Global and Stochastic Analysis
Vol. 12 No. 3 (May, 2025)

Received: 10th January 2025 *Revised: 28th February 2025* *Accepted: 10th April 2025*

PRIME GRACEFUL LABELING IN DIRECTED GRAPHS AND APPLICATIONS: A CRYPTOGRAPHIC APPROACHAQ

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Abstract. This paper introduces prime graceful digraphs, a novel class of directed graphs with a labeling approach that integrates injectivity and prime-related constraints. For a digraph with p vertices and q edges, vertex labels are uniquely assigned from the set $\{1, 2, \ldots, k\}$, where $k = min(2p, 2q)$, and edge labels are determined by the absolute difference of vertex labels modulo $k + 1$. Additionally, adjacent vertex labels must have a greatest common divisor of 1, emphasizing prime-related properties.

We apply this framework to cryptographic systems by representing ciphertexts as labeled digraphs. Combining one-time pad encryption with prime graceful labeling and cipher graph visualization enhances the security and interpretability of encrypted messages. Through examples of diverse graph structures, this work links graph theory and number theory, offering insights for applications in combinatorial optimization, cryptography, and secure communication.

1. Introduction

Graph labeling is a well-established and continually evolving area of study within graph theory. One of the key concepts in graph labeling is graceful labeling, which was introduced by Rosa in 1967 [5].

A graceful labeling of a graph $G = (V, E)$ with p vertices and q edges is a one-to-one mapping γ of the vertex set $V(G)$ into the set $\{1, 2, \ldots, n\}$.This mapping satisfies the condition that for each edge $e = u, v \in E(G)$, the value $\gamma(e) = |\gamma(u) - \gamma(v)|$ is a one-to-one mapping of the edge set $E(G)$ onto the set $\{1, 2, \ldots, p\}$. A graph is termed graceful if it admits such a graceful labeling.

The concept of prime labeling, which ensures that the greatest common divisor (gcd) of the labels of adjacent vertices is 1, has been a topic of interest in graph theory since its introduction by Tout, Dabboucy, and Howalla in 1982[8]. we adapt the following known results:

Theorem 1.1[4] Alternating path(A$\overrightarrow{P_n}$) admits prime pair labeling.

Theorem 1.2[4] Alternating cycle (A$\overrightarrow{C_n}$) admits prime pair labeling.

Theorem 1.3[6] Directed cycle (A$\overrightarrow{C_n}(n \geq 3)$) admits indegree prime labeling.

Theorem 1.4[6] Instar $k_{1,n}$ $(n \geq 3)$ admits indegree Prime Labeling.

T.M.Selvarajan, R. Subramoniam[9] introduced the concept prime graceful graph.

2000 *Mathematics Subject Classification.* Primary 05C78; Secondary 94A60.

Key words and phrases. digraph, prime labeling, graceful labeling, prime graceful labeling, cryptography, cyper clue, cyper text.

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A graph G with p vertices and q edges is considered to have a prime graceful labeling if there exists a vertex labeling function γ,mapping the vertices of G to the set $\{1, 2, \ldots, k\}$ where $k = min(2p, 2q)$.This labeling satisfies the condition that gcd of $\gamma(v_i)$ and $\gamma(v_j)$ for adjacent vertices v_i and v_j is 1. Furthermore,an injective function γ',mapping the edges of G to $\{1, 2, \ldots, k-1\}$ is defined by $\gamma'(v_iv_j) = \gamma(v_i) - \gamma(v_j)$, ensuring all edge labels are distinct.

In this paper, we extend the concept of prime graceful labeling to directed graphs (digraphs) by introducing prime graceful digraphs using the above definition.

Prime graceful digraphs combine the principles of graceful labeling with conditions related to prime numbers, adding a new dimension to the study of digraphs.
By integrating this prime labeling constraint with graceful labeling, we define prime graceful digraphs as follows:

Definition 1.1:
A **prime graceful digraph** $D = (V, E)$ with p vertices and q edges is a one to one mapping γ of the vertex set $V(D)$ into the set $\{1, 2, \ldots, k\}$ where $k = min(2p, 2q)$ with the following properties:
a) For each edge $e = \overrightarrow{uv} \in E(D)$, The function $\gamma : E(D) \to \{1, 2, \ldots, k-1\}$ is defined as $\gamma(e) = (\gamma(u) - \gamma(v))mod(k+1)$, and all edge labels are distinct.
b) For every pair of vertices $u, v \in V(D)$, $gcd(\gamma(u), \gamma(v)) = 1$.

