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FEW MORE RELATIONS PERTAINING DIFFERENTIAL IDENTITIES AND h- FUNCTIONS

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ABSTRACT. Ramanujan's lost notebook contains a wealth of mathematical discoveries, many of which broaden our understanding of modular forms, special functions, and infinite series. Among the notable concepts introduced are the Ramanujan-type Eisenstein series, a category of infinite series that exhibit extraordinary properties and often reveal deep connections to modular forms and number theory. This work builds on Ramanujan's foundational contributions to unveil novel differential identities, creating a richer understanding of η-functions, h-functions, and their connections to modular forms and modern mathematics. By connecting the sum of the Ramanujan-type Eisenstein series to the Class one infinite series, we bridge the gap between the two representations. This creates a fundamental tool for mathematical analysis by enabling us to infer the convergence of one from the convergence of the other.

1. Introduction

Differential equations are crucial in applied mathematics, acting as fundamental tools that drive advancements across various fields, including medicine, engineering, physics, chemistry, and beyond. Differential equations involving theta functions, particularly Jacobi theta functions, are used in fluid mechanics to solve problems with periodicity or in scenarios requiring solutions that satisfy specific boundary conditions. These applications often arise in contexts where temperature, velocity, or pressure variations follow periodic patterns or where boundary constraints lead to elliptic differential equations. In honor of Ramanujan's contributions, we aim to develop ordinary differential equations by leveraging the derived relationships of Eisenstein series at various levels. Extensive research has been devoted to the ordinary differential equations satisfied by modular forms.

In his notebook, Ramanujan [1] focused significantly on Eisenstein series, particularly P, Q, and R, and presented many intriguing differential identities that involve infinite series and theta functions. A thorough study of h-functions and various modular equations for h has been carried out by M. S. M. Naika et al.[7]. In their research, S. Cooper and D. Ye [4] performed an extensive analysis of the h-function and showcased its application in the development of differential equations. Additionally, S. Cooper [3] has recorded specific relationships between Eisenstein series of different levels and h-functions. In recent work, H. C. Vidya and B.

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Ashwath Rao [5, 10] established sophisticated connections between Eisenstein series and theta functions. They utilized these relationships to derive differential identities and to evaluate specific convolutional identities. Furthermore, H. C. Vidya and B. R. Srivatsa Kumar [9], derived differential equations encompassing identities related to the η-function. They highlighted the significance of these equations in enabling the formulation of incomplete integrals involving η-functions. Moreover, on page 188 of his lost notebook [8], Ramanujan documented specific formulas that connect the class-one infinite series $T_{2r}(q)$, for $r= 1, 2, \ldots, 6$, with the Eisenstein series P, Q, and R. The primary proof of six formulas for $T_{2r}(q)$ was provided in a research paper by B. C. Berndt and A. J. Yee [2] . An alternative proof of these formulas was later presented in Liu's paper [6]. In this paper, we have extended the work, conducted by H. C. Vidya and Ashwath Rao B.[11]. They have supplied elegant relations connecting, the Eisenstein series, Cooper's r-functions (r_a, r_b, r_c), and Ramanujan's Class one infinite series and h- functions.

Section 2 is dedicated to presenting the preliminary findings that contribute to achieving the main objectives. The focus is on deriving and analyzing a series of differential identities associated with η-functions, which are modular forms with deep connections to number theory. Additionally, the paper investigates the properties and applications of the h-functions, expressed as infinite products, which serve as essential tools in linking different mathematical structures. In the initial part of the paper, we draw upon the foundational results established in [3] to further our understanding of the intricate relationship between Ramanujan-type Eisenstein series and h-functions. This involves carefully extracting relevant identities and properties from the referenced work and extending their implications within the framework of our study. The detailed derivation and analysis of these results are systematically presented in Section 3, laying the groundwork for exploring broader applications and deeper theoretical insights in the subsequent sections. The derived relations are then applied in Sections 4 and 5 to formulate intriguing differential equations involving h-functions and to establish connections between two elegant infinite series.

2. Preliminaries

Ramanujan provided the following definition of a general theta function in his notebook[1]. Let a, b and q, be any complex number with $|ab| < 1$,

$$f(a, b) := \sum_{i=-\infty}^{\infty} a^{i(i+1)/2} b^{i(i-1)/2}$$

$$= (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty,$$

where,

$$(a;q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i), \quad |q| < 1.$$

The following instances are perticular cases of theta functions as defined by Ramanujan [1]:

$$f(-q) := f(-q, -q^2) = \sum_{i=-\infty}^{\infty} (-1)^i q^{i(3i-1)/2} = (q;q)_\infty = q^{-1/24}\eta(\tau), \qquad (2.1)$$

where $q = e^{2\pi i\tau}$. We denote $f(-q^n) = f_n$.

