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SOME q-CONTINUED FRACTIONS OF ORDER TWENTY-TWO AND FORTY-FOUR, THETA-FUNCTION IDENTITIES AND APPLICATIONS TO PARTITION THEORY

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Abstract. Five continued fractions of orders twenty-two and forty-four are derived from Ramanujan's general continued fraction identity. Additionally, the corresponding theta functions for these continued fractions were obtained, along with several partition identities that were established using the colored partition of integers.

1. Introduction

One of Ramanujan's notable contributions lies in the area of q continued fractions. The most famous among them is Rogers-Ramanujan continued fraction $R(q)$. This concept was initially presented by Rogers [12] in 1894. In 1912, Ramanujan reexamined the continued fraction. He documented several explicit values of $R(q)$ in his notebooks, as well as in his first correspondence with Hardy, which were later validated by Watson and Ramanathan [14, 15]. For further information, one may refer [3, 5, 6] for more details. In 2017, Surekha [13] established modular relations related to continued fractions of order sixteen, drawing parallels to the Rogers-Ramanujan continued fraction. Following this, Saikia and Rajkhowa [8, 9] made significant contributions by developing continued fractions of various orders that resemble the Rogers-Ramanujan continued fractions and by formulating modular identities for these fractions. Additionally, they applied these concepts to develop color partition identities based on partition theory.

1.1. Preliminaries. Throughout this paper, we consider for any complex numbers a and q, define the q-product $(\delta;q)_\infty$ as

$$(\delta;q)_\infty := \prod_{t=0}^{\infty}(1-\delta q^t), \qquad |q|<1. \tag{1.1}$$

For simplicity, we often write

$$(\delta_1;q)_\infty(\delta_2;q)_\infty(\delta_3;q)_\infty...(\delta_m;q)_\infty = (\delta_1,\delta_2,\delta_3,...,\delta_m;q)_\infty.$$

The general theta function $f(l,m)$ interms of Ramanujan [4, p. 34] is defined as

$$f(l,m) = \sum_{t=-\infty}^{\infty} l^{t(t+1)/2} m^{t(t-1)/2}. \tag{1.2}$$

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The $f(l,m)$ interms of Jacobi's triple product identity [4, p. 35, Entry 19] can be stated as

$$f(l,m) = (-l;lm)_\infty(-m;lm)_\infty(lm;lm)_\infty = (-l,-m,lm;lm)_\infty. \tag{1.3}$$

The special cases of $f(l,m)$ are the theta-functions $\phi(q)$,$\psi(q)$ and $f(-q)$ [4, p. 36, Entry 22 (i)-(iii)] are given by

$$\phi(q) := f(q,q) = \sum_{t=-\infty}^{\infty} q^{t^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.4}$$

$$\psi(q) := f(q,q^3) = \sum_{t=0}^{\infty} q^{t(t+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.5}$$

$$f(-q) := f(-q,-q^2) = \sum_{t=-\infty}^{\infty} (-1)^t q^{t(3t-1)/2} = (q;q)_\infty. \tag{1.6}$$

After Ramanujan, define

$$\chi(q) = (-q;q^2)_\infty. \tag{1.7}$$

Ramanujan documented numerous continued fractions in his notebooks, with the most renowned being the Rogers-Ramanujan continued fraction of order 5. $R(q)$ defined by

$$R(q) := q^{1/5}\frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty} = q^{1/5}\frac{f(-q,-q^4)}{f(-q^2,-q^3)} = \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+...}}}}, \qquad |q|<1. \tag{1.8}$$

Ramanujan documented several general identities related to continued fractions in his notebook. One such identity that he noted is the following general continued fraction identity [4, p. 24, Entry 12]. Suppose that w,m and q are complex numbers with $|wm|<1$ and $|q|<1$, or that $w = m^{2t+1}$ for some integer t. Then,

$$\frac{(w^2q^3;q^4)_\infty(m^2q^3;q^4)_\infty}{(w^2q;q^4)_\infty(m^2q;q^4)_\infty} = \cfrac{1}{1-wm+\cfrac{(w-mq)(m-wq)}{(1-wm)(q^2+1)+\cfrac{(w-mq^3)(m-wq^3)}{(1-wm)(q^4+1)+...}}}. \tag{1.9}$$

