an integer m the $m-$th prolongation of $\mathcal{R}_k$ is the subset $\rho_m(\mathcal{R}_k) = J_m(\mathcal{R}_k) \cap J_{k+m}(\mathcal{E})$ of $J_{k+m}(\mathcal{E})$, where the natural inclusion of $J_{k+m}(\mathcal{E})$ in $J_m(\mathcal{E})$ is used.

As an example we consider $\mathcal{R}_{Lap}$ defined by $X = R^2, \mathcal{E} = R^2 \times R$ and

$$\mathcal{R}_{Lap} = \{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) | u_{xx} + u_{yy} = 0\}.$$

Let us use coordinates $(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$ for $\mathcal{R}_{Lap}$ and note that since $\mathcal{R}_{Lap} \subset J_2(\mathcal{E})$, then

$$(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \mapsto (x, y, u, u_x, u_y, u_{xx}, u_{xy}, -u_{xx}). \tag{2.1}$$

One can define prolongations project to subsets of prolongations. Given $\mathcal{R}_k \subset J_k(\mathcal{E})$ we have $\pi^{k+l}_{k+j}(\rho_l(\mathcal{R}_k)) \subset \rho_j(\mathcal{R}_k)$ for integer k, l, j with $j \leq l$. In addition $\hat{\pi}^{k+l}_{k+j} : \rho_l(\mathcal{R}_k) \to \rho_j(\mathcal{R}_k)$, $j \leq l$ and $\hat{\pi}^{k+l}(\mathcal{R}_k) \to X$ are the canonical projections.

Let us use coordinates $(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}$ for $\mathcal{R}_{Lap}$ and note that the inclusion of $\mathcal{R}_{Lap}$ into $J_2(\mathcal{E})$ is then given by

$$(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) \mapsto (x, y, u, u_x, u_y, u_{xx}, u_{xy}, -u_{xx}) \tag{2.2}$$

Let us denote by $\rho_1(\mathcal{R}_{Lap})$ the first prolongation of $\mathcal{R}_{Lap}$. To determine this we note that coordinates for $J_1(\mathcal{R}_2)$ are denoted by

$$(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{,x}, u_{,y}, u_{x,x}, u_{x,y}, u_{y,x}, u_{y,y} u_{xx,x}, u_{xx,y},$$

$$u_{xy,x}, u_{xy,y}).$$

Here the indices to the right of the commas mean partial differentiation of the fibre variables of $J_1(\mathcal{R}_2)$. If we think of $J_1(\mathcal{R}_2)$ as a subset of $J_1(J_2(\mathcal{E}))$ using the inclusion (2.2), this subset is given by

$$\{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{,x}, u_{,y}, u_{x,x}, u_{x,y}, u_{y,x}, u_{y,y}, \tag{2.3}$$

$$u_{xx,x}, u_{xx,y}, u_{xy,x}, u_{xy,y}, u_{yy,x}, u_{yy,y}) | u_{yy} = -u_{xx}, u_{yy,x} =$$

$$-u_{xx,x}, u_{yy,y} = -u_{xx,y}\}$$

Now, the inclusion of $J_3(\mathcal{E})$ in $J_1(J_2(\mathcal{E}))$ is given by

$$((x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxx}, u_{xxy}, u_{xyy}, u_{yyy}) \to ((x, y, u, u_x, u_y,$$

$$u_{xx}, u_{xy}, u_{yy}, u_x, u_y, u_{xx}, u_{yx}, u_{yy}, u_{xxx}, u_{xxy}, u_{xxy}, u_{xyy}, u_{xyy}, u_{yyy})).$$

Thus $J_3(\mathcal{E})$ is the subset of $J_1(J_2(\mathcal{E}))$ given by

$$\{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{,x}, u_{,y}, u_{x,x}, u_{x,y}, u_{y,x}, u_{y,y}, \tag{2.4}$$

$$u_{xx,x}, u_{xx,y}, u_{xy,x}, u_{xy,y}, u_{yy,x}, u_{yy,y})|u_{,x} = u_x, u_{,y} = u_y,$$

$$u_{x,x} = u_{xx}, u_{x,y} = u_{xy} = u_{y,x}, u_{y,y} = -u_{yy}, u_{xy,x} = u_{xx,y}, u_{xy,y} = -u_{yy,x}\}.$$

As a result we receive that $J_1(\mathcal{R}_2) \cap J_3(\mathcal{E})$ is the subset of $J_1(J_2(\mathcal{E}))$ derived by combining (2.3) and (2.4)

$$\rho_1(\mathcal{R}_{Lap}) = \{(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{,x}, u_{,y}, u_{x,x}, u_{x,y}, u_{y,x}, u_{y,y},$$

$$u_{xx,x}, u_{xx,y}, u_{xy,x}, u_{xy,y}, u_{yy,x}, u_{yy,y})|u_{,x} = u_x, u_{,y} = u_y,$$

$$u_{x,x} = u_{xx}, u_{x,y} = u_{xy} = u_{y,x}, u_{y,y} = -u_{yy}, u_{xy,x} = u_{xx,y}, u_{xy,y} = -u_{yy,x}\}.$$

This is the first prolongation of $\mathcal{R}_{Lap}$ thought of as a subset of $J_1(J_2(\mathcal{E}))$. At the other hand we can think of this as a subset of $J_3(\mathcal{E})$ via (2.4)

$$\rho_1(\mathcal{R}_{Lap}) = \{x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, u_{xxx}, u_{xxy}, u_{xyx}, u_{xyy}, u_{yyy})|$$

$$u_{xx} + u_{yy} = 0, u_{xxx} + u_{yyx} = 0, u_{xxy} + u_{yyy} = 0\}.$$

3 Probabilistic approach to quasilinear and semilinear parabolic equations

Let d be an integer, $a(x) \in R^d, A(x) \in R^d \times R^d$, $x \in R^d$, $x \cdot y \equiv \langle x, y \rangle$ be the inner product in R^d, $\|x\|$ be the norm in R^d and $|A| = (TrA^*A)^{\frac{1}{2}}$ be the matrix norm. Let $C^k(R^d, R^d)$ denote the set of continuous bounded functions $f : R^d \to R^d$ with continuous bounded derivatives up to the order k, we say in this time that u is k times differentiable. Let $C^{1,k}([0, T] \times R^d, R^d)$ denote the set of continuous bounded functions $u : [0, T] \times R^d R^d$ such that $u(s, x)$ is differentiable in t and k times differentiable in x. We use below the short notations $f \in C^k$, $u \in C^{1,k}$ and denote by $\|u\|_k = \sup_x \|u^{(k)}(x)\|$ the sup-norm in C^k.

Given a probability space $(\Omega, \mathcal{F}, P)$ and a Wiener process $w(t) \in R^d$ we denote by $\mathcal{F}_t$ a stochastic flow adapted to $w(t)$ and consider a stochastic differential equation

$$d\xi(t) = a(\xi(t))dt + A(\xi(t))dw(t), \quad \xi(s) = x. \tag{3.1}$$

We state conditions on coefficients we will need in the sequel.

Condition **C 3.1**. Functions $a(x) \in R^d, A(x) \in M^d$ are nonrandom and there are exist positive constants K, L such that

$$\|a(x)\|^2 + |A(x)|^2 \leq K[1 + \|x\|^2],$$

$$\|a(x) - a(y)\|^2 + |A(x) - A(y)|^2 \le L\|x - y\|^2, \quad x, y \in R^d.$$

A classical existence and uniqueness theorem reads.

Theorem 3.1. *Let* **C 2.1** *hold. Then there exists a unique solution $\xi_{s,x}(t) \in R^d$ of (2.3) possessing the Markov property.*

Next we state conditions to ensure that the solution $\xi_{s,x}(t)$ of (3.1) depends on initial data smoothly.

Condition **C 3.2**. Assume that in addition to **C 3.1** $a(x)$ and $A(x)$ are twice differentiable.

Theorem 3.2. *Let* **C 3.2** *hold, then there exist processes*

$$\eta(t) = \nabla\xi_{s,x}(t) \in R^d \times R^d$$

and

$$\gamma(t) = \nabla^2\xi_{s,x}(t) \in R^d \times R^d \times R^d$$

that satisfy the stochastic equations

$$d\eta(t) = \nabla a(\xi(t))\eta(t)dt + \nabla A(\xi(t))(\eta(t), dw(t)), \eta(s) = I,$$

$$d\gamma(t) = \nabla a(\xi(t))\gamma(t)dt + \nabla A(\xi(t))(\gamma(t), dw(t))+$$

$$\nabla^2 a(\xi(t))(\eta(t), \eta(t))dt + \nabla^2 A(\xi(t))(\eta(t), \eta(t), dw(t)), \gamma(s) = 0.$$

In addition there exist positive constants C_1, C_2 such that

$$E\|\eta(t)h\|^2 \le C\|h\|^2, \quad E\|\gamma(t)(h,h)\|^2 \le C_2\|h\|^2 \text{ for } h \in R^d.$$

As a result one can prove the following assertion [4].

Theorem 3.3. *Let* **C 2.2** *hold and $u_0(x) \in R^1$ be bounded and twice differentiable. Then the function*

$$u(s,x) = E[u_0(\xi_{s,x}(T))] \tag{3.2}$$

is a unique classical solution of the Cauchy problem

$$u_s + \mathcal{A}u = 0, \quad u(T,x) = u_0(x), \tag{3.3}$$

where

$$\mathcal{A}u = a(x)\cdot\nabla u + \frac{1}{2}TrA^*(x)\nabla^2 uA(x). \tag{3.4}$$

Note that condition **C 3.1** allows to construct the function $u(s,x)$ via (3.2) but we need **C 3.2** to prove that $u \in C^{1,2}$ and solves the Cauchy problem (3.3).

To get a flavor of that how these results can be extended to a quasilinear case we consider the Cauchy problem for a scalar nonlinear parabolic equation

$$\frac{\partial f}{\partial t} + \langle a(x,f), \nabla\rangle f + \frac{1}{2}TrA^*(x,f)f''A(x,f) = 0, \quad f(0,x) = f_0(x). \tag{3.5}$$

Here $a(x,f) \in R^d$ and $A(x,f) \in L(R^d), x \in R^d, f \in R^1, .$

In the sequel we use C, K, L to denote absolute constants (if a constant depends on a parameter f we denote it by C_f, K_f).