Definition 2.1. Cooper [6] documents certain relations involving η-functions in his notebook, as described below.

$$s_a(q) = \frac{\eta_3^3\eta_4}{\eta_1\eta_{12}^3}, \quad s_b(q) = \frac{\eta_2^2\eta_6^4}{\eta_1\eta_3\eta_{12}^4}.$$

Definition 2.2. Ramanujan[8] presented the Class one infinite series,

$$T_{2k}(q) := 1 + \sum_{i=1}^{\infty} (-1)^i \left[(6i-1)^{2k} q^{\frac{i(3i-1)}{2}} + (6i+1)^{2k} q^{\frac{i(3i+1)}{2}}\right],$$

and expressed $T_{2k}(q)$ for $k = 1, 2, ..., 6$ in terms of Ramanujan-type Eisenstein series:

$$P(q) := 1 - 24\sum_{j=1}^{\infty} \frac{jq^j}{1-q^j}.$$

Throughout, we denote $P(q^n) = P_n$.
In addition, a fascinating relationship was made by B. C. Berndt [2],

$$T_2(q) = (q;q)_\infty P(q). \qquad (2.2)$$

Definition 2.3. [3] For $|q| < 1$, *the h-function is defined by*

$$h := h(q) := q\prod_{i=1}^{\infty} \frac{(1-q^{12i-1})(1-q^{12i-11})}{(1-q^{12i-5})(1-q^{12i-7})}.$$

The weight two modular form y_{12} in terms of h-function is defined by

$$y_{12} = q\frac{d}{dq} log h = 1 - \sum_{s=1}^{\infty} \chi_{12}(s)\frac{sq^s}{1-q^s},$$

where

$$\chi_{12}(s) = \begin{cases} 1 & \text{if } s = 1 \text{ or } 11 (\text{mod} 12), \\ -1 & \text{if } s = 5 \text{ or } 7 (\text{mod} 12), \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.4 [3] The relation among an infinite series and *h-functions hold:*

$$\begin{pmatrix} P(q) \\ P(q^2) \\ P(q^3) \\ p(q^4) \\ P(q^6) \\ P(q^{12}) \end{pmatrix} = \begin{pmatrix} 6 & 2 & 3 & -6 & 2 & 0 \\ 3 & 2 & 0 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 2 & -2 & \frac{5}{3} & \frac{2}{3} & 0 & \frac{2}{3} \\ \frac{3}{2} & \frac{5}{4} & -\frac{5}{3} & \frac{3}{4} & \frac{1}{2} & 0 \\ 1 & 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{6} & \frac{5}{12} & 0 & \frac{1}{6} \end{pmatrix} \begin{pmatrix} h\frac{dy_{12}}{dh} \\ \frac{(1-h^2)}{(1+h^2)} y_{12} \\ \frac{(1-h^2)}{(1-h+h^2)} y_{12} \\ \frac{(1-h^2)}{(1-4h+h^2)} y_{12} \\ \frac{(1-h^2)}{(1-2h+h^2)} y_{12} \\ \frac{(1-h^2)}{(1+2h+h^2)} y_{12} \end{pmatrix}.$$

3. Relations among Eisenstein series and h functions

Theorem 3.1. *The connections among the Ramanujan-type Eisenstein series and h-function holds:*

$$(i) \quad P_1 - 4P_2 + 3P_3 - 24P_6 + 48P_{12} = \frac{24uv}{u^2-4} y_{12},$$

$$(ii) \quad P_1 - 9P_3 - 4P_4 + 36P_{12} = \frac{24v}{u} y_{12},$$

$$(iii) \quad 3P_1 - 14P_2 - 3P_3 + 8P_4 + 6P_6 + 24P_{12} = \frac{24v}{u+2} y_{12},$$

$$(iv) \quad 3P_1 - 6P_2 - 9P_3 + 8P_4 + 54P_6 - 72P_{12} = -\frac{24v}{u-2} y_{12},$$

$$(v) \quad P_2 - 4P_4 - 3P_6 + 12P_{12} = \frac{6v}{u-1} y_{12},$$

$$(vi) \quad 3P_1 - 4P_2 - 3P_3 + 4P_4 + 12P_6 - 36P_{12} = -\frac{24v}{u-4} y_{12},$$

$$(vii) \quad -2P_1 + 4P_2 + 6P_3 + 4P_4 + 24P_6 - 84P_{12} = -\frac{48v(u^2-2)}{u(u^2-4)} y_{12},$$

$$(viii) \quad -2P_1 + 9P_2 - 4P_4 + 9P_6 - 36P_{12} = -\frac{6v(u-1)}{u^2-4} y_{12},$$

$$(ix) \quad -P_2 - 6P_3 + 4P_4 + 39P_6 - 60P_{12} = -\frac{6v(u+1)}{u^2-4} y_{12},$$

$$(x) \quad -P_1 - 3P_3 + 16P_4 + 36P_6 - 96P_{12} = -\frac{12v(4u^2-u-8)}{(u-1)(u^2-4)} y_{12},$$