By selecting specific values for w and m, along with an appropriate choice for q, it is possible to derive a q-continued fraction of a particular order that adheres to theta function identities similar to those associated with $R(q)$. This paper focuses on continued fractions of orders twenty-two and forty-four. By replacing q by $q^{11/2}$ in (1.9), setting $\{w = q^{1/4}, m = q^{21/4}\}$, $\{w = q^{3/4}, m = q^{19/4}\}$, $\{w = q^{5/4}, m = q^{17/4}\}$, $\{w = q^{7/4}, m = q^{15/4}\}$ and $\{w = q^{9/4}, m = q^{13/4}\}$ and simplifying using the results $\{(q^{27};q^{22})_\infty = (q^5;q^{22}_\infty)/(1-q^5)\}$, $\{(q^{26};q^{22})_\infty = (q^4;q^{22}_\infty)/(1-q^4)\}$, $\{(q^{25};q^{22})_\infty = (q^3;q^{22}_\infty)/(1-q^3)\}$, $\{(q^{24};q^{22})_\infty = (q^2;q^{22}_\infty)/(1-q^2)\}$, and

$\{(q^{23};q^{22})_\infty = (q;q^{22}_\infty)/(1-q)\}$, we obtain the following continued fractions of order twenty two respectively.

$$\begin{aligned} R_1(q) =& q^{1/4}\frac{f(-q^5,-q^{17})}{f(-q^6,-q^{16})} \\ =& \cfrac{q^{1/4}(1-q^5)}{(1-q^{11/2})+\cfrac{q^{11/2}(1-q^{1/2})(1-q^{21/2})}{(1-q^{11/2})(1+q^{11})+\cfrac{q^{11/2}(1-q^{23/2})(1-q^{43/2})}{(1-q^{11/2})(1+q^{22})+\dots}}}. \end{aligned} \tag{1.10}$$

$$\begin{aligned} R_2(q) =& q^{3/4}\frac{f(-q^4,-q^{18})}{f(-q^7,-q^{15})} \\ =& \cfrac{q^{3/4}(1-q^4)}{(1-q^{11/2})+\cfrac{q^{11/2}(1-q^{3/2})(1-q^{19/2})}{(1-q^{11/2})(1+q^{11})+\cfrac{q^{11/2}(1-q^{25/2})(1-q^{41/2})}{(1-q^{11/2})(1+q^{22})\dots}}}. \end{aligned} \tag{1.11}$$

$$\begin{aligned} R_3(q) =& q^{5/4}\frac{f(-q^3,-q^{19})}{f(-q^8,-q^{14})} \\ =& \cfrac{q^{5/4}(1-q^3)}{(1-q^{11/2})+\cfrac{q^{11/2}(1-q^{5/2})(1-q^{17/2})}{(1-q^{11/2})(1+q^{11})+\cfrac{q^{11/2}(1-q^{27/2})(1-q^{39/2})}{(1-q^{11/2})(1+q^{22})+\dots}}}. \end{aligned} \tag{1.12}$$

$$\begin{aligned} R_4(q) =& q^{7/4}\frac{f(-q^2,-q^{20})}{f(-q^9,-q^{13})} \\ =& \cfrac{q^{7/4}(1-q^2)}{(1-q^{11/2})+\cfrac{q^{11/2}(1-q^{7/2})(1-q^{15/2})}{(1-q^{11/2})(1+q^{11})+\cfrac{q^{11/2}(1-q^{29/2})(1-q^{37/2})}{(1-q^{11/2})(1+q^{22})+\dots}}}. \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} R_5(q) =& q^{9/4}\frac{f(-q,-q^{21})}{f(-q^{10},-q^{12})} \\ =& \cfrac{q^{9/4}(1-q)}{(1-q^{11/2})+\cfrac{q^{11/2}(1-q^{9/2})(1-q^{13/2})}{(1-q^{11/2})(1+q^{11})+\cfrac{q^{11/2}(1-q^{31/2})(1-q^{35/2})}{(1-q^{11/2})(1+q^{22})+...}}}. \end{aligned} \tag{1.14}$$