Condition **C 3.3.**

The functions $a(x,f) \in R^d$, $A(x,f) \in R^d \times R^d$, $x \in R^d, f \in R^1$ satisfy the estimates

$$\|a(x,f) - a(x_1,f_1)\|^2 + |A(x,f) - A(x_1,f_1)| \leq L\|x - x_1\|^2 + L_1|f - f_1|^2,$$

$$\|a(x,f)\|^2 + |A(x,f)|^2 \leq K[1 + \|x\|^2 + |f|^{2m}],$$

where L_1 depend on $\max(|f|, |f_1|)$, $f_0(x) \in R^1$, is bounded and differentiable and m is an integer. $f, f_1 \in R^d$, $|A| = \sum_{k=1}^d \|Ae_k\|^2$, $\{e_k\}_{k=1}^d$ is an orthonormal basis in R^d, and

$$sup_x\|f_0(x)\| \leq K_0, \quad sup_x\|\nabla f_0(x)\| \leq K_0^1.$$

To construct the solutions to (3.5) we reduce it to the stochastic system

$$d\xi = a(\xi(\tau), f(\tau, \xi(\tau)))d\tau + A(\xi(\tau), f(\tau, \xi(\tau)))dw(\tau), \quad \xi(s) = x, \qquad (3.6)$$

$$f(s,x) = E_{s,x}f_0(\xi(t)). \qquad (3.7)$$

Here $E_{s,x}f_0(\xi(t)) = E[f_0(\xi(t))|\xi(s) = x]$. We construct the solution to (3.6), (3.7) by the successive approximation method. Consider the stochastic equations

$$d\xi^k(\tau) = -a(\xi^k(\tau), f^k(\tau, \xi^k(\tau)))d\tau + A(\xi^k(\tau), f^k(\tau, \xi^k(\tau)))dw(\tau), \qquad (3.8)$$

$$\xi^k(0) = x$$

and the functions

$$f^0(t,x) = f_0(x), \quad f^{k+1}(t,x) = Ef_0(\xi^k(t)). \qquad (3.9)$$

To prove the convergence of (3.8), (3.9) to a limit $\xi(t), f(t,x)$ as $k \to \infty$ we need a number of auxiliary estimates.

Let $\mathcal{L}$ be the subspace of the space $C(R^1 \times R^d, R^1)$ of continuous bounded functions consisting of Lipschitz continuous functions f equipped with the uniform norm $\|f\|_{\mathcal{L}} = sup_{x \in R^d}|f(x)|$ for $f \in \mathcal{L}$.

Denote by $L_f(t)$ and $K_f(t)$ minimal constants such that inequalities

$$|f(t,x) - f(t,y)| \leq L_f(t)\|x - y\|, \quad \|f(t)\|_{\mathcal{L}} \leq K_f(t)$$

hold.

Let $v(s,x)$ be a scalar function such that $\|v(s,\cdot)\|_{\mathcal{L}} \le K_v(s) < \infty$, $|v(s,x) - v(s,y)| \le L_v(s)\|x-y\|$, where $L_v(s) < \infty$ for $s \in [0,T]$.

Consider the stochastic equation

$$\xi(t) = x - \int_s^t a(\xi(\tau), v(\tau, \xi(\tau)))d\tau + \int_s^t A(\xi(\tau), v(\tau, \xi(\tau))))dw(\tau). \quad (3.10)$$

We use the notation $\xi_{s,x,v}(t)$ for the solution of this equation if we are interested in the particular dependence of the process $\xi(t)$ on these parameters and fix some constant T such that $0 \le s \le t < T$.

Lemma 3.4. *Assume that* **C 3.3** *holds. Then the solution $\xi_{x,v}(t)$ of (3.10) satisfies the following estimates*

$$E\|\xi(t)\|^2 \le 3[\|x\|^2 + (T+1)\int_s^t [C_0 + C_1 K_v^{2m}(\tau)d\tau],$$

$$E\|\xi_{x,v}(t) - \xi_{y,v}(t)\|^2 \le 3\|x-y\|^2 e^{3Q\int_s^t L_v^2(\tau)d\tau}, \quad (3.11)$$

$$E\|\xi_{x,v}(t) - \xi_{x,v_1}(t)\|^2 \le$$

$$2Q\int_s^t \|v(\tau) - v_1(\tau)\|^2_{\mathcal{L}} d\tau e^{2Q\int_s^t L_v^2(\tau)]d\tau} \quad (3.12)$$

where $Q = (T+1)L_{(v,v)}$. In addition, for $f(s,x) = Ef_0(\xi_{s,x,v}(t))$ the estimates

$$\|f(t-s)\|_{\mathcal{L}} \le K_0$$

and

$$|f(s,x) - f(s,y)|^2 \le 3[K_0^1]^2\|x-y\|^2 e^{3Q\int_s^t L_v^2(\tau)d\tau} \quad (3.13)$$

hold.

Proof. The proof of these estimates is standard and based on the properties of stochastic integrals (see [5]). We show only the proof of (3.11). By stochastic integral properties and coefficient estimates in **C 3.1** we have

$$E\|\xi_x(t) - \xi_y(t)\|^2 \le 3\|x-y\|^2 +$$

$$3(T+1)\int_s^t L_{(v,v)}\|v(\tau, \xi_x(\tau)) - v(t-\tau, \xi_y(\tau)\|^2 d\tau \le$$

$$3Q\int_s^t L_v(\tau)\|\xi_x(\tau) - \xi_y(\tau)\|^2 d\tau.$$

Finally, by Gronwall's lemma we get

$$E\|\xi_x(t) - \xi_y(t)\|^2 \le 3\|x-y\|^2 e^{3Q\int_s^t L_v^2(t-\tau)d\tau}.$$

Lemma 3.5 *Let* **C 3.3** *hold. Then there exists an interval* $\Delta_1 = [T_1, T]$ *and functions* $\alpha(s)$, $\beta(s)$ *bounded for* $s \in \Delta_1$. *In addition, for all* $s \in \Delta_1$ *if* $\|v(s)\|_{\mathcal{L}} \leq \alpha(s)$ *and* $|v(s,x) - v(s,y)| \leq \beta(s)\|x - y\|$, *then*

$$\|f(s)\|_{\mathcal{L}} \leq \alpha(s), \quad |f(s,x) - f(s,y)| \leq \beta(s)\|x - y\|. \tag{3.14}$$

Proof. Under **C 3.3** we can choose $\alpha(s) = K_0$ to get $\|f(s)\|_{\mathcal{L}} \leq \alpha(s)$. To prove (3.14) we notice that the estimate

$$L_f(s) \leq 3[K_0^1]^2 e^{3Q\int_s^T L_v(\tau)d\tau} \tag{3.15}$$

results from (3.11).

We choose for β the solution to the equation

$$\beta(s) = 3[K_0^1]^2 e^{3Q\int_s^T \beta(\tau)d\tau}. \tag{3.16}$$

and notice that β solves the following Cauchy problem

$$\frac{d\beta(s)}{ds} = -3Q\beta^2(s), \quad \beta(T) = 3[K_0^1]^2$$

and admits the explicit representation

$$\beta(s) = \frac{3[K_0^1]^2}{1 - 9Q[K_0^1]^2(T - s)}. \tag{3.17}$$

We see thus that $\beta(s)$ is bounded on the interval $\Delta_1 = [T_1, T]$ with

$$|T - T_1| < \frac{1}{9Q[K_0^1]^2} \tag{3.18}$$

and meets the demands of the lemma.

Coming back to the successive approximation system (3.6) we can prove the following statement.

Theorem 3.7 *Assume that* **C 3.1** *holds. Then there exists an interval* $[T_1, T]$ *such that for all* $s \in [T_1, T]$ *there exists a unique solution to* $\xi(t), f(s,x)$ *the the system (3.3),(3.4). The process* $\xi(t)$ *is a Markov process in* R^d, *while* $f(s,x)$ *is a bounded and Lipschitz continuous scalar function.*

Proof. By lemma 3.5. we know that the mapping

$$\Phi(s,x,v) = Ef_0(\xi_{s,x,v}(t))$$

acts in the space $\mathcal{L}$. Let

$$r^k(s,x) = |f^{k+1}(s,x) - f^k(s,x)|^2 \text{ and } \zeta^k(s) = sup_x r^k(s,x).$$

By the estimates of lemma 3.4 we have

$$r^k(s,x) \leq$$

$$2[K_0^1]^2(T+1)\int_s^T L_{(f^{k+1},f^k)}\|f^k(\tau)-f^{k-1}(\tau)\|_{\mathcal{L}}^2 d\tau e^{Q_1\int_s^T \beta(\tau)d\tau}$$

and hence

$$\zeta^k(s) \leq \delta^k \int_s^T \dots \int_{\tau_2}^T \|f^1(\tau_1)-f^0\|^2 d\tau_1 \dots d\tau_k$$

holds with $\delta = 2[K_0^1]^2(T+1)e^{Q_1\int_s^t \beta(\tau)d\tau}$ and $Q_1 = 2L_{(f_k,f_{k-1})}(T+1)$. Notice that Q_1 depends on sup-norm of functions $f_k(t), f_{k-1}(t)$ that are bounded for $t \in \Delta_1$ due to estimates of Lemma 3.2

Since f^k are uniformly bounded by K_0 and

$$\|f^1(s,\cdot)-f^0(\cdot)\|_{\mathcal{L}} \leq const < \infty,$$

we get

$$\|f^k(s,\cdot)-f^{k-1}(s,\cdot)\|_{\mathcal{L}} \leq \frac{N^k}{k!}const$$

where $N = T\delta$ and $T \geq T_1$ is fixed. Hence we obtain that for each $s \in (T_1, T]$ the family $f^k(s,\cdot)$ uniformly converges to a limiting function $f(s,\cdot)$ for all $s \in [T_1, T]$ for T_1 satisfying (3.18) In addition, it is easy to check that $f(s,x)$ is Lipschitz continuous in x. In fact by lemma 3.5 for each $s \in [T_1, T]$ we have

$$|f^k(s,x)-f^k(s,y)| \leq \beta(s)\|x-y\|$$

where $\beta(s)$ is given by (3.17) and the estimate is uniform in k.

To prove that the above constructed solution is unique we assume on the contrary that there exist two solutions $f_1(s,x)$, $f_2(s,x)$ to (3.6), (3.7) possessing the same initial data $f_1(0,x) = f_2(0,x) = f_0(x)$. It results from lemma 3.4 that there exists a constant C such that

$$\|f_1(s,\cdot)-f_2(s,\cdot)\|_{\mathcal{L}} \leq C\int_s^T \|f_1(\tau,\cdot)-f_2(\tau,\cdot)\|_{\mathcal{L}} d\tau$$

and hence by the Gronwall lemma $\|f_1(s,\cdot)-f_2(s,\cdot)\|_{\mathcal{L}} = 0$.

Finally, we know that a stochastic equation with Lipschitz coefficients has a unique solution of the Cauchy problem. This yields the uniqueness of the solution to the system (3.6),(3.7).

The Markov property of the process $\xi(t)$ can be deduced in the standard way since as soon as the Lipschitz continuous function $f(s,x)$ is constructed we found ourselves in the framework of the classical theory of SDEs.

When the function $f(s,x)$ given by (3.7) possesses two continuous bounded derivatives one can easily check using the Ito formula that $f(s,x)$ is a unique solution to (3.5).

The above results can be easily extended to the case

$$u_s + a(x,u)\cdot\nabla u + \frac{1}{2}TrA^*(x,u)\nabla^2 uA(x,u) + f(x,u) = 0, \quad u(T,x) = u_0(x). \tag{3.19}$$

To construct a solution to (3.19) we reduce it to a stochastic problem

$$d\xi(t) = a(\xi(t), u(t,\xi(t)))dt + A(\xi(t), u(t,\xi(t)))dw(t), \quad \xi(s) = x. \tag{3.20}$$

$$u(s,x) = E\left[u_0(\xi_{s,x}(T)) + \int_s^T f(\xi_{s,x}(\theta), u(\theta, \xi_{s,x}(\theta)))d\theta\right]. \tag{3.21}$$

To prove the existence and uniqueness theorem for the solution of (3.20), (3.21) we have to add to **C 3.3** and assumption that $g(x,u)$ is sublinear in x, has a polynomial growth in u and Lipschitz continuous in both x and u. We say in this case that **C 3.3'** holds.