(xi) $P_1 - 3P_3 + 2P_4 + 18P_6 - 42P_{12} = -\frac{6v(3u^2 - 8u - 4)}{(u-4)(u^2-4)} y_{12},$

(xii) $-2P_1 + 7P_2 + 6P_3 - 2P_4 - 3P_6 - 30P_{12} = -\frac{12v(u+1)}{u(u+2)} y_{12},$

(xiii) $-P_2 + 2P_4 + 9P_6 - 18P_{12} = -\frac{8v(u-1)}{u(u-2)} y_{12},$

(xiv) $-P_1 - 4P_2 + 9P_3 + 20P_4 + 12P_6 - 84P_{12} = -\frac{48v(u-\frac{1}{2})}{u(u-1)} y_{12},$

(xv) $P_1 - 2P_2 + 3P_3 + 4P_4 + 6P_6 - 36P_{12} = -\frac{24v(u-2)}{u(u-4)} y_{12},$

(xvi) $P_1 - 4P_2 + 3P_3 - 24P_6 + 48P_{12} = \frac{24uv}{u^2-4} y_{12},$

(xvii) $-3P_1 + 10P_2 + 3P_3 + 8P_4 + 6P_6 - 72P_{12} = -\frac{24v(u+1)}{(u+1)(u+4)} y_{12},$

(xviii) $5P_2 - 2P_4 + 3P_6 - 30P_{12} = -\frac{24v(u+\frac{1}{2})}{(u-1)(u+2)} y_{12},$

(xix) $P_1 - 10P_2 - 9P_3 + 24P_4 + 66P_6 - 120P_{12} = -\frac{48v(u-\frac{1}{2})}{(u-1)(u-2)} y_{12},$

(xx) $-2P_1 - 5P_2 - 6P_3 + 6P_4 + 33P_6 - 54P_{12} = -\frac{24v(u-3)}{(u-2)(u-4)} y_{12},$

(xxi) $3P_1 - 8P_2 - 3P_3 + 20P_4 + 24P_6 - 840P_{12} = -\frac{24v(2u-5)}{(u-1)(u-4)} y_{12},$

(xxii) $P_1 - 2P_2 - 9P_3 + 18P_6 = -\frac{24v(u-4)}{u(u-2)} y_{12},$

(xxiii) $P_3 - 2P_4 - 9P_6 + 18P_{12} = \frac{4v}{u-2} y_{12},$

where, $h + \frac{1}{h}$=u and $-h + \frac{1}{h}$=v.

Proof. By applying Lemma 2.4 and performing the subsequent simplifications using Maple, we derive the necessary relationships. □

4. Relations connecting differential identities with h- functions

In the subsequent Theorems 4.1–4.4, we derive differential equations arising from Eisenstein series of various levels, which link h-functions with the derivative modular forms.

Theorem 4.1. *If*

$$X(q) = s_a(q),$$

then the following differential identity holds:

$$q\frac{dX}{dq} + \left[\frac{v}{u} y_{12}\right] X = 0,$$

where $h+\frac{1}{h}$=u, $-h+\frac{1}{h}$=v.

Proof. We note from equation (2.1) that, $X(q)$ may be reformulated in terms of η-functions given by

$$X(q)=\frac{\eta_3^3\eta_4}{\eta_1\eta_{12}^3}.$$

Rewriting the expression for $S(q)$ using theta-function,

$$X(q)=\frac{1}{q}\frac{f_3^3f_4}{f_1f_{12}^3}$$

Further expressing the above in terms of q-series notation and then logarithmically differentiating with respect to q, we reach the following outcome:

$$\frac{1}{X}\frac{dX}{dq}=-\frac{1}{q}+\frac{1}{q}\left[\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\sum_{r=1}^{\infty}\frac{9rq^{3r}}{1-q^{3r}}-\sum_{r=1}^{\infty}\frac{4rq^{4r}}{1-q^{4r}}+\sum_{r=1}^{\infty}\frac{36rq^{12r}}{1-q^{12r}}\right].$$

By utilizing the definition of the Eisenstein series and performing simplifications, we obtain the following result:

$$\frac{q}{X}\frac{dX}{dq}=\frac{1}{24}[-P_1+9P_3+4P_4-36P_{12}].$$

Finally, incorporating Lemma 2.4, and representing Eisenstein series in terms of h-functions, followed by simplification, we derive

$$\frac{q}{X}\frac{dX}{dq}=-\frac{(1-h^2)}{(1+h^2)}y_{12}.$$

Further, denoting $h+\frac{1}{h}$=u, $-h+\frac{1}{h}$=v and then simplifying, we derive the required expression.
By applying the same methodology used to derive the previously mentioned result, we can formulate the following set of differential equations:

(*i*) If $X(q)=s_b(q)=\dfrac{f_2^2f_6^4}{qf_1f_3f_{12}^4}$, then $q\dfrac{dX}{dq}+\left[\dfrac{uv}{(u^2-4)}y_{12}\right]X=0$,