In the same way, we have the following five continued fractions of order forty four respectively from (1.9) given by

$$\begin{aligned} P_1(q) =& q^{5}\frac{f(-q,-q^{43})}{f(-q^{21},-q^{23})} \\ =& \cfrac{q^{5}(1-q)}{(1-q^{11})+\cfrac{q^{11}(1-q^{10})(1-q^{12})}{(1-q^{11})(1+q^{22})+\cfrac{q^{11}(1-q^{32})(1-q^{34})}{(1-q^{11})(1+q^{44})...}}}. \end{aligned} \tag{1.15}$$

$$\begin{aligned} P_2(q) =& q^{4}\frac{f(-q^3,-q^{41})}{f(-q^{19},-q^{25})} \\ =& \cfrac{q^{4}(1-q^3)}{(1-q^{11})+\cfrac{q^{11}(1-q^{8})(1-q^{14})}{(1-q^{11})(1+q^{22})+\cfrac{q^{11}(1-q^{30})(1-q^{36})}{(1-q^{11})(1+q^{44})...}}}. \end{aligned} \tag{1.16}$$

$$\begin{aligned} P_3(q) =& q^{3}\frac{f(-q^5,-q^{39})}{f(-q^{17},-q^{27})} \\ =& \cfrac{q^{3}(1-q^5)}{(1-q^{11})+\cfrac{q^{11}(1-q^{6})(1-q^{16})}{(1-q^{11})(1+q^{22})+\cfrac{q^{11}(1-q^{28})(1-q^{38})}{(1-q^{11})(1+q^{44})...}}}. \end{aligned} \tag{1.17}$$

$$\begin{aligned} P_4(q) =& q^{2}\frac{f(-q^7,-q^{37})}{f(-q^{15},-q^{29})} \\ =& \cfrac{q^{2}(1-q^7)}{(1-q^{11})+\cfrac{q^{11}(1-q^{4})(1-q^{18})}{(1-q^{11})(1+q^{22})+\cfrac{q^{11}(1-q^{26})(1-q^{40})}{(1-q^{11})(1+q^{44})...}}}. \end{aligned} \tag{1.18}$$

and

$$\begin{aligned} P_5(q) =& q\frac{f(-q^9,-q^{35})}{f(-q^{13},-q^{31})} \\ =& \cfrac{q(1-q^9)}{(1-q^{11})+\cfrac{q^{11}(1-q^2)(1-q^{20})}{(1-q^{11})(1+q^{22})+\cfrac{q^{11}(1-q^{24})(1-q^{42})}{(1-q^{11})(1+q^{44})...}}}. \end{aligned} \tag{1.19}$$

To obtain the above continued fraction, we replace q by q^{11} in (1.9), then set the values $\{w = q^5, m = q^6\}$,$\{w = q^4, m = q^7\}$, $\{w = q^3, m = q^8\}$, $\{w = q^2, m = q^9\}$, $\{w = q, m = q^{10}\}$ and then simplifying using the results $\{(q^{45}; q^{44})_\infty = (q; q^{44})_\infty/(1 - q)\}$, $\{(q^{47}; q^{44})_\infty = (q^3; q^{44})_\infty/(1 - q^3)\}$, $\{(q^{49}; q^{44})_\infty = (q^5; q^{44})_\infty/(1 - q^5)\}$, $\{(q^{51}; q^{44})_\infty = (q^7; q^{44})_\infty/(1 - q^7)\}$, $\{(q^{53}; q^{44})_\infty = (q^9; q^{44})_\infty/(1 - q^9)\}$, respectively. In Section 2, we prove some theta function identities for the above given continued fraction of order twenty two and forty four. Using colour partition of integers, we deduce some partition theoretic results from the theta function identities in Section 3.

2. Theta-function identities for $R_i(q)$ and $P_i(q)$

In this section, we prove some theta-function identities for the continued fractions $R_t(q)$ and $P_t(q)$ for $t = 1, 2, 3, 4$ and 5.