To prove the required smoothness of $f(s,x)$ given by (3.7 we need higher smoothness of the Cauchy problem data.

Condition **C 3.3; k.** In addition to **C 3.3** we assume that $a(x,f)$, $A(x,f)$ are k-times differentiable in $x \in R^d, u \in R^1$ and $f_0(x))$ is k-times differentiable and bounded. Under these conditions we can state the following assertion.

Theorem 3.5. *Let* **C 3.3** *and* **C 3.3; 2+** ε *hold. Then there exists an interval* $[T_2, T] \subset [T_1, T]$ *such that the function* $f(s,x)$ *given by (3.7) is a unique classical solution of (3.5), defined on the interval* $[T_2, T]$*, that is* $T_2 \leq s \leq t \leq T$.

Note that to check the required smoothness of $f(s,x)$ we consider the second order differential prolongation of both the Cauchy problem (3.5) and the stochastic system (3.6),(3.7). It will immediately leads to the problem to construct a probabilistic approach to the Cauchy problem for (non-diagonal) systems of PDEs.

Obviously one can extend the results concerning the solution of (3.5) to a diagonal system of the form

$$\frac{\partial u_k}{\partial s} + \mathcal{A}^u u_k + f_k = 0, \quad u_k(T,x) = u_{0k}(x), \quad k = 1,\ldots,d_1, \tag{3.22}$$

where $\mathcal{A}^u u_k(s,x) = a(x,u(s,x))\cdot\nabla u_k + \frac{1}{2}TrA^*(x,u(s,x))\nabla^2 u_k A(x,u(s,x))$.

To extend this approach to study the Cauchy problem for a non-diagonal system of the form

$$\frac{\partial u_k}{\partial s} + \mathcal{A}^u u_k + \sum_{i=1}^{d}\sum_{l=1}^{d_1} B^i_{kl}(x,u(s,x))\nabla_i u_l + \sum_{l=1}^{d_1} c_{kl}(x,u(s,x))u_l + f_k = 0, \tag{3.23}$$

$$u_k(T,x) = u_{0k}(x), \quad k = 1, \dots d_1$$

we consider a stochastic system of the form

$$d\xi(t) = a(\xi(t), u(t,\xi(t)))dt + A(\xi(t), u(t,\xi(t)))dw(t),$$

$$\xi(s) = x \in R^d, \tag{3.24}$$

$$d\eta(t) = c(\xi(t), u(t,\xi(t)))\eta(t)dt + C(\xi(t), u(t,\xi(t)))(\eta(t), dw(t)),$$

$$\eta(s) = h \in R^{d_1}, \tag{3.25}$$

$$\langle h, u(s,x)\rangle = E\left[\langle \eta(T), u_0(\xi_{s,x}(T))\rangle + \int_s^T \langle \eta(\theta), f(\xi_{s,x}(\theta), u(\theta, \xi_{s,x}(\theta)))\rangle d\theta\right], \tag{3.26}$$

where $B^i_{kl} = \sum_{j=1}^d C^j_{kl} A^i_j, i = 1, \dots, d,\ l = 1, \dots, d_1,\ \langle h, u\rangle = \sum_{k=1}^{d_1} u_k h_k$.

We assume that the following conditions hold.

Condition **C 3.4.** Let **C 3.3** hold with $u_0, f \in R^{d_1}$ and functions $c(x,u) \in R^{d_1} \times R^{d_1}$, $C(x,u)y \in R^{d_1} \times R^{d_1}$, $x \in R^d, u \in R^{d_1}, y \in R^d$ satisfy the estimates

$$|c(x,u) - c(x_1,u_1)|^2 \le L\|x - x_1\|^2 + L_1\|u - u_1\|^2,$$

$$|C(x,u)y - C(x_1,u_1)y|^2 \le [L\|x - x_1\|^2 + L_1\|u - u_1\|^2]\|y\|^2,$$

$$\langle h, c(x,u)h\rangle \le [K_0 + K_1\|u\|^p]\|h\|^2, \quad |C(x,u)y|^2 \le K[1 + \|u\|^{2p}]\|y\|^2,$$

where $L_1 > 0$ depend on $\max(\|u\|, \|u_1\|)$, L, K_0, K_1 are constants and $L, K_1 > 0$.

Besides we will need more restrictions on the coefficient smoothness.

Condition **C 3.4;k.** Let **C 3.3;k** hold and the functions $c(x,u)$, $C(x,u)$, $f(x,u)$ be k times differentiable in $x \in R^d, u \in R^1$ and $u_0(x)$ be bounded and k times differentiable.

Under these conditions we can state the following assertion.

Theorem 3.6. *Let* **C 3.3** *and* **C 3.3;1** *hold. Then there exists an interval $[T_1, T]$ and a unique solution $(\xi_{s,x}(t), u(s,x))$ of the system (3.6), (3.7) defined on this interval, that is $0 \le T_1 \le s \le t \le T$. The process $\xi_{s,x}(t) \in R^d$ possesses the Markov property, $\eta(t) \in R^{d_1}$ defines a multiplicative operator functional of the process $\xi_{s,x}(t)$ while $u(s,x) \in R^{d_1}$ is bounded and Lipschitz continuous in x if $s \in [T_1, T]$.*

Theorem 3.7. *Let* **C 3.4** *and* **C 3.4;**$2+\varepsilon$ *hold. Then there exists an interval $[T_2, T] \subset [T_1, T]$ such that the function $u(s,x)$ given by (3.18) is a unique classical solution of (3.11), defined on a smaller interval $[T_2, T]$.*

Note that one can easily reduce the system (3.12) to a scalar equation with respect to a function $\Phi(s,x,h) = \langle h, u(s,x)\rangle$ and check that conditions **C 3.4** imply conditions **C 3.3** for coefficients of this new scalar equation under the assumptions that $\|\Phi(s)\| = \sup_{x\in R^d, \|h\|=1} |\Phi(s,x,h)|$ and hence the proof of theorems 3.5 and 3.6 is similar to the proof of theorems 3.3 and 3.4.

The detailed proofs of these assertions one may find in [4].

At the next step we consider the Cauchy problem for a semilinear parabolic equation of the form

$$\frac{\partial u}{\partial s} + a(x,u,\nabla u)\cdot\nabla u + \frac{1}{2}F(x,u,\nabla u)\diamond\nabla^2 u = 0, \tag{3.27}$$

$$u(T,x) = u_0(x),$$

where $A^*A = F$ and $\sum_{i,j=1}^d F_{ij}\nabla^2_{ji}u = F\diamond\nabla^2 u$.

To construct a solution of (3.27) we consider its differential prolongations, that is we differentiate (3.27) in x variable and consider the solution u of (3.27) as a component of a solution $V=(u,p)$ of a system consisting of and its differential prolongation. Here $p=\nabla u$ and we result in a system with coefficients depending only on the unknown function V and independent on its derivatives.

To this end assuming that coefficients in (3.27) are smooth enough we denote by $p(s,x)=\nabla u(s,x), q(s,x)=\nabla^2 u(s,x)$ and twice differentiate this equation in x variable. We use notations a_x, a_u, a_p to denote partial derivatives of the function a in x,u and p respectively. As a result we obtain

$$\frac{\partial p}{\partial s} + a(x,u,p)\cdot\nabla p + \frac{1}{2}F(x,u,p)\diamond\nabla^2 p + \alpha(a)p + \alpha(F)q = 0, \tag{3.28}$$

$$v(T,x) = \nabla u_0(x) = p_0(x),$$

where $\alpha(a) = a_x + a_u p + a_p\nabla p$,

$$\frac{\partial q}{\partial s} + a(x,u,p)\cdot\nabla q + \frac{1}{2}F(x,u,p)\diamond\nabla^2 q + [\frac{1}{2}\beta(F)+\alpha(a)]\nabla p + \alpha(F)\nabla q + \tag{3.29}$$

$$\alpha(a)q + \beta(a)p = 0, \quad q(T,x) = \nabla^2 u_0(x) = q_0(x),$$

where

$$\beta(a) = a_{xx} + 2(a_{xu}p + a_{xp}\nabla p + a_{up}pq) + a_u\nabla p + a_p\nabla q + a_{uu}p + a_{pp}(\nabla p)^2$$

Denote by $b(x,V) = a(x,u,p), G(x,V) = F(x,u,p)$. Then we we can rewrite the system (3.27)-(3.29) as a quasilinear system of parabolic equations

$$\frac{\partial V^k}{\partial s} + b(x,V)\cdot\nabla V^k + \frac{1}{2}G(x,V)\diamond\nabla^2 V^k + D^k(x,V)\diamond\nabla V + \tag{3.30}$$

$$\sum_{l=1}^{3} c_l^k(x,V)V^l + G(x,V) = 0,$$

where $V = (V^1, V^2, V^3) = (u, p, q)$. The matrix c and the tensor D aa well as the vector function G can be easily obtained from (3.27)-(3.29).

$$V(T,x) = V_0(x) = (u_0(x), \nabla u_0(x), \nabla^2 u_0(x)).$$

Next we consider a stochastic problem

$$d\xi(t) = b(\xi(t), V(t,\xi(t)))dt + B(\xi(t), V(t,\xi(t)))dw(t), \quad \xi(s) = x, \tag{3.31}$$

$$d\eta(t) = c(\xi(t), V(t,\xi(t)))\eta(t)dt + C(\xi(t), V(t,\xi(t)))(\eta(t), dw(t)), \quad \eta(s) = h, \tag{3.32}$$

$$\langle h, V(s,x)\rangle = E\left[\langle \eta(T), V_0(\xi_{s,x}(T))\rangle + \int_s^T \langle \eta(\theta), G(\xi_{s,x}(\theta), V(\theta, \xi_{s,x}(\theta)))\rangle d\theta\right] \tag{3.33}$$

corresponding to (3.30), where

$$B(x,V) = A(x,u,p),\ D(x,V) = C(x,V)B(x,V), V = (u,p,q).$$

In [5], [7] there were stated conditions that allows to prove the existence and uniqueness of the solution to (3.31)– (3.33) on a small interval $(T_1, T]$ with the length depending on functions F and u_0 in (1.11). This immediately leads to the construction of the solution to (1.11), namely the first component V_1 of the function V is a unique classical solution of the Cauchy problem(1.11) with

$$f(x,u,p,q) = a(x,u,p)\cdot p + \frac{1}{2} Tr A^*(x,u,p) q A(x,u,p).$$

The similar approach works for a fully nonlinear parabolic equation of the form (1.11). To illustrate the construction we consider (1.11) for the case $d = d_1 = 1$ and $F(x,u,p,q) = F(q)$ still preserving the previous notations for spatial derivatives.