(*ii*) If $X(q)=s_a(q)+2=\dfrac{f_2^7f_3}{qf_1^3f_4^2f_6f_{12}^2}$, then $q\dfrac{dX}{dq}+\left[\dfrac{v}{(u+2)}y_{12}\right]X=0$,

(*iii*) If $X(q)=s_a(q)-2=\dfrac{f_1f_4^2f_6^9}{qf_2^3f_3^3f_{12}^6}$, then $q\dfrac{dX}{dq}+\left[\dfrac{v}{(u-2)}y_{12}\right]X=0$,

(*iv*) If $X(q)=s_a(q)-1=\dfrac{f_4^4f_6^2}{qf_2^2f_{12}^4}$, then $q\dfrac{dX}{dq}-\left[\dfrac{v}{(u-1)}y_{12}\right]X=0$,

(*v*) If $X(q)=s_a(q)-4=\dfrac{f_1^3f_4f_6^2}{qf_2^2f_3f_{12}^3}$, then $q\dfrac{dX}{dq}+\left[\dfrac{v}{(u-4)}y_{12}\right]X=0$,

where $h+\frac{1}{h}$=u, $-h+\frac{1}{h}$=v. Therefore the proof is completed. □

Theorem 4.2. *If*

$$X(q)=s_a(q))s_b(q),$$

then the following differential identity holds:

$$q\frac{dX}{dq}+2\left[\frac{v(u^2-2)}{u(u^2-4)}y_{12}\right]X=0,$$

where $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$.

Proof. We note from 2.1, that, $X(q)$ may be reformulated in terms of η-functions given by

$$X(q)=\frac{1}{q^2}\frac{\eta_2^2\eta_3^2\eta_4\eta_6^4}{\eta_1^2\eta_{12}^7}.$$

Rewriting the expression for $S(q)$ using theta-function ,

$$X(q)=\frac{1}{q^2}\frac{f_2^2f_3^2f_4f_6^4}{f_1^2f_{12}^7}.$$

Further expressing the above by q-series notation and then by logarithmic differentiation with respect to q, we get the following result:

$$\frac{1}{X}\frac{dX}{dq}=-\frac{2}{q}+\frac{2}{q}\left[\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\sum_{r=1}^{\infty}\frac{2rq^{2r}}{1-q^{2r}}-\sum_{r=1}^{\infty}\frac{3rq^{3r}}{1-q^{3r}}-\sum_{r=1}^{\infty}\frac{2rq^{4r}}{1-q^{4r}}-\sum_{r=1}^{\infty}\frac{12rq^{6r}}{1-q^{6r}}+\sum_{r=1}^{\infty}\frac{42rq^{12r}}{1-q^{12r}}\right].$$

By utilizing the definition of the Eisenstein series and performing simplifications, we obtain the following result:

$$\frac{q}{X}\frac{dX}{dq}=\frac{1}{24}[-2P_1+4P_2+6P_3+4P_4+24P_6-84P_{12}].$$

Finally, incorporating Lemma 2.4, and expressing Eisenstein series in terms of h-functions, followed by simplification, we derive

$$\frac{q}{X}\frac{dX}{dq}=-\frac{48(1-h^2)(1+h^4)}{(1+h^2)(1-2h+h^2)(1+2h+h^2)}y_{12}.$$

Further, using the notation $h+\frac{1}{h}=u$, $-h+\frac{1}{h}=v$ and then simplifying, we derive expression (i).
Applying the same methodology used to derive the aforementioned result, we can construct the following set of differential equations:

For $h+\frac{1}{h}$=u, $-h+\frac{1}{h}$=v ,

(i) If $X(q)=s_a(q)(s_a(q)+2)=\dfrac{f_3^4f_2^7}{q^2f_1^4f_4f_6f_{12}^5}$,

then $q\dfrac{dX}{dq}+\left[\dfrac{2v(u+1)}{u(u+2)}y_{12}\right]X=0.$

(ii) If $X(q)=s_a(q)(s_a(q)-2)=\dfrac{f_3^3f_4^5f_6^2}{q^2f_1f_2^2f_{12}^7}$,

then $q\dfrac{dX}{dq}+\left[\dfrac{2v(u-\frac{1}{2})}{u(u-1)}y_{12}\right]X=0.$

(iii) If $X(q)=(s_a(q)-1)s_b(q)=\dfrac{f_4^4f_6^6}{q^2f_1f_3f_{12}^8}$,

then $q\dfrac{dX}{dq}+\left[\dfrac{4u^2-u-8}{2(u-1)(u^2-4)}y_{12}\right]X=0.$

(iv) If $X(q)=(s_a(q)-4)s_b(q)=\dfrac{f_1^2f_4f_6^6}{q^2f_3^2f_{12}^7}$,

then $q\dfrac{dX}{dq}+\left[\dfrac{v(3u^2-8u-4)}{2(u-4)(u^2-4)}y_{12}\right]X=0.$