A graph is called prime graceful if it has a prime graceful labeling.

The various common digraphs, such as path digraphs, star digraphs, cycle digraphs, bistar digraphs, friendship digraphs, triangular snake digraphs, and complete bipartite digraphs, are shown to possess for certain orientation is prime graceful labeling. These classes of digraphs exhibit the versatility and broad applicability of prime graceful labeling techniques.

2. Prime Graceful digraph

For standard terminology and notations related to graph we refer [1],[2],[3] and [7].
In the following theorem, we demonstrate that path digraphs can be labeled in a prime graceful manner, thereby enhancing our understanding of graph labeling techniques and the unique characteristics of directed graphs.

Theorem 2.1. *The unidirectional path digraph $\overrightarrow{P_p}$ is prime graceful.*

Proof. Let $D = \overrightarrow{P_p}$ be a directed path digraph with p vertices and $q = p - 1$ edges. The vertices of the path are labeled $\alpha_1, \alpha_2, \ldots, \alpha_p$, and the directed edges are $e_1 = \overrightarrow{\alpha_1\alpha_2}$, $e_2 = \overrightarrow{\alpha_2\alpha_3}$,$\ldots$,$e_{p-1} = \overrightarrow{\alpha_{p-1}\alpha_p}$.

We define a mapping γ as a one-to-one function from the vertex set $V(D)$ to the set $\{1, 2, \ldots, k\}$, where $k = min(2p, 2q)$.
As $q = p - 1$,we have: $k = min(2p, 2(p-1)) = 2p - 2$.
Thus, the vertices of D must be mapped to distinct values in the set $\{1, 2, \ldots, 2q - 2\}$.
For each directed edge $e = \overrightarrow{uv} \in E(D)$, we define the edge label as $\gamma(e) = (\gamma(u) - \gamma(v)) mod(k+1)$.
In this case, $k + 1 = 2p - 1$. The function $\gamma(e)$ assigns distinct values to the edges by computing the absolute difference between the labels of the adjacent vertices and taking the modulus with respect to $2p - 1$.

Furthermore, the mapping γ must satisfy the condition that for any two vertices $u, v \in V(D)$, the greatest common divisor $gcd(\gamma(u), \gamma(v)) = 1$. This condition ensures that the labels assigned to the vertices of the path digraph are pairwise coprime (i.e., relatively prime).

To construct a valid labeling for the directed path digraph D, we assign labels to the vertices $\alpha_1, \alpha_2, \ldots, \alpha_p$.
For each directed edge $e_i = \overrightarrow{\alpha_i \alpha_{i+1}}$, we compute the edge label: $\alpha(e_i) = (\alpha(v_i) - \alpha(v_{i+1})) mod(2p - 1)$, ensure that all edge labels are distinct.

Therefore, by constructing a prime graceful labeling that by satisfies both the gcd condition and the distinct edge labeling condition, we conclude that a path digraph is indeed prime graceful. □

The following theorem demonstrates that directed cycle, denoted as $\overrightarrow{C}_p$ is prime graceful.

Theorem 2.2. *The unidirectional digraph $\overrightarrow{C}_p$ is prime graceful.*

Proof. Let $\overrightarrow{C}_p$ be a directed cycle digraph with p vertices labeled $v_1, v_2, \ldots, v_p$ and edges e_i=$\overrightarrow{v_i v_{i+1}}$ for $i = 1, 2, \ldots, p-1$, and the final edge $e_p = \overrightarrow{v_p v_1}$ closing the cycle.