A differential prolongations of the Cauchy problem

$$\frac{\partial u}{\partial s} + f(\nabla^2 u) = 0, \quad u(T,x) = u_0(x) \tag{3.34}$$

has the form

$$\frac{\partial p}{\partial s} + f'(\nabla p)\nabla^2 p = 0, \quad p(T,x) = p_0(x) = \nabla u_0(x), \tag{3.35}$$

$$\frac{\partial q}{\partial s} + f''(q)(\nabla q, \nabla q) + f'(q)\nabla^2 q = 0, \quad q(T,x) = q_0(x) = \nabla^2 u_0(x) \tag{3.36}$$

and to obtain the equation with coefficients having nonlinear dependence only on an unknown function we have to differentiate (3.28) once more to derive the equation

$$\frac{\partial r}{\partial s} + f'''(q) r^2 \nabla q + 3 f''(q) r \nabla r + f'(q)\nabla^2 r = 0, \quad r(T,x) = \nabla^3 u_0(x). \tag{3.37}$$

Finally we rewrite (3.37) in the form

$$\frac{\partial u}{\partial s} + f(q) + f'(q)\nabla^2 u - f'(q)q = 0, \quad u(T,x) = u_0(x), \tag{3.38}$$

and present the system (3.35)-(3.38) in the form

$$\frac{\partial V^k}{\partial s} + \frac{1}{2}A^2(V)\nabla^2 V^k + a(V)\nabla V^k + B^k_l(V)\nabla V^l + c^k_l V^l + G^k(V) = 0, \tag{3.39}$$

$$V^k(T,x) = V^k_0(x),$$

where $V = (u, \nabla u, \nabla^2 u, \nabla^3 u) = (V^1, V^2, V^3, V^4)$. Here coefficients a, A have the form

$$\frac{1}{2}A^2(V) = f'(q), \quad a(V) \equiv 0, \tag{3.40}$$

the matrices $c = (c_{jk})^4_{j,k=1}$ and $B = (B_{jk})^4_{j,k=1}$ have the form

$$c_{13} = -f'(q), \quad B_{33} = f''(q)r, \quad B_{43} = f'''(q)r^2, \quad B_{44} = 3f''(q)r, \tag{3.41}$$

$c_{jk} = B_{jk} = 0$ for remaining j, k and finally $G^1 = F(q), G^k = 0$ for $k = 2, 3, 4$. Here and below we assume a usual convention of summing up in repeating indices if the contrary is not mentioned.

At the next step we reduce the system (3.35)-(3.38) to a stochastic system of the form (3.28)-(3.30) with coefficients given by (3.40), (3.41). Next we state conditions on these coefficients and u_0 to ensure the existence and uniqueness of the solution of a corresponding stochastic system on a certain time interval. Finally we verify that in this way we have constructed a unique classical solution to (3.34) on a certain time interval. To be more precise, assuming that coefficients of (3.39) satisfy condition **C 3.4** we ensure existence and uniqueness of the solution to (3.28)-(3.30) with coefficients given in (3.40)-(3.41) and the condition **C 3.4;2** allows to verify that the function $u(s,x)$ given by (3.29) is a unique classical solution to (3.34).

In a similar way we can study the Cauchy problem for more general fully nonlinear parabolic equations and systems (see [5], [7]).

$$u_t + f(x, u, u_x, u_{xx}) = 0, \quad u(T) = \Phi(x). \tag{3.42}$$

Let $\Phi : R^d \to R^1$, $F : R^d \times R^1 \times R^d \times M^d \to R^1$ be sufficiently smooth functions. If $\Gamma(x) = f(x, u(x), \nabla u(x), \nabla^2 u(x))$ is a three times differentiable function then one can reduce at least formally the fully nonlinear problem (3.31) to a quasilinear problem

$$v_t + \frac{1}{2}f_{u_{xx}} v_{xx} + G_1 v_x + G_2 v + G_3 = 0, \quad v(T) = V(x), \tag{3.43}$$

where $v = (u, u_x, u_{xx}, u_{xxx})$, $V = (\Phi, \Phi_x, \Phi_{xx}, \Phi_{xxx})$, and functions G_i $i = 1, 2, 3$ depend on the function v. To apply a probabilistic approach in this case we have to assume that $f_{u_{xx}}$ ua a nonnegative matrix

$$d\xi(\theta) = a(\xi(\theta), v(\theta, \xi(\theta))d\theta + A(\xi(\theta), v(\theta, \xi(\theta))dw(\theta), \quad \xi(t) = x, \tag{3.44}$$

$$d\eta(t) = G_2(\xi(\theta), v(\theta, \xi(\theta))\eta(\theta)d\theta + G_1(\xi(\theta), v(\theta, \xi(\theta))\eta(\theta)dw(\theta), \quad \eta(t) = h, \tag{3.45}$$

$$(h, v(t, x)) = E\left[(\eta(T), V_0(\xi(T))) + \int_t^T (\eta(\tau), G_3(\xi(\tau)))d\tau\right]. \tag{3.46}$$

As a final remark in this section we note that all the above results can be naturally extended to the case when coefficients depends on a time variable as well.

4 BSDEs and nonlinear parabolic systems

In this section we introduce necessary notions and notations and recall some results concerning backward stochastic differential equations (BSDE) and their connections with nonlinear parabolic equations [8], [9]. We will need these results later in nest sections to apply them to a fully nonlinear parabolic equation (1.12).

First we consider the Cauchy problem

$$u_s + a(x)\cdot\nabla u + \frac{1}{2}TrA^*(x)\nabla^2 uA(x) + g(x, u, \nabla u) = 0, \quad u(T, x) = u_0(x). \tag{4.1}$$

which is a particular case of (1.11) corresponding to $f(x, u, p, q) = ap + \frac{1}{2}A^*qA + g(x, u, p)$. Here $x \in R^d, u \in R^d, p \in R^d \otimes R^d = M$, $g : R^d \times R^d \times M \to R^d$ and $TrA^*qA \in R^d$.

To construct a solution to (4.1) in this section we apply a BSDE approach and compare it with the approach used in the previous sections. To this end we need some additional notations.

Let $\mathcal{M}^2([0, T]; R^d)$ denote the set of progressively measurable square integrable stochastic processes $\xi(t) \in R^d$, $E\left[\int_0^T \|\xi(\tau)\|^2 d\tau\right] < \infty$, and $\mathcal{S}^2([0, T], R^d)$ be the set of semimartingales $\eta(t) \in R^d$ satisfying the estimate

$$E\left[\sup_{0\le t\le T} \|\eta(t)\|^2\right] < \infty.$$

To explain what sort of a stochastic problem we deal with this time let us consider a stochastic equation

$$d\xi(t) = a(\xi(t))dt + A(\xi(t))dw(t), \quad \xi(s) = x, \tag{4.2}$$

with a Wiener process $w(t) \in R^d$ and a function $u \in C^{1,2}$ satisfying (4.1). By the Ito formula we know that the stochastic differential of a random process $y(t) = u(t, \xi(t))$ has the form

$$dy(t) = [u_t + \nabla u a(\xi(t)) + \frac{1}{2} Tr A(\xi(t)) \nabla^2 u A^*(\xi(t))]dt + \nabla u A(\xi(t))dw(t). \quad (4.3)$$

Since by assumption u satisfies (4.1), we easily deduce from (4.3) that the process $y(t)$ solves the Cauchy problem

$$dy(t) = -g(y(t), \nabla u(\xi(t))dt + \nabla u A(\xi(t))dw(t), \quad y(T) = u_0(\xi(T)). \quad (4.4)$$

Moreover if $\xi_{s,x}(t)$ is a solution to (4.2), then we have $u(s,x) = y^{s,x}(s)$, where $y^{s,x}(t)$ given by

$$y^{s,x}(t) = E\left[u_0(\xi_{s,x}(T)) + \int_t^T g(\xi_{s,x}(\theta), y^{s,x}(\theta), \nabla u(\xi_{s,x}(\theta)))d\theta / \mathcal{F}_t\right] \quad (4.5)$$

solves the BSDE (4.4).

Remark. Let $u_0 : [0,T] \times R^d \to R^1$ and $g(x,p,q) \in R^1, x \in R^d, p \in R^1, q \in R^d$. Then (4.5) yields the integral representation of $u(s,x)$

$$u(s,x) = E\left[u_0(\xi_{s,x}(T)) + \int_s^T g(\xi_{s,x}(\theta), u(\theta, \xi_{s,x}(\theta), \nabla u(\xi_{s,x}(\theta)))d\theta\right] \quad (4.6)$$

which coincides with (3.21) provided coefficients in (3.20) do not depend on u.

Let us recall some general results of the BSDE theory.

Consider a function $f : \Omega \times [0,T] \times R^d \times R^{d \times d} \to R^d$. We say that condition **C 4.1** holds if:

1)$f(\cdot, t, y, z)$ is a progressively measurable random variable $\forall t$ and for any (y, z) valued in $R^d \times M$;

2)$E\|\bar{f}(t)\| < \infty, \quad \bar{f}(t) = f(t, 0, 0)$;

3) there exist a constant μ and positive constants K, L such that
$\|f(t,y,z)\| \leq \bar{f}(t) + K[\|y\| + |z|] \quad \forall t \in [0,T], y \in R^d, z \in M, \ P-$ a.s.;

4)$\|f(t,y,z) - f(t,y,z_1)\| \leq L|z - z_1|$,
$\forall t \in [0,T], y \in R^d, z, z_1 \in M, \ P-$ a.s.;

5)$\langle y - y_1, f(t,y,z) - f(t,y_1,z)\rangle \leq \mu\|y - y_1\|^2$,
$\forall t \in [0,T], y, y_1 \in R^d, z \in M, P-$ a.s.;

6) $\|f(t,y,z) - f(t,y_1,z)\| \leq L\|y - y_1\|$,
$\forall t \in [0,T], y, y_1 \in R^d, z \in M, \ P-$ a.s.

Consider a backward stochastic differential equation

$$dy(t) = -f(t, y(t), z(t))dt + z(t)dw(t), \quad y(T) = \eta, \quad (4.7)$$

where $\eta \in R^d$ is $\mathcal{F}_T$ measurable.

By definition a solution of BSDE (4.7) is a pair of progressively measurable stochastic processes $(y(t), z(t))$ valued in $R^d \times M$ such that

1) $E\int_0^T |z(t)|^2 dt < \infty, \quad \sup_{)\le t\le T} E\|y(t)\|^2 < \infty$;

2) $y(t) = \eta + \int_t^T f(s, y(s), z(s))ds - \int_t^T z(s)dw(s), \quad 0 \le t \le T.$

Now we state conditions that ensures the existence and uniqueness of a solution $(y(t), z(t)) \in R^{d_1} \times M$ to (4.7).