(v) If $X(q)=s_a(q)(s_a(q)-4)=\dfrac{f_1^2f_3^2f_4^2f_6^2}{q^2f_2^2f_{12}^6}$,

then $q\dfrac{dX}{dq}+\left[\dfrac{2v(u-2)}{u(u-4)}y_{12}\right]X=0.$

(vi) If $X(q)=(s_a(q)+2)s_b(q)=\dfrac{f_2^9f_6^3}{q^2f_1^4f_4^2f_{12}^6}$, then

$q\dfrac{dX}{dq}+\left[\dfrac{4v(u-1)}{(u^2-4)}y_{12}\right]X=0.$

(vii) If $X(q)=(s_a(q)-2)s_b(q)=\dfrac{f_4^2f_6^{13}}{q^2f_2f_3^4f_{12}^{10}}$, then

$q\dfrac{dX}{dq}+\left[\dfrac{2v(u+1)}{(u^2-4)}y_{12}\right]X=0.$

Hence the proof. □

Theorem 4.3. *If*

$$X(q)=s_a^2(q)-4,$$

then the following differential identity holds:

$$q\frac{dX}{dq}+\left[\frac{2uv}{(u^2-4)}y_{12}\right]X=0.$$

where $h+\frac{1}{h}$=u, $-h+\frac{1}{h}$=v.

Proof. According to the information provided in [1], it appears that $S(q)$ can be expressed using theta functions as

$$S(q) = \frac{f_2^4 f_6^8}{q^2 f_1^2 f_3^2 f_{12}^8},$$

By reformulating the expression for $S(q)$ in q-series notation and subsequently logarithmically differentiating with respect to q, we arrive at the following result:

$$\frac{1}{S}\frac{dS}{dq} = -\frac{2}{q} + \frac{2}{q}\left[\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r} - \sum_{r=1}^{\infty}\frac{4rq^{2r}}{1-q^{2r}} + \sum_{r=1}^{\infty}\frac{3rq^{3r}}{1-q^{3r}} - \sum_{r=1}^{\infty}\frac{24rq^{6r}}{1-q^{6r}} + \sum_{r=1}^{\infty}\frac{48rq^{12r}}{1-q^{12r}}\right].$$

Upon applying the definition of the Eisenstein series and conducting simplifications, we arrive at the following outcome:

$$\frac{q}{X}\frac{dX}{dq} = \frac{1}{12}[-P_1 + 4P_2 - 3P_3 + 24P_6 - 48P_{12}].$$

Finally, incorporating Lemma 2.4 and representing Eisenstein series in terms of h-functions, followed by simplification, we derive expression (i).
Employing the methodology utilized in deriving the aforementioned results, we can formulate the following set of differential equations:

$$(i)\text{If } X(q) = (s_a(q)+2)(s_a(q)-1) = \frac{f_2^5 f_3 f_4^2 f_6}{q^2 f_1^3 f_{12}^6},$$

$$\text{then } q\frac{dX}{dq} + \left[\frac{v}{(u+2)}y_{12}\right]X = 0,$$

$$(ii)\text{If } X(q) = (s_a(q)+2)(s_a(q)-4) = \frac{f_2^5 f_6}{q^2 f_4 f_{12}^5},$$

$$\text{then } q\frac{dX}{dq} + \left[\frac{2v(u+1/2)}{(u-1)(u+2)}y_{12}\right]X = 0,$$

$$(iii)\text{If } X(q) = (s_a(q)-2)(s_a(q)-1) = \frac{f_1 f_4^6 f_6^{11} f_6}{q^2 f_3^3 f_{12}^{10}},$$

$$\text{then } q\frac{dX}{dq} + \left[\frac{2v(u-1/2)}{(u-1)(u-2)}y_{12}\right]X = 0,$$

$$(iv)\text{If } X(q) = (s_a(q)-2)(s_a(q)-4) = \frac{f_1 f_4^3 f_6^{11} f_6}{q^2 f_3^4 f_{12}^9},$$

$$\text{then } q\frac{dX}{dq} + \left[\frac{2v(u-3)}{(u-2)(u-4)}y_{12}\right]X = 0,$$

$$(v)\text{If } X(q) = (s_a(q)-1)(s_a(q)-4) = \frac{f_1^3 f_4^5 f_6^4}{q^2 f_2^4 f_3 f_{12}^7},$$

$$\text{then } q\frac{dX}{dq} + \left[\frac{v(2u-5)}{(u-1)(u-4)}y_{12}\right]X = 0,$$

where $h + \frac{1}{h} = u$, $-h + \frac{1}{h} = v$.