Theorem 2.1. *For $t = 1, 2, 3, 4$ and 5, We have*

$$\frac{1}{R_t(q)} \pm R_t(q) = \frac{\phi(\mp q^{11/2})f(\pm q^{(2t-1)/2}, \pm q^{(22-(2t-1))/2})}{q^{(2t-1)/4}f(-q^{5-(t-1)}, -q^{5+t})\psi(q^{11})}.$$

Proof. from (1.10), we obtain

$$\frac{1}{\sqrt{R_1(q)}} - \sqrt{R_1(q)} = \frac{f(-q^7, -q^{15}) - q^{3/4}f(-q^4, -q^{18})}{\sqrt{q^{3/4}f(-q^4, -q^{18})f(-q^7, -q^{15})}} \tag{2.1}$$

From [4, p.46, Entry 30(i) and (ii)], we have

$$f(l, m) = f(l^3m, lm^3) + lf(m/l, l^5m^3). \tag{2.2}$$

Setting $(l = -q^{1/4}, m = q^{21/4})$ and $(l = q^{1/4}, m = -q^{21/4})$ in (2.2) we obtain,

$$f(-q^{1/4}, q^{21/4}) = f(-q^6, -q^{16}) - q^{1/4}f(-q^5, -q^{17}) \tag{2.3}$$

and

$$f(q^{1/4}, -q^{21/4}) = f(-q^6, -q^{16}) + q^{1/4}f(-q^5, -q^{17}). \tag{2.4}$$

Employing (2.3) in (2.1), we find that

$$\frac{1}{\sqrt{R_1(q)}} - \sqrt{R_1(q)} = \frac{f(-q^{1/4}, q^{21/4})}{\sqrt{q^{1/4}f(-q^5, -q^{17})f(-q^6, -q^{16})}}. \tag{2.5}$$

Similarly, from (1.10) and applying (2.4), we deduce that

$$\frac{1}{\sqrt{R_1(q)}} + \sqrt{R_1(q)} = \frac{f(q^{1/4}, -q^{21/4})}{\sqrt{q^{1/4}f(-q^5, -q^{17})f(-q^6, -q^{16})}}. \tag{2.6}$$

Combining (2.5) and (2.6), we get

$$\frac{1}{R_1(q)} - R_1(q) = \frac{f(-q^{1/4}, q^{21/4})f(q^{1/4}, -q^{21/4})}{q^{1/4}f(-q^5, -q^{17})f(-q^6, -q^{16})}. \tag{2.7}$$

Again [4, p.46,Entry 30 (i),(iv)] , we note that

$$f(l, lm^2)f(m, l^2m) = f(l, m)\psi(lm) \tag{2.8}$$

and

$$f(l, m)f(-l, -m) = f(-l^2, -m^2)\phi(-lm). \tag{2.9}$$

Setting ($l = -q^5, m = -q^6$) in (2.8) and ($l = -q^{1/4}, m = q^{21/4}$) in (2.9), we obtain

$$f(-q^5, -q^{17})f(-q^6, -q^{16}) = f(-q^5, -q^6)\psi(q^{11}) \tag{2.10}$$

and

$$f(-q^{1/4}, q^{21/4})f(q^{1/4}, -q^{21/4}) = f(-q^{1/2}, -q^{21/2})\phi(q^{11/2}), \tag{2.11}$$

respectively. Employing (2.10) and (2.11) in (2.7), we complete the proof (i). Squaring (2.6), we obtain

$$\frac{1}{R_1(q)} + R_1(q) = \frac{f^2(q^{1/4}, -q^{21/4})}{q^{1/4}f(-q^5, -q^{17})f(-q^6, -q^{16})} - 2. \tag{2.12}$$

From [4, p.46,Entry 30 (v),(vi)], we note that

$$f^2(l, m) = f(l^2, m^2)\phi(lm) + 2af(m/l, l^3m)\psi(l^2m^2). \tag{2.13}$$

Setting ($l = q^{1/4}, m = -q^{21/4}$) in (2.13), we obtain

$$f^2(q^{1/4}, -q^{21/4}) = f(q^{1/2}, q^{21/2})\psi(-q^{11/2}) + 2q^{1/4}f(-q^5, -q^6)\psi(q^{11}). \tag{2.14}$$

Employing (2.14) and (2.10) in (2.12). By simplifying, we get the result. The proofs for the other identities are analogous to the above proof, thus we will not include them. □