To prove that $\overrightarrow{C}_p$ is prime graceful, we need to show that there exists a prime graceful labeling γ on the vertices of $\overrightarrow{C}_p$ that satisfies the following two conditions:

(1) We define the mapping γ that labels the vertex set $V(\overrightarrow{C}_p)$ as a one-to-one function: $\gamma : V(\overrightarrow{C}_p) \to \{1, 2, \ldots, k\}$ where $k = min(2p, 2q)$. For a directed cycle, $p = q$, so $k = 2p$.
(2) For each directed edge $e_i = \overrightarrow{v_i v_{i+1}}$, we define the incident function as: $\gamma(e_i) = (\gamma(v_i) - \gamma(v_{i+1})) mod(k+1)$.This ensures that all the values of $\gamma(e_i)$ are distinct, making the incident function is injective.
(3) Finally, we need to show that for all pairs of adjacent vertices $\overrightarrow{v_i v_{i+1}} \in V(\overrightarrow{C}_p)$, the greatest common divisor $gcd(\gamma(v_i), \gamma(v_{i+1})) = 1$, thereby satisfying the coprime condition.

Thus, The digraph $\overrightarrow{C}_p$ is prime graceful. □

Another important class of digraphs is the star digraph, denoted as $\overrightarrow{K}_{1,p}$. Star digraphs consist of a central vertex connected to p outer vertices or inner vertices with directed edges, forming a hierarchical structure often seen in communication and network models. The following theorem establishes that star digraphs also admit prime graceful labeling.

Theorem 2.3. *The star digraph $\overrightarrow{K}_{1,p}$ is prime graceful.*

Proof. Let $\{v_0, v_1, v_2, ..., v_{p-1}\}$ be the $p+1$ vertices of the star digraph with p edges.
In star digraph one vertex v_0 is adjacent with remaining p vertices.
We label the vertex v_0 with 1 and remaining with $\{2, 3, 4, \ldots, k\}$ where $k = min(2p, 2q) = k = min(2(p+1), 2p) = 2p$.
The gcd of adjacent vertices of each edge is 1.
Here we study the following cases:
1.Out-star The edge labels $1, 2, 3, ..., p$ are distinct with modulo $(k+1)$ where $k = 2p$.
2.In-star The edge labels $k, k-1, ..., p$ are distinct with modulo $(k+1)$ where $k = 2p$.
Hence $\overrightarrow{K}_{1,p}$ is prime graceful digraph. □

A Friendship digraphs, denoted as $\overrightarrow{F}_p$, are characterized by a unique structure where multiple triangles share a common vertex, known as the "friend." This digraph captures the essence of social networks, where one individual may have connections to several others. The following theorem demonstrates that friendship digraphs can also be labeled in a prime graceful.

Theorem 2.4. *The friendship digraph $\overrightarrow{F}_p$ is prime graceful.*

Proof. The friendship digraph $\overrightarrow{F}_p$ has $2(p+1)$ vertices and $3p$ edges.
In friendship digraph, one vertex of degree $2p$ is adjacent to the remaining $2p$ vertices, label the vertex of $2p$ with 1. Choose a vertex from each cycle C_3, label it with $2, 3, ..., p+1$ and label the remaining vertices with $p+2, p+3, ..., k$ where $k = min\{2(p+1), 3p\}$.
All the edge labels are distinct and gcd condition holds.
Hence $\overrightarrow{F}_p$ is prime graceful. □

A triangular snake digraph is a directed graph consisting of a sequence of connected triangles, where each subsequent triangle shares an edge with the previous one. This graph can be represented as a sequence of vertices and directed edges, where each edge is directed either within the triangles or between the triangles, forming a "snake-like" structure. The following proof discuses the prime graceful labeling of the triangular snake digraph.

Theorem 2.5. *The triangular snake digraph $\overrightarrow{T}_p$ is prime graceful.*

Proof. Let $\overrightarrow{T}_p$ represent a triangular snake digraph. The graph consists of $2p+1$ vertices and $3p$ directed edges.

To demonstrate that T_p is prime graceful, we construct a labeling that satisfies the prime graceful conditions as follows:

1. The digraph $\overrightarrow{T}_p$ contains $2p+1$ vertices. We label the vertices of the triangular snake digraph sequentially using integers from the set $1, 2, \ldots, k$, where $k = min(2n, 2m) = min(2(2p+1), 2(3p)) = 4p+2$. Each vertex v_i of T_p is assigned a unique label from this set, ensuring that no two vertices share the same label.
2. For each directed edge $e = \overrightarrow{v_i v_j}$, the edge label is defined as the absolute difference between the labels of its endpoints,modulo $k+1$. Specifically, $\gamma(e) = (\gamma(v_i) - \gamma(v_j)) mod(k+1)$, where $k = 4p+2$.