Therefore the proof is completed. □

Theorem 4.4. *If*

$$X(q) = \frac{s_a(q) - 2}{s_a^2(q)}$$

then the following differential identity holds:

$$q\frac{dX}{dq} - \left[\frac{v(u-4)}{u(u-2)}y_{12}\right]X = 0,$$

where $h + \frac{1}{h} = u$, $-h + \frac{1}{h} = v$.

Proof. Based on the details outlined in [1], it appears that the expression of $S(q)$ can be represented utilizing theta functions as,

$$X(q) = \frac{qf_1^3 f_6^9}{f_2^3 f_3^9}.$$

Through rephrasing the expression for $X(q)$ in q-series notation and subsequently applying logarithmic differentiation with respect to q, we obtain the following outcome:

$$\frac{1}{X}\frac{dX}{dq} = \frac{1}{q} + \frac{3}{q}\left[-\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r} + \sum_{r=1}^{\infty}\frac{2rq^{2r}}{1-q^{2r}} + \sum_{r=1}^{\infty}\frac{9rq^{3r}}{1-q^{3r}} - \sum_{r=1}^{\infty}\frac{18rq^{6r}}{1-q^{6r}}\right].$$

After employing the definition of the Eisenstein series and performing simplifications, we reach the following result:

$$\frac{q}{X}\frac{dX}{dq} = \frac{1}{8}[P_1 - 2P_2 - 9P_3 + 18P_6].$$

Finally, by applying Lemma 2.4 and expressing the Eisenstein series in terms of h-functions, further simplification leads to expression (i).
Using the same methodology applied in deriving the previous results, we can establish the following set of differential equations:
For $h + \frac{1}{h} = u$ and $-h + \frac{1}{h} = v$,

$$(i)\quad \text{If } X(q) = \frac{1}{s_a(q)(s_a(q)-2)} = \frac{q^2 f_2^3 f_{12}^9}{f_4^3 f_6^9}, \quad \text{then } q\frac{dX}{dq} - \left[\frac{v}{6(u-2)}y_{12}\right]X = 0.$$

Hence the proof. □

5. Relations among Class one infinite series and h-functions

H. C. Vidya and B. Ashwath Rao [5] developed a representation of the Eisenstein series using the classical Class one infinite series, inspired by the work of B. C. Berndt [2]. Using this representation, we derive intriguing formulas that connect the Class one infinite series to h-functions. Some of these formulas are presented in Theorem 5.2.

Lemma 5.1. [5]*For every positive integer $n \geq 2$, the following relationship holds between the two distinct series:*

$$P(q^n) = 1 + nq^{n-1}\left[\frac{T_2(q^n)+1}{(q^n;q^n)_\infty} - 1\right]. \tag{5.1}$$

Theorem 5.2. *The relationship between Class one infinite series and h-functions is presented below:*

$$i) - \frac{T_2(q)}{f_1} + 27q^2\frac{T_2(q^3)}{f_3} + 16q^3\frac{T_2(q^4)}{f_4} - 432q^{11}\frac{T_2(q^{12})}{f_{12}} - 27q^2\left(1-\frac{1}{f_3}\right) - 16q^3\left(1-\frac{1}{f_4}\right) + 432q^{11}\left(1-\frac{1}{f_{12}}\right) + \frac{24v}{u}y_{12} - 23 = 0,$$

$$ii) - \frac{T_2(q)}{f_1} + 8q\frac{T_2(q^2)}{f_2} - 9q^2\frac{T_2(q^3)}{f_3} + 144q^5\frac{T_2(q^6)}{f_6} - 576q^{11}\frac{T_2(q^{12})}{f_{12}} - 8q\left(1-\frac{1}{f_2}\right) + 9q^2\left(1-\frac{1}{f_3}\right) - 144q^5\left(1-\frac{1}{f_6}\right) + 576q^{11}\left(1-\frac{1}{f_{12}}\right) + \frac{24uv}{(u^2-4)}y_{12} - 23 = 0,$$

$$iii) - 3\frac{T_2(q)}{f_1} + 28q\frac{T_2(q^2)}{f_2} + 9q^2\frac{T_2(q^3)}{f_3} - 32q^3\frac{T_2(q^4)}{f_4} - 36q^5\frac{T_2(q^6)}{f_6} - 288q^{11}\frac{T_2(q^{12})}{f_{12}} - 28q\left(1-\frac{1}{f_2}\right) - 9q^2\left(1-\frac{1}{f_3}\right) + 32q^3\left(1-\frac{1}{f_4}\right) + 36q^5\left(1-\frac{1}{f_6}\right) + 288q^{11}\left(1-\frac{1}{f_{12}}\right) + \frac{24v}{(u+2)}y_{12} - 21 = 0,$$