Theorem 2.2. *We have,*

$$(i)\frac{1}{P_1(q)} \pm P_1(q) = \frac{\phi(\mp q^{11})f(\pm q^{10}, \pm q^{12})}{q^5\psi(q^{22})f(-q, -q^{21})},$$

$$(ii)\frac{1}{P_2(q)} \pm P_2(q) = \frac{\phi(\mp q^{11})f(\pm q^8, \pm q^{14})}{q^4\psi(q^{22})f(-q^3, -q^{19})},$$

$$(iii)\frac{1}{P_3(q)} \pm P_3(q) = \frac{\phi(\mp q^{11})f(\pm q^6, \pm q^{16})}{q^3\psi(q^{22})f(-q^5, -q^{17})},$$

$$(iv)\frac{1}{P_4(q)} \pm P_4(q) = \frac{\phi(\mp q^{11})f(\pm q^4, \pm q^{18})}{q^2\psi(q^{22})f(-q^7, -q^{15})}$$

and

$$(v)\frac{1}{P_5(q)} \pm P_5(q) = \frac{\phi(\mp q^{11})f(\pm q^2, \pm q^{20})}{q\psi(q^{22})f(-q^9, -q^{13})}.$$

Proof. Proofs of $(i) - (v)$ are identical proofs of Theorem (2.1), so omit the proofs. □

3. Applications to colored partitions

The theta-function identities derived in Theorems 2.1 and 2.2 can be used to derive some colour partition identities. First we give the defnition of colour partition of a positive integer n and its generating function.
"A partition of a positive integer n has l colors if there are l copies of n available and all of them are viewed as distinct objects. Partitions of a positive integer into parts with colors are colored partitions.

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty,$$

where r and s are positive integer and $r < t$. For example, $(q^{2\pm}; q^8)_\infty$ means $(q^2, q^6; q^8)_\infty$ which is $(q^2; q^8)_\infty\ (q^6; q^8)_\infty$.
For example, the colored partitions of 2 are given as 2, $1_r + 1_r$, $1_g + 1_g$ and $1_r + 1_g$. Where we use r (red) and g (green) to distinguish the two colors of 1. Also

$$\frac{1}{(q^a; q^b)^m_\infty},$$

is the generating function where all the parts are congruent to $a \pmod b$ and have m colors."

Theorem 3.1. *Let $X_1(n)$ indicate the number of ways to partition the integer. n divided into segments that are consistent with $\pm 3, \pm 8, \pm 19, \pm 22 \pmod{44}$ with the stipulation that the components congruent to ± 8 and $\pm 22 \pmod{44}$ have two distinct colors. Let $X_2(n)$ indicate the number of ways to partition the integer. n divided into segments that are consistent with ± 3, ± 14, ± 19 or $\pm 22 \pmod{44}$ such that parts congruent to ± 14 and $\pm 22 \pmod{44}$ have two distinct colors. Let $X_3(n)$ indicate the number of ways to partition the integer. n divided into segments that are consistent with ± 8, ± 11 and $\pm 14 \pmod{44}$ have two distinct colors.Then for any positive integer $n \geq 3$,*

$$X_1(n) - X_2(n-3) - X_3(n) = 0.$$

Proof. Employing (1.11), (1.4) and (1.5) and replacing q by q^2 we obtain,

$$\frac{(q^{14\pm}; q^{44})_\infty}{(q^{8\pm}; q^{44})_\infty} - q^3 \frac{(q^{8\pm}; q^{44})_\infty}{(q^{14\pm}; q^{44})_\infty} - \frac{(q^{3\pm}, q^{19\pm}; q^{44})_\infty (q^{22\pm}; q^{44})^2_\infty}{(q^{8\pm}, q^{14\pm}; q^{44})_\infty (q^{11\pm}; q^{44})^2_\infty} = 0. \tag{3.1}$$

Dividing by $(q^{3\pm}, q^{8\pm}, q^{14\pm}, q^{19\pm}; q^{44})_\infty (q^{22\pm}; q^{44})^2_\infty$,we obtain,

$$\frac{1}{(q^{8\pm}, q^{22\pm}; q^{44})^2_\infty (q^{3\pm}, q^{19\pm}; q^{44})_\infty} - q^3 \frac{1}{(q^{14\pm}, q^{22\pm}; q^{44})^2_\infty (q^{3\pm}, q^{19\pm}; q^{44})_\infty} - \frac{1}{(q^{8\pm}, q^{11\pm}, q^{14\pm}; q^{44})^2_\infty} = 0. \tag{3.2}$$