We denote by $\mathcal{H} = \mathcal{S}^2([0,T]; R^d) \times \mathcal{M}^2([0,T]; M)$ and define a mapping $\Phi(u, v) = (y, z)$ that acts in $\mathcal{H}$ as follows. Given $(u(t), v(t))$, we define the process $y(t)$ by

$$y(t) = E\left[\eta + \int_t^T f(s, u(s), v(s))ds|\mathcal{F}_t\right], \quad 0 \le t \le T, \tag{4.8}$$

and the process $z(t)$ by the Ito martingale representation theorem. Namely, since by the above assumptions

$$\kappa = \eta + \int_0^T f(s, u(s), v(s))ds$$

is a square integrable random variable we apply the Ito martingale representation theorem to derive the relation

$$\kappa = E[\kappa] + \int_0^T z(s)dw(s).$$

Since $y(0) = E\left[\eta + \int_0^T f(s, u(s), v(s))ds\right] = E[\kappa]$, we obtain

$$y(0) = \kappa - \int_0^T z(s)dw(s) = \eta + \int_0^T f(s, u(s), v(s))ds - \int_0^T z(s)dw(s). \tag{4.9}$$

Note that $y(0) - \int_0^t f(s, u(s), v(s))ds + \int_0^t z(s)dw(s)$ is an $\mathcal{F}_t$- measurable random variable and

$$y(0) - \int_0^t f(s, u(s), v(s))ds + \int_0^t z(s)dw(s) = \tag{4.10}$$

$$\eta + \int_t^T f(s, u(s), v(s))ds - \int_t^T z(s)dw(s).$$

Applying the conditional expectation $E[\cdot|\mathcal{F}_t]$ to both sides of (4.10) and taking into account (4.8) and (4.9) we get

$$y(t) = \eta + \int_t^T f(s, u(s), v(s))ds - \int_t^T z(s)dw(s). \tag{4.11}$$

Theorem 4.1. *Let* **C 4.1** *hold. Then there exists a unique solution* (y, z) *of (4.7) such that* $(y, z) \in \mathcal{H}$.

Proof. First we prove the uniqueness. Let both $(y(t), z(t))$ and $(y_1(t), z_1(t))$ be solutions to (4.7). Denote by $\bar{y}(t) = y(t) - y_1(t)$ and $\bar{z}(t) = z(\theta) - z_1(\theta)$. By the Ito formula we deduce

$$\delta(t) = E\left[\|\bar{y}(t)\|^2 + \int_t^T |\bar{z}(s)|^2 ds\right] = \tag{4.12}$$

$$2E\int_t^T \langle \bar{y}(s), f(s, y(s), z(s)) - f(s, y_1(s), z_1(s))\rangle ds.$$

By conditions in **C 4.1** on f and an elementary estimate $2ab \leq a^2 + b^2$ we deduce

$$\delta(t) \leq 2E\int_t^T [\mu\|\bar{y}(s)\|^2 + L\|\bar{y}(s)\||\bar{z}(s)|]ds \leq$$

$$(2\mu + L^2)E\int_t^T \|\bar{y}(s)\|^2 ds + E\int_t^T |\bar{z}(s)|^2 ds.$$

Hence for $\bar{y}(t)$ we have an estimate

$$E\|\bar{y}(t)\|^2 \leq (2\mu + L^2)\int_t^T E\|\bar{y}(s)\|^2 ds$$

and by the Gronwall lemma we deduce $E\|\bar{y}(t)\|^2 = 0$ for all $t \in [0, T]$. Coming back to (4.13) we note that when $\bar{y}(t) = 0$, we get $\int_0^t E|\bar{z}(s)|^2 ds = 0$ for all $t \in [0, T]$ and hence $E|z(s) - z_1(s)|^2 = 0$.

The couple $(y, z) \in \mathcal{H}$ satisfies (4.7), if and only if (y, z) is a fixed point of the mapping Φ defined above by (4.8), (4.11). Hence it remains to check that under conditions **C 4.1** the mapping Φ is a contraction.

We present the proof of this fact in the following two lemmas. In the first we will prove apriori estimates of a solution to (4.7) and in the second we prove the contraction property of the map Φ.

Lemma 4.2 *Let* $(y(t), z(t))$ *be a solution of (4.7) and* **C 4.1** *holds. Then there exists a constant* C *depending only on* T, μ *and* L *such that*

$$E\left(sup_{0\leq t\leq T}\|y(t)\|^2 + \int_0^T |z(\theta)|^2 d\theta\right) \leq CE\left(\|\xi\|^2 + \int_0^T \|f(t, 0, 0)\|^2 dt\right).$$

Proof. If (y, z) is a solution to (4.7), then

$$y(t) = y(0) - \int_0^t f(\theta, y(\theta), z(\theta))d\theta + \int_0^t z(\theta)dw(\theta). \tag{4.13}$$

For each integer k we define the stopping time

$$\tau_l^k = \inf_{0\le t\le T}\{t : |y_l(t)| \ge k\},$$

and set $y_l^k(t) = y_l(t\wedge\tau_k)$. Then

$$y_l^k(t) = y_l^k(0) - \int_0^t I_{[0,\tau_k]} f_l(y^k(\theta), z(\theta))d\theta + \int_0^t I_{[0,\tau_k]} \sum_{m=1}^d z_{lm}(\theta)dw_m(\theta).$$

Hence,

$$E\|y^k(t)\|^2 \le 3(\|y^k(0)\|^2 + t\int_0^t E\|f(\theta, y^k(\theta), z(\theta))\|^2 d\theta + \int_0^t E|z(\theta)|^2 d\theta),$$

and by the properties of the function $f(y,z)$ stated in **C 4.1** and the property of the process z which is the component of the solution to (4.7) we deduce that $E\|y^k(t)\|^2 \le C[1 + \int_0^t E\|y^k(\theta)\|^2 d\theta]$. Applying the Gronwall lemma we obtain an estimate $E\|y^k(t)\|^2 \le Ce^{Ct}$ and by the Fatou lemma we get

$$E\|y(t)\|^2 \le Ce^{Ct}.$$

Applying the Burkholder inequality, we deduce from (4.7) and the above estimates that

$$E[sup_{0\le t\le T}\|y(t)\|^2] < \infty$$

and by Ito's formula we have

$$E\left[\|y(t)\|^2 + \int_t^T |z(\theta)|^2 d\theta\right] = E\left[\|\xi\|^2 + 2\int_t^T \langle y(\theta), f(\theta, y(\theta), z(\theta))\rangle d\theta\right] \le$$

$$E\|\xi\|^2 + K\int_t^T E[\|y(\theta)\|^2 + |z(\theta)|^2]d\theta + E\int_0^T \|f(t,0,0)\|^2 dt,$$

where the constant depends on T, μ, L. Finally computing the sup of both sides in the last inequality and applying the Gronwall lemma we obtain

$$sup_{0\le t\le T} E\|y(t)\|^2 + \int_0^T E|z(\theta)|^2 d\theta \le Ce^{KT}$$

where $C = E\|\xi\|^2 + E\int_0^T \|f(t,0,0)\|^2 dt$ and the required estimate follows from the Burkholder-Davis-Gundy inequality.

Lemma 4.3. *Under the condition 1)-4) and 6) from* **C 4.1** *the map* Φ *is a contraction in* $\mathcal{H}$.

Proof. Given $(u,v) \in \mathcal{H}$, $(u_1,v_1) \in \mathcal{H}$ and $(y,z) = \Phi(u,v), (y_1,z_1) = \Phi(u_1,v_1)$, we denote by $(\bar{u},\bar{v}) = (u-u_1, v-v_1)$, $(\bar{y},\bar{z}) = (y-y_1, z-z_1)$. By the Ito formula we deduce that for any constant C we have

$$e^{Ct}E\|\bar{y}(t)\|^2 + \int_t^T e^{C\tau}[C\|\bar{y}(\tau)\|^2 + |\bar{z}(\tau)|^2]d\tau \leq$$

$$2KE\int_t^T e^{C\tau}\|\bar{y}(\tau)\|[\|\bar{u}(\tau) + |\bar{v}(\tau)|]d\tau \leq$$

$$4K^2E\int_t^T e^{C\tau}\|\bar{y}(\tau)\|^2d\tau + \frac{1}{2}E\int_t^T e^{C\tau}[\|\bar{u}(\tau)\|^2 + |\bar{v}(\tau)|^2]d\tau.$$

Choosing $C = 1 + 4K^2$ we get

$$E\int_t^T e^{C\tau}[\,\|\bar{y}(\tau)\|^2 + |\bar{z}(\tau)|^2]d\tau \leq \frac{1}{2}E\int_t^T e^{C\tau}[\|\bar{u}(\tau)\|^2 + |\bar{v}(\tau)|^2]d\tau$$

which results that Φ is a contraction in $\mathcal{H}$ equipped with the norm

$$\|(y,z)\|_{\mathcal{H}} = \left(E\int_t^T e^{C\tau}[\,\|y(\tau)\|^2 + |z(\tau)|^2]d\tau\right)^{\frac{1}{2}}.$$

By the fixed point theorem we get that Φ has a unique fixed point and hence there exists a unique solution to (4.7).

The same assertion is true under the condition 1)-5) from **C 4.1** [8].

Let the random function $f(t,x,y,z) = g(\xi(t),y,z)$ satisfies **C 4.1**, where $\xi(t)$ is a diffusion process that solves an SDE

$$d\xi(t) = a(\xi(t))dt + A(\xi(t))dw(t), \quad \xi(s) = x. \tag{4.14}$$

Then due to theorem 4.1 there exists a unique solution $(y(t),z(t))$ of the BSDE

$$dy(t) = -g(\xi(t),y(t),z(t))dt + z(t)dw(t), \quad y(T) = \eta = u_0(\xi(T)). \tag{4.15}$$

Next we have to explain in what sense the function $u(s,x) = y(s)$ given by (4.15) is a solution to the Cauchy problem

$$u^i_s + a(x)\cdot\nabla u^i + \frac{1}{2}TrA^*(x)\nabla^2u^iA(x) + g(x,u,A(x)\nabla u^i) = 0, \tag{4.16}$$

$$u^i(T,x) = u^i_0(x), i = 1,\dots,d.$$

First note that the following useful assertion holds.

Lemma 4.4. *Assume that $(\hat{s},\hat{x}) \in G_T$ and $u^i \in C(G_T)$, then*

$$\mathcal{P}^{2,+}_G u^i(\hat{s},\hat{x}) = \{(\phi_s(\hat{s},\hat{x}), \nabla\phi(\hat{s},\hat{x}), \nabla^2\phi(\hat{s},\hat{x})) : \phi \in C^{1,2} \tag{4.17}$$

$$\text{and } u^i - \phi \text{ has a local maximum at } (\hat{s}, \hat{x}) \in G_T\},$$

and

$$\mathcal{P}_G^{2,-} u^i(\hat{s}, \hat{x}) = \{(\phi_s(\hat{s}, \hat{x}), \nabla\phi(\hat{s}, \hat{x}), \nabla^2\phi(\hat{s}, \hat{x})) : \phi \in C^{1,2} \tag{4.18}$$
$$\text{and } u^i - \phi \text{ has a local minimum at } (\hat{s}, \hat{x}) \in G_T\}.$$

We use this assertion to check that the solution of the BSDE (4.15) constructed above gives rise to a viscous solution of (4.16).

In other words to prove that $u(s, x)$ is a subsolution of (4.16), we have to verify that $u^i(T, x) \leq u_0^i(x)$ and for any $\phi \in C^{1,2}([0, T] \times R^d)$ and a point $(s, x) \in [0, T] \times R^d$ which is a local maximum of the function $u^i - \phi$ the following inequality holds

$$\phi_s + f(x, u, \nabla\phi, \nabla^2\phi) \geq 0. \tag{4.19}$$

Similarly $u(s, x)$ is a supersolution of (4.16) when $u(T, x) \geq u_0(x)$ and for any $\phi \in C^{1,2}([0, T] \times R^d)$ and a point $(s, x) \in [0, T] \times R^d$ which is a local minimum of the function $u^i - \phi$ the following inequality holds

$$\phi_s + f(x, u, \nabla\phi, \nabla^2\phi) \leq 0. \tag{4.20}$$

Finally, if u is both a supersolution and a subsolution of (4.16), then it is called a viscous solution of (4.16).