$$iv)\frac{T_2(q)}{f_1} - 12q\frac{T_2(q^2)}{f_2} - 27q^2\frac{T_2(q^3)}{f_3} + 32q^3\frac{T_2(q^4)}{f_4} + 324q^5\frac{T_2(q^6)}{f_6} - 864q^{11}\frac{T_2(q^{12})}{f_{12}} + 12q\left(1-\frac{1}{f_2}\right) + 27q^2\left(1-\frac{1}{f_3}\right) - 32q^3\left(1-\frac{1}{f_4}\right) - 324q^5\left(1-\frac{1}{f_6}\right) + 864q^{11}\left(1-\frac{1}{f_{12}}\right) + \frac{24v}{(u-2)}y_{12} - 25 = 0,$$

$$v)2q\frac{T_2(q^2)}{f_2} - `16q^3\frac{T_2(q^4)}{f_4} + 18q^5\frac{T_2(q^6)}{f_6} - 144q^{11}\frac{T_2(q^{12})}{f_{12}} 2q\left(1-\frac{1}{f_2}\right) - 16q^3\left(1-\frac{1}{f_4}\right) - 18q^5\left(1-\frac{1}{f_6}\right) + 144^{11}\left(1-\frac{1}{f_{12}}\right) + \frac{6v}{(u-1)}y_{12} - 6 = 0,$$

$$vi)3\frac{T_2(q)}{f_1} - 8q\frac{T_2(q^2)}{f_2} - 9q^2\frac{T_2(q^3)}{f_3} + 16q^3\frac{T_2(q^4)}{f_4} + 72q^5\frac{T_2(q^6)}{f_6}$$

$$- 432q^{11}\frac{T_2(q^{12})}{f_{12}} + 8q\left(1-\frac{1}{f_2}\right) + 9q^2\left(1-\frac{1}{f_3}\right) - 16q^3\left(1-\frac{1}{f_4}\right)$$
$$- 72q^5\left(1-\frac{1}{f_6}\right) + 432q^{11}\left(1-\frac{1}{f_{12}}\right) + \frac{24v}{(u-4)}y_{12} - 27 = 0,$$

$$vii) - 2\frac{T_2(q)}{f_1} + 8q\frac{T_2(q^2)}{f_2} + 18q^2\frac{T_2(q^3)}{f_3} + 16q^3\frac{T_2(q^4)}{f_4} + 144q^5\frac{T_2(q^6)}{f_6}$$
$$- 1008q^{11}\frac{T_2(q^{12})}{f_{12}} - 8q\left(1-\frac{1}{f_2}\right) - 18q^2\left(1-\frac{1}{f_3}\right) - 16q^3\left(1-\frac{1}{f_4}\right)$$
$$- 144q^5\left(1-\frac{1}{f_6}\right) + 1008q^{11}\left(1-\frac{1}{f_{12}}\right) + \frac{48v(u^2-2)}{u(u^2-4)}y_{12} - 46 = 0,$$

$$viii) - 2\frac{T_2(q)}{f_1} + 18q\frac{T_2(q^2)}{f_2} - 16q^3\frac{T_2(q^4)}{f_4} + 54q^5\frac{T_2(q^6)}{f_6} - 432q^{11}\frac{T_2(q^{12})}{f_{12}}$$
$$- 18q\left(1-\frac{1}{f_2}\right) + 16q^3\left(1-\frac{1}{f_4}\right) - 54q^5\left(1-\frac{1}{f_6}\right)$$
$$+ 432q^{11}\left(1-\frac{1}{f_{12}}\right) + \frac{6v(u-1)}{(u^2-4)}y_{12} - 22 = 0,$$

$$ix) - 2q\frac{T_2(q^2)}{f_2} - 18q^2\frac{T_2(q^3)}{f_3} + 16q^3\frac{T_2(q^4)}{f_4} + 234q^5\frac{T_2(q^6)}{f_6} - 720q^{11}\frac{T_2(q^{12})}{f_{12}}$$
$$+ 2q\left(1-\frac{1}{f_2}\right) + 18q^2\left(1-\frac{1}{f_3}\right) - 16q^3\left(1-\frac{1}{f_4}\right) - 234q^5\left(1-\frac{1}{f_6}\right)$$
$$+ 720q^{11}\left(1-\frac{1}{f_{12}}\right) + \frac{6v(u+1)}{(u^2-4)}y_{12} - 24 = 0,$$

$$x) - \frac{T_2(q)}{f_1} - 9q^2\frac{T_2(q^3)}{f_3} + 64q^3\frac{T_2(q^4)}{f_4} + 234q^5\frac{T_2(q^6)}{f_6} - 1152q^{11}\frac{T_2(q^{12})}{f_{12}}$$
$$+ 9q^2\left(1-\frac{1}{f_3}\right) - 64q^3\left(1-\frac{1}{f_4}\right) - 234q^5\left(1-\frac{1}{f_6}\right)$$
$$+ 1152q^{11}\left(1-\frac{1}{f_{12}}\right) + \frac{12v(4u^2-u+8)}{(u-1)(u^2-4)}y_{12} - 47 = 0,$$