The quotients mentioned above serve as the generating functions for $X_1(n)$, $X_2(n)$ and $X_3(n)$, respectively. Hence, is equivalent to

$$\sum_{n=0}^{\infty} X_1(n) q^n - q \sum_{n=0}^{\infty} X_2(n-3) q^n - \sum_{n=0}^{\infty} X_3(n) q^n = 0, \tag{3.3}$$

where we set $X_1(0) = X_2(0) = X_3(0) = 1$. By equating the coefficients of q^n on both sides, we obtain the desired outcome.
Theorem 3.1 is illustrated in the table 1 below: □

TABLE 1. The case n=3 for the above theorem

$X_1(3) = 1$	$X_2(0) = 1$	$X_3(3) = 0$
3_r		

Theorem 3.2. *Let $Y_1(n)$ indicate the number of ways to partition the integer. n divided into segments that are consistent with ± 5,± 6,± 17 or $\pm 22 \pmod{44}$ with the stipulation that the components congruent to ± 6 and $\pm 22 \pmod{44}$ have two colors. Let $Y_2(n)$ indicate the number of ways to partition the integer. n divided into segments that are consistent with ± 5,± 16,± 17 or $\pm 22 \pmod{44}$ such that parts congruent to ± 16 and $\pm 22 \pmod{44}$ have two colors. Let $Y_3(n)$ indicate the number of ways to partition the integer. n divided into segments that are consistent with ± 6,± 11 and $\pm 16 \pmod{44}$ have two colors. Then for any positive integer $n \geq 5$,*

$$Y_1(n) - Y_2(n-5) - Y_3(n) = 0.$$

Proof. Employing (1.12), (1.4) and (1.5) and replacing q by q^2 we obtain,

$$\frac{(q^{16\pm}; q^{44})_\infty}{(q^{6\pm}; q^{44})_\infty} - q^5 \frac{(q^{6\pm}; q^{44})_\infty}{(q^{16\pm}; q^{44})_\infty} - \frac{(q^{5\pm}, q^{17\pm}; q^{44})_\infty (q^{22\pm}; q^{44})^2_\infty}{(q^{6\pm}, q^{16\pm}; q^{44})_\infty (q^{11\pm}; q^{44})^2_\infty} = 0. \tag{3.4}$$

Dividing by $(q^{5\pm}, q^{6\pm}, q^{16\pm}, q^{17\pm}; q^{44})_\infty (q^{22\pm}; q^{44})^2_\infty$,we obtain,

$$\frac{1}{(q^{6\pm}, q^{22\pm}; q^{44})^2_\infty (q^{5\pm}, q^{17\pm}; q^{44})_\infty} - q^5 \frac{1}{(q^{16\pm}, q^{22\pm}; q^{44})^2_\infty (q^{5\pm}, q^{17\pm}; q^{44})_\infty} - \frac{1}{(q^{6\pm}, q^{11\pm}, q^{16\pm}; q^{44})^2_\infty} = 0. \tag{3.5}$$

The quotients mentioned above serve as the generating functions for $Y_1(n), Y_2(n)$ and $Y_3(n)$, respectively. Hence, is equivalent to

$$\sum_{n=0}^{\infty} Y_1(n) q^n - q \sum_{n=0}^{\infty} Y_2(n-5) q^n - \sum_{n=0}^{\infty} Y_3(n) q^n = 0, \tag{3.6}$$

where we set $Y_1(0) = Y_2(0) = Y_3(0) = 1$. By equating the coefficients of q^n on both sides, we obtain the desired outcome.
Theorem 3.2 is illustrated in the table 2 below: □

TABLE 2. The case n=5 for the above theorem

$X_1(5) = 1$	$X_2(0) = 1$	$X_3(5) = 0$
3_r		

Conclusion: The other theta function identities can also be verified using the theory of partition. So omit the proof of other identities.

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