Theorem 4.5. *Let* (y, z) *solve (4.15) with* $\eta = u(T, \xi(T)) = u_0(\xi(T))$*, then the function* $u(s, x) = y(s)$ *is a continuous viscous solution of the Cauchy problem (4.16).*

Proof. Continuity of $u(s, x)$ is a consequence of the square mean continuity of the solution to (4.15).

To show that $u(s, x)$ is a viscosity solution we choose a function $\phi \in C^{1,2}$ and a point $(s, x) \in [0, T] \times R^d$ such that at (s, x) the function $u^i - \phi$ has a local maximum. Without loss of generality we assume that $u^i(s, x) = \phi(s, x)$. It yields that at any stopping time τ we have

$$u^i(\tau, \xi_{s,x}(\tau)) - \phi(\tau, \xi_{s,x}(\tau)) \leq 0. \tag{4.21}$$

From the Ito formula we deduce

$$\phi(\tau, \xi(\tau)) = \phi(s, x) + \int_s^\tau [\phi_\theta + \mathcal{A}\phi](\theta, \xi(\theta))d\theta + \int_s^\tau \nabla\phi(\theta, \xi(\theta))A(\xi(\theta))dw(\theta).$$

At the other hand due to the martingale representation theorem we have a representation

$$y^i(s) = u_0^i(\xi(T)) + \int_s^T g^i(\xi(\theta), y(\theta), z^i(\theta))d\theta - \int_s^T \langle z^i(\theta), dw(\theta)\rangle.$$

Since the solution $y(t) = u(t, \xi(t))$ of the equation (4.7) is unique we deduce

$$u^i(s,x) = y^i(s) = y^i(\tau) + \int_s^\tau g^i(\xi(\theta), y(\theta), z^i(\theta))d\theta - \int_s^\tau \langle z^i(\theta), dw(\theta)\rangle =$$

$$u^i(\tau, \xi(\tau)) + \int_s^\tau g^i(\xi(\theta), y(\theta), z^i(\theta))d\theta - \int_s^\tau \langle z^i(\theta), dw(\theta)\rangle.$$

Substituting the last relation into (4.21) we derive

$$0 \geq u^i(\tau, \xi(\tau)) - \phi(\tau, \xi(\tau)) = u^i(s,x) - \phi(s,x) - \int_s^\tau [\frac{\partial \phi}{\partial \theta} + \mathcal{A}\phi](\theta, \xi(\theta))d\theta - \tag{4.22}$$

$$\int_s^\tau g^i(\xi(\theta), y(\theta), z^i(\theta))d\theta + \int_s^\tau \sum_{j=1}^d [z_{ij}(\theta) - \sum_{m=1}^d \nabla_{x_m}\phi A_{mj}]dw_j(\theta).$$

Since by assumption the equality $u^i(s,x) - \phi(s,x) = 0$ holds at the point (s,x) we can compute the expectation of the both sides of (4.22) to obtain the inequality

$$E\left(\int_s^\tau \Lambda_l(\theta, \xi(\theta), y(\theta), z(\theta))d\theta\right) \geq 0, \quad l = 1, \dots d_1, \tag{4.23}$$

where

$$\Lambda^i(s,x,y,z) = \left[\frac{\partial \phi}{\partial s} + \mathcal{A}\phi\right](s,x) + g^i(x,y,z^i).$$

To verify that $u^i(s,x)$ is a subsolution of (4.1) we have to show that

$$\Lambda^i(t,x,y,z^i) \geq 0.$$

Assume on the contrary that there exists $\delta_0 < 0$ such that $\Lambda^i(t,x,y,z) < \delta_0$ for some l and set

$$\tau_1 = \inf\{\theta > s : \Lambda^i(\theta, \xi(\theta), y(\theta), z(\theta)) \geq \delta_0\} \wedge T.$$

By definition inequality (4.21) holds for any stopping time and hence for the stopping time $\tau_1 > s$. Thus we come to a contradiction

$$0 > \delta_0 E(\tau_1 - s) \geq E\left(\int_s^{\tau_1} \Lambda^i(\xi(\theta), y(\theta), z(\theta))d\theta\right) \geq 0,$$

and hence $u(s,x)$ is a subsolution of (4.16). In a similar way we can check that $u(s,x)$ is a supersolution of (4.16) and hence is a viscosity solution of (4.16).

5 Stochastic problems associated with fully nonlinear parabolic equations

In this section we come back to the Cauchy problem for a fully nonlinear parabolic equation

$$u_s^k + g^k(x, u, \nabla u^k, \nabla^2 u^k) = 0, \; k = 1, \dots d, \tag{5.1}$$

$$u(T, x) = u_0(x) \in R^d, x \in R^d, s \in [0, T].$$

We derive a differential prolongation to reduce (5.1) to a semilinear PDE and then apply the BSDE approach to construct both a solution of an associated fully coupled FBSDEs and a viscosity solution of (5.1).

Assume that $g(x, u, p, q) \in R^d, \quad x, u \in R^d, p \in R^d \otimes R^d = M, q \in M \otimes R^d = M_1$ is a differentiable function in all arguments and consider along with the Cauchy problem (5.1) its first differential prolongation

$$p_s + g_1(x, u, \nabla u, \nabla^2 u) + g_2(x, u, \nabla u, \nabla^2 u)v + g_3(x, u, \nabla u, \nabla^2 u)\nabla p + \tag{5.2}$$

$$g_4(x, u, \nabla u, \nabla^2 u)\nabla^2 p = 0, \quad p(T, x) = \nabla u_0(x).$$

Here

$$p = \nabla u, \quad g_1 = \frac{\partial g}{\partial x}, \quad g_2 = \frac{\partial g}{\partial u}, \quad g_3 = \frac{\partial g}{\partial p}, \quad g_4 = \frac{\partial g}{\partial q}.$$

We say that condition **C 5.1** holds if $g : R^d \times R^d \times M \times M_1 \to R^d$ has the form $g^i(x, u, p, q) = g^i(x, u, p^i, q^i), i = 1, \dots, d$ and is differentiable in all arguments. Besides for each component g^i of the function g the derivative g_3^i is a positive definite matrix.

Set $V(s, x) = (u(s, x), p(s, x))$ and rewrite (5.1), (5.2) in the form

$$u_s + g(x, u, p, \nabla p) + g_3(x, u, p, \nabla p)\nabla u + g_4(x, u, p, \nabla p)\nabla^2 u - \tag{5.3}$$

$$g_3(x, u, p, \nabla p)p - g_4(x, u, p, \nabla p)\nabla p = 0,$$

$$p_s + g_1(x, u, p, \nabla p) + g_2(x, u, p, \nabla p)p + g_3(x, u, p, \nabla p)\nabla p + \tag{5.4}$$

$$g_4(x, u, p, \nabla p)\nabla^2 p = 0$$

or in the form

$$V_s + \mathcal{B}V - G(X, V, \nabla V) = 0. \tag{5.5}$$

where

$$\mathcal{B}V = \frac{1}{2} Tr B^*(X, V, \nabla V)\nabla^2 V B(X, V, \nabla V) + b(X, V, \nabla V)\nabla V. \tag{5.6}$$

Here $b = \begin{pmatrix} g_3 \\ g_3 \end{pmatrix}$, $[B^*B]_{jk} = 2\begin{pmatrix} \frac{\partial g}{\partial q_{jk}} & 0 \\ 0 & \frac{\partial g}{\partial q_{jk}} \end{pmatrix}$, $X = (x, I)$, where I is the identity matrix, $V = (u, p)$ and

$$G(X, V, \nabla V) = \begin{pmatrix} g(x,u,p,\nabla p) - g_3(x,u,p,\nabla p)v - g_4(x,u,p,\nabla p)\nabla p \\ g_1(x,u,p,\nabla p) + g_2(x,u,p,\nabla p)p \end{pmatrix}.$$

Under the assumption **5.1** the equation (5.5) has the form

$$V^i_s + \mathcal{B}V^i - G(X, V, \nabla V^i) = 0. \tag{5.7}$$

Denote by $H_1 = R^d \times M$, $H_2 = M \times M_1$ and set

$$X(t) = (\xi(t), \eta(t)) \in H_1,\ Y(t) = (y(t), p(t)) \in H_1,\ Z(t) = (z(t), q(t)) \in H_2.$$

Consider an FBSDE consisting of FSDEs

$$d\xi = a(\xi(t), Y(t), Z(t))dt + A(\xi(t), Y(t), Z(t))dw(t), \quad \xi(s) = x \in R^d, \tag{5.8}$$

$$d\eta = \nabla a(\xi(t), Y(t), Z(t))\eta(t)dt + \nabla A(\xi(t), Y(t), Z(t))(\eta(t), dw(t)) \tag{5.9}$$

with $\eta(s) = I \in M$, $\nabla a(x, y(x), z(x)) = a_x + a_y\nabla y + a_z\nabla z$ and BSDEs

$$dy(t) = -f(\xi(t), y(t), p(t), q(t))dt + p(t)dw, \quad y(T) = \eta \in R^d, \tag{5.10}$$

$$dp(t) = -G(\xi(t), \eta(t), y(t), p(t), q(t))dt + q(t)dw, \quad p(T) = \zeta \in M. \tag{5.11}$$

We can rewrite this system in the form of an FBSDE

$$dX(t) = b(X(t), Y(t), Z(t))dt + B(Y(t), Z(t))dw(t),\ \ X(s) = (x, I) = \chi \in H_1 \tag{5.12}$$

$$dY(t) = -F(X(t), Y(t), Z(t))dt + Z(t)dW, \quad \kappa = Y(T) = (\eta, \zeta) \in H_1. \tag{5.13}$$

Here $F = (f, G) \in H_1$,

$$f(x, y, p, q) = g(x, y, p, q) + g_2(x, y, p, q)p + g_3(x, y, p, q)q,$$

$G(x, y, p, q) = g_1(x, y, p, q)p$, $W(t) = (w(t), w(t)) \in R^d \times R^d$, $Y(t) \in H_1$, $Z(t) \in H_2 = M \times M_1$ are $\mathcal{F}_t$-measurable random processes, $Y(T) \in \mathcal{M}^2([0, T]; H_1)$ and $Z(T) \in \mathcal{M}^2([0, T]; H_2)$, $b(x, (u, p), (p, q)) = a(x, u, p, q)$.

We say that condition **C 5.2** holds if a random function

$$G(t, X, Y, Z) = F(X(t), Y, Z) \in H_1$$

satisfies condition **C 4.1**.

Let

$$\mathcal{H}_1 = \{Y(t) \in H_1 : E \sup_{t\in[0,T]} \|Y(t)\|^2 < \infty\},$$

$$\mathcal{H}_2 = \{Z(t) \in H_2 : E\int_0^T \|Z(t)\|^2 dt < \infty\}.$$

Denote by $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_1 \times \mathcal{H}_2$ and let $\|\cdot\|_{\mathcal{H}}$ denote the norm in $\mathcal{H}$, that is for $\alpha = (X, Y, Z) \in \mathcal{H}$

$$\|\alpha\|^2_{\mathcal{H}} = E\left[\sup_{[0,T]} \|X(t)\|^2 + \sup_{[0,T]} \|Y(t)\|^2 + \int_0^T \|Z(t)\|^2 dt\right].$$

Denote by $D = H_1 \times H_1 \times H_2$ and let

$$\Upsilon(\Theta) = (-F(\Theta), b(\Theta), B(\Theta)) \text{ for } \Theta = (X, Y, Z) \in D.$$

Let $\mathcal{M}^2(0, T; D)$ denote the set of all D-valued $\mathcal{F}_t$ progressively measurable processes $\Theta(t)$ and $\mathcal{D} = \mathcal{M}^2(0, T; D) \cap \mathcal{H}$.