$$xi)\frac{T_2(q)}{f_1} - 9q^2\frac{T_2(q^3)}{f_3} + 4q^3\frac{T_2(q^4)}{f_4} + 108q^5\frac{T_2(q^6)}{f_6} - 504q^{11}\frac{T_2(q^{12})}{f_{12}}$$
$$+ 9q^2\left(1-\frac{1}{f_2}\right) - 4q^3\left(1-\frac{1}{f_4}\right) - 108q^5\left(1-\frac{1}{f_6}\right)$$
$$+ 504q^{11}\left(1-\frac{1}{f_{12}}\right) + \frac{6v(3u^2-8u-4)}{(u-4)(u^2-4)}y_{12} - 25 = 0,$$

$$xii) - 2\frac{T_2(q)}{f_1} + 14q\frac{T_2(q^2)}{f_2} + 18q^2\frac{T_2(q^3)}{f_3} - 8q^3\frac{T_2(q^4)}{f_4} - 18q^5\frac{T_2(q^6)}{f_6}$$
$$- 360q^{11}\frac{T_2(q^{12})}{f_{12}} - 14q\left(1 - \frac{1}{f_2}\right) - 18q^2\left(1 - \frac{1}{f_3}\right) + 8q^3\left(1 - \frac{1}{f_4}\right)$$
$$+ 18q^5\left(1 - \frac{1}{f_6}\right) + 360q^{11}\left(1 - \frac{1}{f_{12}}\right) + \frac{12v(u+2)}{u(u+2)}y_{12} - 22 = 0,$$

$$xiii) - 2q\frac{T_2(q^2)}{f_2} + 8q^3\frac{T_2(q^4)}{f_4} + 54q^5\frac{T_2(q^6)}{f_6} - 216q^{11}\frac{T_2(q^{12})}{f_{12}}$$
$$+ 2q\left(1 - \frac{1}{f_2}\right) + 8q^3\left(1 - \frac{1}{f_4}\right) - 54q^5\left(1 - \frac{1}{f_6}\right)$$
$$+ 216q^{11}\left(1 - \frac{1}{f_{12}}\right) + \frac{8v(u-1)}{u(u-2)}y_{12} - 8 = 0,$$

$$xiv) - \frac{T_2(q)}{f_1} - 8q\frac{T_2(q^2)}{f_2} + 27q^2\frac{T_2(q^3)}{f_3} + 80q^3\frac{T_2(q^4)}{f_4} - 72q^5\frac{T_2(q^6)}{f_6}$$
$$- 1008q^{11}\frac{T_2(q^{12})}{f_{12}} + 8q\left(1 - \frac{1}{f_2}\right) - 27q^2\left(1 - \frac{1}{f_3}\right) - 80q^3\left(1 - \frac{1}{f_4}\right)$$
$$+ 72q^5\left(1 - \frac{1}{f_6}\right) + 1008q^{11}\left(1 - \frac{1}{f_{12}}\right) + \frac{48v(u-1/2)}{u(u-1)}y_{12} - 47 = 0,$$

$$xv)\frac{T_2(q)}{f_1} - 4q\frac{T_2(q^2)}{f_2} + 9q^2\frac{T_2(q^3)}{f_3} + 16q^3\frac{T_2(q^4)}{f_4} + 36q^5\frac{T_2(q^6)}{f_6} - 432q^{11}\frac{T_2(q^{12})}{f_{12}}$$
$$+ 4q\left(1 - \frac{1}{f_2}\right) - 9q^2\left(1 - \frac{1}{f_3}\right) - 16q^3\left(1 - \frac{1}{f_4}\right) - 36q^5\left(1 - \frac{1}{f_6}\right)$$
$$+ 432q^{11}\left(1 - \frac{1}{f_{12}}\right) + \frac{24v(u-2)}{u(u-4)}y_{12} - 25 = 0,$$

where $h + \frac{1}{h} = u$, $-h + \frac{1}{h} = v$.

Proof. By applying equations (2.2) and (5.1) in the framework of Theorem 3.1 (i) through (xv), and simplifying the resulting expressions, we derive relationships for $P(q^n)$ in terms of $T_2(q^n)$ and h-functions, as described. □

6. Conclusions

The function $h = h(q)$ has recently been investigated by the researchers, who have developed continued fractions for the h-function and established various modular equations associated with it. Significantly, we succeeded in deriving several differential identities involving h-functions using Ramanujan-type Eisenstein series. These identities may be instrumental in developing new mathematical functions or constructing incomplete elliptic integrals.These new differential identities could lay the groundwork for further research and discoveries in mathematics. Furthermore, we successfully established a connection between two distinct series derived by Srinivasa Ramanujan. By connecting the Ramanujan-type series with a Class one infinite series, we demonstrated that the sum of the Class one infinite series converges to an infinite product involving h-functions. This work, by relating these series, enhances the understanding of Ramanujan's contributions and their connections to contemporary mathematical theories.

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