We must ensure that this edge labeling is injective, meaning each edge receives a unique label. Since the triangular snake structure is regular, with directed edges forming a repeating pattern of triangles, the injectivity of edge labels can be maintained by assigning distinct mod values within and between the triangles.

3. We must check that for every pair of adjacent vertices u and v, the greatest common divisor $gcd(\gamma(u), \gamma(v)) = 1$. Since the vertex labels are consecutive integers, they are coprime. That is, for any two adjacent vertices v_i and v_{i+1}, we have:
$gcd(\gamma(v_i), \gamma(v_{i+1})) = gcd(i, i+1) = 1$. Thus, the GCD condition is satisfied for all adjacent vertices in the triangular snake digraph.

Thus, the triangular snake digraph $\overrightarrow{T}_p$ is prime graceful because it satisfies all the conditions for prime graceful labeling. □

The complete bipartite digraph $\overrightarrow{K}_{m,n}$ consists of two sets of $m+n$ vertices, with every vertex in one set is connected to every vertex in the other. In the following proof, we demonstrate that $\overrightarrow{K}_{m,n}$ admits a prime graceful labeling by satisfying the injectivity and gcd conditions for both vertices and edges.

Theorem 2.6. *The Complete bipartite digraph $\overrightarrow{K}_{m,n}$ is prime graceful.*

Proof. Let $\overrightarrow{K}_{m,n}$ be the complete bipartite digraph with vertex sets
$U = u_1, u_2, u_3, \ldots, u_m$ and $V = v_1, v_2, v_3, \ldots, v_n$, where each u_i is connected to each v_j by directed edges. The digraph has $m+n$ vertices and the total number of edges is mn.
Label the vertices using integers from the set $\{1, 2, \ldots, k\}$, where $k = min(2(m+n), 2mn)$
For each directed edge $e = \overrightarrow{u_i v_j}$, define the edge label as
$\gamma(e) = (\gamma(u_i) - \gamma(v_j)) mod(k+1)$. This produces distinct edge labels since the

values are distinct under modulo $(k+1)$.

For adjacent vertices u_i and v_j, we check that $gcd(\gamma(u_i), \gamma(v_j)) = 1$, ensuring all vertex pairs are coprime.
Thus, $K_{m,n}$ satisfies the conditions of a prime graceful digraph. □

Theorem 2.7. *The Bi-Star $\overrightarrow{B}(m,n)$ is prime graceful.*

Proof. let $D = \overrightarrow{B}(m,n)$ represent the Bi-Star digraph, where $V(D) = \{v_i, u_j | 0 \leq i \leq m, 0 \leq j \leq n\}$ is the vertex set, with v_0 and u_0 are the apex vertices, and v_i, u_j as the pendent vertices. Total vertices are $p = m+n+2$.

The edge set is given by: $E(D) = \{\overrightarrow{v_0 v_i} | 1 \leq i \leq m\} \cup \{\overrightarrow{u_0 u_j} | 1 \leq j \leq n\} \cup \{\overrightarrow{u_0 v_0}\}$. Total edges are $q = m+n+1$.

Now, define a labeling function $f : |V(D) \cup E(D)| \rightarrow \{1,2,3,...,k\}$, where $k = min(2p, 2q)$ with $f(v_0) = 1, f(u_0) = 3$.

We label the vertices and edges as follows:

If all v_i's are labeled with even numbers and u_j's with odd numbers then $f(\overrightarrow{u_0 v_0}) = k+1$, $f(v_i) = 2i+1$ and $f(u_j) = 2m+2j$ $f(\overrightarrow{v_0 v_i}) = 2i, f(\overrightarrow{u_0 u_j}) = 2m+2j+1; i = 2,3,\ldots,m; j = 1,2,3,\ldots,n$

To prove that the Bi-Star is prime graceful labeling, we check the coprimeness of the vertex and edge labels. We verify that $f(\overrightarrow{uv})$ pair wise relatively prime labeling, using the following conditions
$f(v_0) \perp f(u_0)$, $f(v_0) \perp f(v_i)$,$f(u_0) \perp f(u_j)$, $f(v_0) \perp f(\overrightarrow{v_0 v_i})$, and $f(u_0) \perp f(\overrightarrow{u_0 u_j})$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.
Hence, Bi-Star is prime graceful digraph. □