Assume that there exists a constant $C > 0$ such that $\Upsilon : D \to D$ and $u_0 : R^d \to R^d$ satisfy the estimates

$$\|\Upsilon(\Theta) - \Upsilon(\Theta_1)\|_D \leq C\|\Theta - \Theta_1\|_D, \quad \forall \Theta, \Theta_1 \in D, \quad P - \text{a.e.}$$

$$\|V_0(X) - V_0(X_1)\| \leq C\|X - X_1\|, \quad \forall X, X_1 \in H_1 \quad P - \text{a.e.}.$$

We say in this case that condition **C 5.3** holds.

We say that condition **C 5.4** holds if there exist a positive constant C_1, $C_2 > 0$ such that

$$\langle\langle \Upsilon(\Theta) - \Upsilon(\Theta_1), \Theta - \Theta_1 \rangle\rangle \leq -C_1\|X - X_1\|^2, \quad \forall X, X_1 \in H_1, P - \text{a.e.},$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is an inner product in D and

$$\langle V_0(X) - V_0(X_1), X - X_1 \rangle \geq C_1\|X - X_1\|^2 \quad X, X_1 \in H_1, P - \text{a.e.}.$$

We say that condition **C 5.4'** holds if there exist positive constants C_2 such that

$$\langle\langle \Upsilon(\Theta) - \Upsilon(\Theta_1), \Theta - \Theta_1 \rangle\rangle \leq -C_1\|Y - Y_1\|^2, \quad \forall Y, Y_1 \in H_1, P - \text{a.e.},$$

and

$$\langle V_0(X) - V_0(X_1), x - x_1 \rangle \geq C_1\|X - X_1\|^2 \quad X, X_1 \in H_1, P - \text{a.e.}.$$

Set $\kappa = (\eta, \zeta) \in H_1$, $\chi = (x, I) \in H_1$, $Y(t) = (y(t), p(t)) \in H_1$, $Z(t) = (p(t), q(t)) \in H_2$, $F(X, Y, Z) = (f(X, Y, Z), G(X, Y, Z))$.

The solution of the FBSDE (5.12), (5.13) is a triple

$$\Theta(t) = (X(t), Y(t), Z(t)) \in \mathcal{D},$$

such that P-a.s.

$$X(t)=\chi+\int_0^t b(X(\theta),Y(\theta),Z(\theta))d\theta+\int_0^t B(X(\theta t),Y(\theta),Z(\theta))dW(\theta), \quad (5.14)$$

$$Y(t)=\kappa+\int_t^T F(X(\theta),Y(\theta),Z(\theta))d\theta-\int_t^T Z(\theta)dW(\theta), \quad (5.15)$$

where $\kappa \in H_1$ is an $\mathcal{F}_T$-measurable random variable.

First we note that the system can have at most one solution.

Lemma 5.1. *Let assumptions* **C 5.1**- **C 5.4** *hold . Then then there exists at most one solution of the system (5.14), (5.15)*

Proof. Assume on the contrary that there exist two solutions $\Theta_1=(X_1,Y_1,Z_1)$ and $\Theta_2=(X_2,Y_2,Z_2)$ of the system (5.14), (5.15). Denote $\bar{X}=X_1-X_2, \bar{Y}=Y_1-Y_2, \bar{Z}=Z_1-Z_2$ and let

$$\bar{b}=b(\Theta_1)-b(\Theta_2), \quad \bar{B}=B(\Theta_1)-B(\Theta_2), \quad \bar{F}=F(\Theta_1)-F(\Theta_2).$$

From assumption **C 5.2** we deduce that $\bar{X}(t),\bar{Y}(t)$ are continuous and the estimates $E\sup_{t\in[0,T]}\|X(t)\|^2<\infty$, $E\sup_{t\in[0,T]}\|Y(t)\|^2<\infty$ are valid.

Next by the Ito formula we deduce that

$$\langle\bar{Y}(T),\bar{X}(T)\rangle=\int_0^T \langle\langle\Upsilon(\Theta_1(t))-\Upsilon(\Theta_2(t)),\Theta_1(t)-\Theta_2(t)\rangle\rangle dt+$$

$$\int_0^T \langle\bar{X}(t),\bar{Z}(t)dW(t)\rangle+\int_0^T\langle\bar{Y}(t),\bar{B}dW(t)\rangle$$

and hence

$$E\langle V_0(X_1(T))-V_0(X_2(T),\bar{X}(T)\rangle=$$

$$E\left[\int_0^T \langle\langle\Upsilon(\Theta_1(t))-\Upsilon(\Theta_2(t)),\Theta_1(t)-\Theta_2(t)\rangle\rangle dt\right]$$

By assumptions **C 5.2- C 5.4** we deduce

$$C_2\|X_1(T)-X_2(T)\|^2\leq\langle V_0(X_1(T))-V_0(X_2(T),\bar{X}(T)\rangle\leq$$

$$-C_2E\int_0^T \|\Theta_1(t)-\Theta_2(t)\|^2dt$$

that yields $\Theta_1=\Theta_2$.

To construct a solution of the fully coupled FBSDE (5.14), (5.15) following the papers [11], [12] we apply the homotopy continuation method that reads as follows.

Assume that we are looking for a solution of an equation $g(x)=0, x\in R^d$. Include the equation $g(x)=0$ in the family of equations $\Phi(\mu,x)=0$, where

$\Phi(\mu,x)=\mu g(x)+(1-\mu)A(x-x_0)$ with a constant matrix A and construct the path connecting zeros of $\Phi(x,\mu)$ from the value $\mu=0$ where they are supposed to be known to the value $\mu=1$.

We construct a solution of (5.14), (5.15) as a homotopy continuation of a solution to a certain simple FBSDE which will be studied in lemma 5.1 below.

Lemma 5.2. *Assume* $(b^0,F^0,B^0)\in\mathcal{D}$, $\kappa^0\in L^2(\Omega,\mathcal{F}_T,P)$. *Then the linear FBSDE*

$$dX(t)=-[Y(t)-b^0(t)]dt-[Z(t)-B^0(t)]dw(t),\quad X(0)=\chi, \tag{5.16}$$

$$dY(t)=-[X(t)-F^0(t)]dt+Z(t)dw(t),\quad Y(T)=X(T)+\kappa,\quad 0\le t\le T. \tag{5.17}$$

has a unique adapted solution $(X,Y,Z)\in\mathcal{D}$.

Proof. Consider a BSDE

$$\hat Y(t)=\kappa-\int_t^T[\hat Y(\theta)+F^0(\theta)-b^0(\theta)]d\theta$$

$$-\int_t^T[2\hat Z(\theta)-B^0(\theta)]dW(\theta),\quad 0\le t\le T. \tag{5.18}$$

By theorem 4.1 we know that this equation has a unique adapted solution $(\hat Y,\hat Z)\in\mathcal{M}^2(H)\times\mathcal{M}^2(H_1)$.

Consider next the forward equation

$$\hat X(t)=\chi-\int_0^t[\hat X(\theta)+\hat Y(\theta)-b^0(\theta)]d\theta-\int_0^t[\hat Z(\theta)-B^0(\theta)]dw(\theta),\quad 0\le t\le T. \tag{5.19}$$

Setting $Y(t)=\hat Y(t)+\hat X(t), Z(t)=\hat Z(t)$ we can easily check that

$$dY(t)=d\hat Y(t)+d\hat X(t)=[-\hat Y(t)-F^0(t)+b^0(t)+\hat X(t)+\hat Y(t)-b^0(t)]dt+$$

$$[2\hat Z(t)-B^0(t)-\hat Z(t)+B^0(t)]dW(t)=-[F^0(t)-\hat X(t)]dt+\hat Z(t)dW(t)$$

and

$$d\hat X(t)=-[Y(t)-b^0(t)]dt-[\hat Z(t)-B^0(t)]dw(t).$$

Hence we see that $\hat X(t)=X(t)$ and the triple (X,Y,Z) solves (5.16)-(5.17). The uniqueness of the solution to (5.16)-(5.17) obviously results from the uniqueness of (5.18)-(5.19).

At the next step for $\mu\in[0,1]$ we consider

$$b^\mu(X,Y,Z)=(1-\mu)Y-\mu b(X,Y,Z),\quad B^\mu(X,Y,Z)=(1-\mu)z-\mu B(x,y,z),$$

$$F^\mu(X,Y,Z)=(1-\mu)X-\mu F(X,Y,Z),\quad V_0^\mu(X)=\mu V_0(X)+(1-\mu)X.$$

It results from lemma 5.1 that there exists a unique solution of the system

$$X(t) = \chi - \int_0^t [b^\mu(\Theta(\tau)) - b^0(\tau)]d\tau - \int_0^t [B^\mu(\Theta(\tau)) - B^0(\tau)]dW(\tau), \quad (5.20)$$

$$Y(t) = (V_0^\mu(X(T)) + \kappa^0) - \int_t^T [F^\mu(\Theta(\tau)) - F^0(\tau)]d\tau - \int_t^T Z(\tau)dW(\tau) \quad (5.21)$$

at least for $\mu = 0$.

Next we prove that if we know that there exists a solution to (5.20), (5.21) for some $\mu = \mu^0 \in [0, 1)$ then we can prove that there exists $\delta_0 > 0$ such that the solution to (5.20), (5.21) exists as well for $\mu = \mu^0 + \delta$ for each $\delta \in [0, \delta_0]$.

Let $\Upsilon^\mu(\Theta) = (-F^\mu(\Theta), b^\mu(\Theta), B^\mu(\Theta)) \in D$ for $\Theta = (X, Y, Z) \in D$.