Theorem 2.8. *The digraph $\overrightarrow{P}_n \cup \overrightarrow{C}_4$ is a prime graceful digraph.*

Proof. Let $\overrightarrow{P}_n$ represent the directed path graph with n vertices, and let $\overrightarrow{C}_4$ denote the directed cycle graph with 4 vertices.
The vertex set for P_n is $\{v_1, v_2, \ldots, v_n\}$ and vertex set for $\overrightarrow{C}_4$ is $\{u_1, u_2, u_3, u_4\}$. The combined set of
$\overrightarrow{P}_n \cup \overrightarrow{C}_4$ is $\{v_1, v_2, \ldots, v_n, u_1, u_2, u_3, u_4\}$.
We check that the vertex labels satisfy the coprimality conditions:

(1) For any two adjacent vertices in $\overrightarrow{P}_n$ (e.g., v_i and v_{i+1}): $gcd(f(v_i), f(v_{i+1})) = gcd(i, i+1) = 1$.
(2) For edges connecting u_i to u_{i+1} in C_4: $gcd(f(v_i), f(v_{i+1})) = 1$.

(3) Similarly, we check for other pairs as needed.

Since the labeling function f assigns distinct values to vertices and satisfies the coprimality conditions for all pairs of adjacent vertices and edges, we conclude that the digraph $\overrightarrow{P}_n \cup \overrightarrow{C}_4$ is a prime graceful digraph. $\square$

In today's digital world, secure communication is essential to protect sensitive information from cyber threats. Ensuring that only authorized individuals can access confidential data is critical across various sectors, including finance, government, and healthcare. One proven method of achieving unbreakable encryption is the One-Time Pad (OTP), a cipher that is theoretically secure when used correctly.

In [10], Jaya Shruthy V. N. et al. discussed the encryption process using graph labeling techniques. This article explores the integration of One-Time Pad encryption with digraph labeling methods, presenting an innovative approach to securing messages. Beyond traditional encryption methods, it investigates how digraph-based structures can represent ciphertexts, adding additional layers of complexity and security.

3. Secure Communication with One-Time Pad and Graph Labeling Integration

The integration of One-Time Pad encryption with graph labeling methods offers a powerful way to secure communications. This combination ensures that messages are not only encrypted but also visually represented through graphs, making it easier for the intended recipient to decipher the message. In this section, we will walk through the key steps in the process, starting with the transformation of the plaintext into numerical form, followed by the encryption process using the OTP. We will then explore how the ciphertext can be represented using graph structures and how cipher clues assist in decryption.

3.1. Transformation of Text to Numerical Code. To begin the encryption process, it is crucial that both the sender and the recipient have access to the same One-Time Pad (OTP). The first step in this encryption technique is to transform the plaintext message into a numerical sequence called the plaincode. This conversion is typically performed using a predefined checkerboard system, which is optimized for the English alphabet and common punctuation marks.

While many checkerboard systems can be used, the one employed in this method specifically maps each letter of the English alphabet and common symbols to a unique numeric value, ensuring that each character in the message can be represented numerically. For example, the letter 'a' might be mapped to the number 1, and 'b' to 60, and so on. Once the plaintext is converted into plaincode, the message can be encrypted. However, it is important to note that this transformation alone does not provide enough security. Without further encryption, the numerical representation of the message is still vulnerable to interception.

3.2. Encrypting the Numerical Code Using OTP. The plaincode is now ready to be encrypted using the One-Time Pad (OTP). The OTP method is a symmetric-key encryption system, meaning the same key is used for both encryption and decryption. To encrypt the plaincode, we combine each digit of the

	a 1	**e** 2	**i** 3	**o** 4	**u** 5	
b 60	**c** 61	**d** 62	**f** 63	**g** 64	**h** 65	**j** 66
k 67	**l** 68	**m** 69	**n** 70	**p** 71	**q** 72	**r** 73
s 74	**t** 75	**v** 76	**w** 77	**x** 78	**y** 79	**z** 80
0 81	**1** 82	**2** 83	**3** 84	**4** 85	**5** 86	**6** 87
7 88	**8** 89	**9** 90	**!** 91	**.** 92	**,** 93	**;** 94
	: 95	**” ”** 96	**( )** 97	**?** 98	**Spc** 99	

TABLE 1. Numerical mapping of English alphabet and symbols

plaincode with the corresponding digit from the OTP key using modular arithmetic (mod 10).