Lemma 5.3. *Assume that* **C5.1-C 5.3** *and* **C5.4** *or* **C5.4'** *hold and given $\mu = \mu_0 \in [0, 1]$ the system (5.20), (5.21) has an adapted solution $\Theta^\mu = (X^\mu, Y^\mu, Z^\mu) \in \mathcal{D}$ for any $(b^\mu, B^\mu, F^\mu) \in \mathcal{D}$ Then there exists a constant $\delta_0 \in [0, 1)$ depending only on C_1, C_2 and T such that for any $\delta \in [0, \delta_0]$ and the system (5.20), (5.21) has an adapted solution $(X^\mu, Y^\mu, Z^\mu) \in \mathcal{D}$ for $\mu = \mu_0 + \delta$.*

Proof. Let us note first that

$$b^{\mu_0+\delta}(\Theta) = b^{\mu_0}(\Theta) + \delta(Y + b(\Theta)), B^{\mu_0+\delta}(\Theta) = B^{\mu_0}(\Theta) + \delta(Z + B(\Theta)),$$

$$F^{\mu_0+\delta}(\Theta) = b^{\mu_0}(\Theta) + \delta(X + F(\Theta)), V_0^{\mu_0+\delta}(X) = V_0^{\mu_0}(X) + \delta(-X + V_0(X)).$$

Next we set $\Theta_0 = (X_0, Y_0, Z_0) = 0$ and consider a system of successive approximations

$$X_{k+1}(t) = \chi + \int_0^t [b^{\mu_0}(\Theta_{k+1}(\theta)) + \delta[Y_k(\theta) + b(\Theta_k(\theta))] - b^0(\theta)]d\theta + \quad (5.22)$$

$$\int_0^t [B^{\mu_0}(\Theta_{k+1}(\theta)) + \delta[Z_k(\theta) + B(\Theta_k(\theta))] - B^0(\theta)]dW(\theta),$$

$$Y_{k+1}(t) = [V_0^{\mu_0}(X_{k+1}(T)) + \delta[(-X_k(T) + V_0(X_k(T)))] - \int_t^T F^{\mu_0}(\Theta_{k+1}(\theta))d\theta +$$

$$(5.23)$$

$$\int_t^T [\delta[X_k(\theta) + F(\Theta_k(\theta))] - F^0(\theta)]d\theta - \int_t^T Z_{k+1}(\theta)dW(\theta).$$

Applying the Ito formula to $\langle \bar{Y}_{k+1}(t), \bar{X}_{k+1}(t)\rangle$, where $\bar{Y}_{k+1} = Y_{k+1} - Y_k$, $\bar{X}_{k+1} = X_{k+1} - X_k$, we obtain

$$E\langle\langle V_0^{\mu_0}(X_{k+1}(T)) - V_0^{\mu_0}(X_k(T)), \bar{X}_{k+1}(T)\rangle\rangle =$$

$$\delta E\langle\langle \bar{X}_k(T) - (V_0(X_k(T)) - V_0(X_{k-1}(T))), \bar{X}_{k+1}(T)\rangle\rangle +$$

$$E\int_0^T \langle\langle \Upsilon^{\mu_0}(\Theta_{k+1}(\tau)) - \Upsilon^{\mu_0}(\Theta_k(\tau)), \bar{\Theta}_{k+1}(\tau)\rangle\rangle d\tau +$$
$$\delta E\int_0^T \langle\langle \Theta_k(\tau) + \Upsilon(\Theta_k(\tau)) - \Upsilon(\Theta_{k-1}(\tau)), \bar{\Theta}_{k+1}(\tau)\rangle\rangle d\tau.$$

Taking into account the estimates in **C 5.3**, **C 5.4** we deduce the inequality

$$E\|\bar{X}_{k+1}(T)\|^2 + E\int_0^T \|\bar{\Theta}_{k+1}(\tau)\|^2 d\tau \le \frac{\delta(1+C)}{C_2}\left[E\|\bar{X}_k(T)\|\|\bar{X}_{k+1}(T)\| + \int_0^T E[\|\bar{\Theta}_{k+1}(\theta)\|\|\bar{\Theta}_k(\theta)\|]d\theta\right], \tag{5.24}$$

where $C_2 = \min(1, C)$.

By the elementary inequality $ab \le \frac{a^2}{2\epsilon} + \frac{b^2\epsilon}{2}$ choosing $\epsilon = \frac{C_2}{\delta(1+C)}$ we get

$$E\|\bar{X}_{k+1}(T)\|^2 + E\int_0^T \|\bar{\Theta}_{k+1}(\tau)\|^2 d\tau \le$$
$$\left[\frac{\delta(1+C)}{C_2}\right]^2 \left[E\|X_k(T)\|^2 + E\int_0^T \|\Theta_k(t)\|^2 dt\right].$$

Given

$$\bar{X}_k(T) =$$
$$\int_0^T [b^{\mu_0}(\Theta_k(\theta)) - b^{\mu_0}(\Theta_{k-1}(\theta)) + \delta(\bar{Y}_{k-1}(\theta) + b(\Theta_{k-1}(\theta)) - b(\Theta_{k-2}(\theta))]d\theta +$$
$$\int_0^T [B^{\mu_0}(\Theta_k(\theta)) - B^{\mu_0}(\Theta_{k-1}(\theta)) + \delta(Z_{k-1}(\theta) + B(\Theta_{k-1}(\theta))) - B(\Theta_{k-2}(\theta))]dW(\theta)$$

by standard estimates taking into account **C 5.3** and **C 5.4** we show that there exists a constant K such that

$$E\|\bar{X}_k(T)\|^2 \le K\left[\int_0^T E[\|\bar{\Theta}_k(\theta))\|^2 + \|\bar{\Theta}_{k-1}(\theta))\|^2]d\theta\right].$$

Hence from this inequality and (5.24) we obtain

$$\int_0^T E\|\bar{\Theta}_{k+1}(\theta))\|^2 d\theta \le K_1\delta^2\left[\int_0^T E[\|\bar{\Theta}_k(\theta))\|^2 + \|\bar{\Theta}_{k-1}(\theta))\|^2]d\theta\right].$$

Finally we can find $\delta_0 \in (0,1)$ which depends only on C, C_1 and T such that when $0 < \delta \le \delta_0$

$$\int_0^T E\|\bar{\Theta}_{k+1}(\theta))\|^2 d\theta \le \frac{1}{4}\int_0^T E\|\bar{\Theta}_k(\theta))\|^2 d\theta + \frac{1}{8}\int_0^T E\|\bar{\Theta}_{k-1}(\theta))\|^2 d\theta \tag{5.25}$$

for any $k \geq 1$.

Note that given a real sequence $\{a_k\}_{k=1}^{\infty}$ of positive numbers satisfying the estimate

$$a_{k+1} \leq \frac{1}{4}a_k + \frac{1}{8}a_{k-1}, \quad k \geq 1,$$

we can deduce that there exists a positive constant L such that $a_k \leq L2^{-k}$. As a result we show that $\Theta_k(\theta)$ is a Cauchy sequence in $\mathcal{D}$ and denote its limit by $\Theta = (X, Y, Z)$. Now if we pass to the limit $k \to \infty$ in equations (5.22) - (5.23) we show that V solves (5.20), (5.21) for $\mu = \mu_0 + \delta$, where $0 < \delta \leq \delta_0$.

Lemmas 5.1 and 5.3 immediately yield the following result.

Theorem 5.4. *Let* **C 5.1** - **C 5.4** *hold, then there exists a unique adapted solution* (X, Y, Z) *of (5.11)-(5.12).*

Finally let us come back to PDEs.

Theorem 5.5. *Let* **C 5.1**-**C 5.4** *holds and* $\Theta = (X, Y, Z)$ *solves (5.14), (5.15) with* $\kappa = V(T, X(T)) = V_0(X(T))$*, then the function* $V(s, \chi) = Y(s)$ *is a continuous viscous solution of the Cauchy problem (5.7).*

Proof. Continuity of $V(s, \chi)$ is a consequence of the square mean continuity of the solution to (5.14)-(5.15).

To show that $V(s, \chi)$ is a viscosity solution we choose a function $\Phi(s, \chi) \in R^1$ belonging to the class $C^{1,2}$ and a point $(s, \chi) \in [0, T] \times H$ such that at (s, χ) the functions $V^i - \Phi$, $i = 1, \dots, d + d^2$ have a local maximum. Without loss of generality we assume that $V^i(s, \chi) = \Phi(s, \chi)$. It yields that at any stopping time τ we have

$$V^i(\tau, X_{s,\chi}(\tau)) - \Phi(\tau, X_{s,\chi}(\tau)) \leq 0, \quad l = 1, \dots d_1. \tag{5.26}$$

From the Ito formula we deduce

$$\Phi(\tau, X(\tau)) = \Phi(s, \chi) + \int_s^\tau [\Phi_\theta + \mathcal{B}\Phi](\theta, X(\theta))d\theta + \int_s^\tau \langle \nabla\Phi(\theta, X(\theta)), BdW(\theta)\rangle.$$

At the other hand due to martingale representation theorem we have a representation

$$Y^i(s) =$$

$$V_0^i(X(T)) - \int_s^T F^i(X(\theta), Y(\theta), Z^i(\theta))d\theta - \int_s^T \langle Z^i(\theta), dW(\theta)\rangle.$$

Since the solution $Y(t) = V(t, X(t))$ of the equation (5.15) is unique we deduce

$$V^i(s, \chi) = Y^i(s) =$$

$$Y^i(\tau) - \int_s^\tau F^i(X(\theta), Y(\theta), Z^i(\theta))d\theta - \int_s^\tau \langle Z^i(\theta), dW(\theta)\rangle =$$

$$V^i(\tau, X_{s,\chi}(\tau)) - \int_s^\tau F^i(X(\theta), Y(\theta), Z^i(\theta))d\theta - \int_s^\tau \langle Z^i(\theta), dW(\theta)\rangle.$$

Substituting the last relation into (5.26) we derive for $i = 1, \ldots, d + d^2$

$$0 \geq V^i(\tau, X_{s,\chi}(\tau)) - \Phi(\tau, X_{s,\chi}(\tau)) =$$

$$V^i(s,\chi) - \Phi(s,\chi) - \int_s^\tau [\frac{\partial\Phi}{\partial\theta} + \mathcal{B}\Phi](\theta, X_{s,\chi}(\theta))d\theta -$$

$$\int_s^\tau F^i(X_{s,\chi}(\theta), Y(\theta), Z^i(\theta))d\theta + \int_s^\tau \langle Z^i(\theta), dW(\theta)\rangle$$

$$- \int_s^\tau \langle \nabla\Phi(\theta, X_{s,\chi}(\theta)), BdW(\theta)\rangle.$$

Recall that by assumption $V^i(s,\chi) - \Phi(s,\chi) = 0$ and compute the expectation of the both sides of the last relation. As a result we obtain the inequality

$$E\left(\int_s^\tau \Lambda^i(\theta, X_{s,\chi}(\theta), Y(\theta), Z^i(\theta))d\theta\right) \geq 0, \tag{5.27}$$

where

$$\Lambda^i(s, X, Y, Z^i) = [\frac{\partial\Phi}{\partial\theta} + \mathcal{B}\Phi](s, X) + F^i(X, Y, Z^i).$$

To verify that $V(s,\chi)$ is a subsolution of (5.5) we have to show that

$$\Lambda^i(s, X, Y, Z^i) \geq 0.$$

Assume on the contrary that there exists $\delta_0 < 0$ such that $\Lambda^i(s, X, Y, Z^i) < \delta_0$ and set $\tau_1 = \inf\{\theta > s : \Lambda^i(\theta, X(\theta), Y(\theta), Z^i(\theta)) \geq \delta_0\} \wedge T$.

By definition inequality (5.27) holds for any stopping time and hence for the stopping time $\tau_1 > s$. Thus we come to a contradiction

$$0 > \delta_0 E(\tau_1 - s) \geq E\left(\int_s^{\tau_1} \Lambda^i(\theta, X(\theta), Y(\theta), Z^i(\theta))d\theta\right) \geq 0,$$

and hence $V(s,\chi)$ is a subsolution of (5.7). In a similar way we can check that $V(s,\chi)$ is a supersolution of (5.7) and hence is a viscosity solution of (5.7).

As a result we deduce that the first component u of the function

$$V(s,\chi) = (u(s,x), \nabla u(s,x))$$

is a viscosity solution of (5.1).

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