Before proceeding, the plaincode is grouped into blocks of six digits to facilitate the encryption process. If any group contains fewer than six digits, it is padded with additional spaces to complete the block. The OTP key is selected from a pre-prepared table of random numbers. The key is matched to the sequence of digits in the plaincode, and for each group, the corresponding digits from the OTP key are added to the plaincode digits. The result of this addition is then reduced modulo 10 to produce the ciphertext.

It is important to emphasize that the first group of the OTP key acts as a "key indicator." This key indicator identifies the OTP sheet being used but does not participate in the arithmetic operation itself. The inclusion of a key indicator ensures that both the sender and recipient are using the same OTP sheet. The resulting ciphertext, now a sequence of numbers, can be safely transmitted to the receiver.

762315	591027	483726	185903	736418	950732
214869	430816	902547	381674	728506	541869
849105	657320	923417	316875	104983	587294
430762	983620	152847	274581	563809	741690
358102	816473	904231	712690	325671	468920
539870	290674	783452	510938	120647	637258
894731	348917	253608	509183	146805	765493
514672	208341	729364	872905	193746	420985
356481	974312	568203	734895	267403	952681
134509	348760	473619	205873	782941	106253

TABLE 2. Random 6-Digit OTP Key Mapping

3.3. Graphical Representation of Ciphertext. Once the message has been encrypted, it can be visualized using a graph-based structure known as the Cipher Graph. This graph consists of nodes and directed edges, with each edge representing a label derived from the ciphertext. The digraph is constructed using predefined graph structures, such as Cycle Graphs, Friendship Graphs, Fan Graphs, and Star Graphs. These structures employ Prime Graceful Digraphs, which are graph constructions that allow labels to be assigned in a way that preserves mathematical properties beneficial for encryption.

The Cipher Graph allows the receiver to visually interpret the ciphertext, making it easier to decode when the proper cipher clues are provided. The receiver interprets the edge labels of the Cipher Graph based on the provided cipher clues, which describe how the labels map back to the numerical ciphertext.

3.4. Decoding Assistance Through Cipher Clues. Cipher clues are an essential component of the decryption process. These clues provide important information that allows the receiver to decipher the encrypted message. Along with the OTP key, the receiver receives a set of cipher clues, which might include:

(1) Abbreviations for frequently used words or symbols.
(2) Structural hints about the cipher digraph used (e.g., which digraph structure was employed).
(3) Information about the connectivity of vertices and edges within the digraph.
(4) Information about the modulo 10 operation: if 0 appears, take the cipher clue as *.

By using these clues, the receiver can accurately map the labels from the Cipher Graph back to the numerical ciphertext and reverse the encryption process.

3.5. Decrypting Ciphertext Back to Plaintext. The decryption process involves reversing the steps taken during encryption. To decrypt the ciphertext, the receiver subtracts the digits of the ciphertext from the corresponding OTP key digits, applying modular arithmetic (mod 10) in the process. After decryption, the resulting numerical sequence is grouped into sets of six digits, allowing the original plaintext to be reconstructed.

This process ensures that the message is securely encrypted during transmission and can only be decrypted by the intended recipient who has access to both the ciphertext and the cipher clues.

To better understand the application of the One-Time Pad (OTP) encryption and decryption process, we now present a case study. In this case study, we will walk through the encryption and decryption of a simple message using the OTP technique. The example will demonstrate how the message is transformed into a numerical sequence (plaincode), how it is encrypted with a randomly generated OTP key, and how the ciphertext is decrypted back to its original plaintext. This practical example will illustrate the effectiveness and security of the OTP method in real-world communication.

The decryption process involves reversing the steps taken during encryption. To decrypt the ciphertext, the receiver subtracts the digits of the ciphertext from the corresponding OTP key digits, applying modular arithmetic (mod 10) in the process. After decryption, the resulting numerical sequence is grouped into sets of six digits, allowing the original plaintext to be reconstructed.

This process ensures that the message is securely encrypted during transmission and can only be decrypted by the intended recipient who has access to both the ciphertext and the cipher clues.

Case Study: One-Time Pad Encryption Using Conversion Table

Let the plaintext be **The eagle has landed successfully.**, by converting the plaintext to plaincode using Table-1, we obtain the OTP encryption of the plaincode to ciphertext, as shown in the table below

Plain Code	KeyID	756529	921646	829965	174996	817062	262997	456161	274746	356868	799299
OTP Key	762315	591027	483726	185903	736418	950732	214869	430816	902547	381674	728506
Cipher Text	762315	247546	304362	904868	800304	767794	476756	886977	176283	637432	417795

Our cipher graph fig 1. is a combination of the Fan digraph $\overrightarrow{F_4}$ and the star digraph $\overrightarrow{K_{1,9}}$, both of which are prime graceful. The receiver uses prime graceful labeling on this combined cipher graph to determine the edge labels, with the aid of the cipher clue.

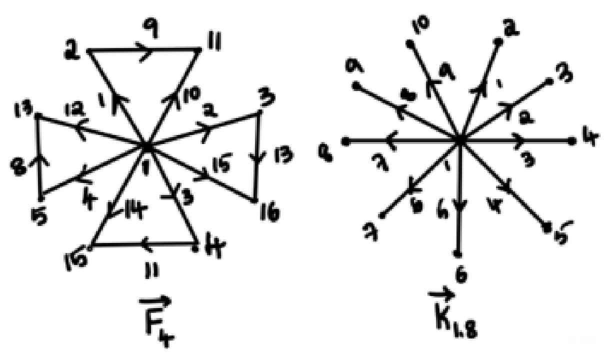

FIGURE 1. Combination of $\overrightarrow{F_4}$ and $\overrightarrow{K_{1,9}}$ for message transmission to the receiver

Using the cipher clue given to the receiver, the ciphertext can be determined.

$\overrightarrow{F_4^{1,3}}, \overrightarrow{K_{1,8}^{1,5}}, \overrightarrow{K_{1,8}^{1,8}}, \overrightarrow{K_{1,8}^{1,6}}, \overrightarrow{F_4^{1,5}}, \overrightarrow{K_{1,8}^{1,7}}$	$\overrightarrow{K_{1,8}^{1,7}}, *, \overrightarrow{K_{1,8}^{1,7}}, \overrightarrow{F_4^{2,11}}, \overrightarrow{K_{1,8}^{1,8}}, *, \overrightarrow{F_4^{5,13}}$	$\overrightarrow{K_{1,8}^{1,4}}, *, \overrightarrow{F_4^{1,5}}, \overrightarrow{K_{1,8}^{1,4}}, \overrightarrow{K_{1,8}^{1,7}}, \overrightarrow{F_4^{1,3}}$
$\overrightarrow{F_4^{2,11}}, *, \overrightarrow{K_{1,8}^{1,5}}, \overrightarrow{F_4^{5,13}}, \overrightarrow{K_{1,8}^{1,7}}, \overrightarrow{F_4^{5,13}}$	$\overrightarrow{F_4^{5,13}}, *, *, \overrightarrow{K_{1,8}^{1,3}}, *, \overrightarrow{F_4^{1,5}}$	$\overrightarrow{K_{1,8}^{1,8}}, \overrightarrow{K_{1,8}^{1,7}}, \overrightarrow{K_{1,8}^{1,8}}, \overrightarrow{K_{1,8}^{1,8}}, \overrightarrow{F_4^{2,11}}, \overrightarrow{K_{1,8}^{1,5}}$
$\overrightarrow{F_4^{1,5}}, \overrightarrow{K_{1,8}^{1,8}}, \overrightarrow{K_{1,8}^{1,7}}, \overrightarrow{K_{1,8}^{1,8}}, \overrightarrow{K_{1,8}^{1,6}}, \overrightarrow{K_{1,8}^{1,7}}$	$\overrightarrow{F_4^{5,13}}, \overrightarrow{K_{1,8}^{1,9}}, \overrightarrow{K_{1,8}^{1,7}}, \overrightarrow{F_4^{2,11}}, \overrightarrow{K_{1,8}^{1,8}}, \overrightarrow{K_{1,8}^{1,8}}$	$\overrightarrow{F_4^{1,2}}, \overrightarrow{K_{1,8}^{1,8}}, \overrightarrow{K_{1,8}^{1,7}}, \overrightarrow{F_4^{1,3}}, \overrightarrow{F_4^{5,13}}, \overrightarrow{K_{1,8}^{1,4}}$
$\overrightarrow{K_{1,8}^{1,7}}, \overrightarrow{F_4^{1,4}}, \overrightarrow{K_{1,8}^{1,8}}, \overrightarrow{F_4^{1,5}}, \overrightarrow{K_{1,8}^{1,4}}, \overrightarrow{F_4^{1,3}}$	$\overrightarrow{F_4^{1,5}}, \overrightarrow{K_{1,8}^{1,2}}, \overrightarrow{K_{1,8}^{1,7}}, \overrightarrow{K_{1,8}^{1,7}}, \overrightarrow{F_4^{2,11}}, \overrightarrow{K_{1,8}^{1,6}}$	

Here, $\overrightarrow{F^{i,j}}$ represents the Friendship graph, and $\overrightarrow{K^{i,j}}$ denotes the Star graph, where (i, j) specifies the edge label connecting vertices i and j based on the prime graceful digraph labeling technique. Consequently, the ciphertext corresponding to these edge labels is as follows:

Cipher Text	762315	247546	304362	904868	800304	767794	476756	886977	176283	637432	417795

By subtracting the ciphertext from the OTP key and applying modulo 10, the plaincode is obtained.

Cipher Text	762315	247546	304362	904868	800304	767794	476756	886977	176283	637432	417795
OTP Key	762315	591027	483726	185903	736418	950732	214869	430816	902547	381674	728506
Plain Code	KeyID	756529	921646	829965	174996	817062	262997	456161	274746	356868	799299

By referring to Table 1, we can convert the Plaincode back into Plaintext.

The process begins with the first Plaincode value, which is 7. Since it is greater than 5, it is treated as a double-digit number, 75 (combined with the following digit, 5), corresponding to the letter 'T'. The next digit, 6, also exceeds 5, so it is combined with the subsequent digit to form 65, representing 'h'. A digit like 2, being less than 5, is taken as a single digit, which maps to 'e'. The code 99 denotes a space, and the sequence continues accordingly.

As a result, the Plaintext reads: The eagle has landed successfully.

4. conclusion

This paper introduces the concept of prime graceful digraphs, a novel approach to the labeling of directed graphs that bridges graph theory and number theory. Through a well-defined labeling function, we have assigned distinct vertex labels from a specific set and imposed prime-related constraints on adjacent vertex pairs. This framework paves the way for the identification of prime graceful digraphs, creating new opportunities for further exploration in graph theory.

The examples provided, such as path digraphs, star digraphs, cycle digraphs, bistar digraphs, friendship digraphs, triangular snake digraphs, and complete bipartite digraphs, illustrate the broad applicability of prime graceful labeling in various graph structures. These constructions demonstrate how prime graceful labeling provides a unique structure that can potentially enhance combinatorial optimization problems and contribute to the development of efficient cryptographic protocols.

In particular, the integration of prime graceful digraphs in cryptographic methods, such as one-time pad encryption and ciphertext graph representation, has the potential to improve security by introducing complex, prime-based relationships within cryptographic systems. Additionally, the encryption techniques explored in this work, such as the use of OTP (One-Time Pad) keys and graph-based representations of ciphertext, offer new possibilities for secure communication systems.

The versatility of prime graceful labeling and its potential in cryptography opens doors for future research. Specifically, it offers the possibility of developing more sophisticated encryption schemes that blend graph theory, number theory, and computational methods. Further studies could explore the development of efficient algorithms for prime graceful labeling, investigate new applications in secure communication, and provide a deeper understanding of the interplay between graph structures and cryptographic security.

Acknowledgment. I sincerely thank all the authors, the NMAM Institute of Technology Nitte Research Center, and Visvesvaraya Technological University, Belgaum, for their support.

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