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BOUNDS ON NUMBER OF CUTVERTICES AND BLOCKS OF A GRAPH

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Abstract. In this paper it is observed that blocks behave like an edge of a graph with multiple vertices. This intutive notion of analogy between blocks and edges of a graph motivated to define block paths in a graph. It is observed that every graph is a block tree (B-Tree). Varieties of block-degrees and expressions for sum of block degrees are obtained. New graphs, semitotal-block-cutvertex graph, total-block-cutvertex graph and semitotal block-vertex-edge graph arising from the given graph are defined and expressions for the number of edges in the new graphs are derived. Several bounds for number of blocks and cutvertices in a graph are obtained. A new class of graphs called block regular graphs are introduced and their properties are studied.

1. Introduction

Throughout the discussion in this paper by a graph G, we mean a finite, undirected, simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The terminologies not presented here can be found in [6]. A vertex $v \in V$ is a *cutvertex* if $G - v$ is disconnected and such an edge is a *bridge*. G is *separable* if it has a cutvertex otherwise it is *nonseparable*. A maximal connected nonseparable subgraph is a block of G. A maximal complete subgraph is a *clique*. Let $B(G)$ and $C(G)$ denote the set of all blocks and cutvertices of G respectively. All through the discussion, we consider $|V(G)| = p$ and $|E(G)| = q$ called the order and size of the graph, while $|B(G)| = m$ and $|C(G)| = n$. If a block $b \in B(G)$ contains a cutvertex $c \in C(G)$ then we say that b and c incident to each other. Two blocks in G are adjacent if there is a common cutvertex incident on them. On the other hand, two cutvertices are adjacent if there is a common block incident on them. A block-graph $B_G(G)$ is a graph with vertex set $B(G)$ and any two vertices in $B_G(G)$ are adjacent if and only if the corresponding blocks are adjacent in G. Similarly, a cutvertex graph $C_G(G)$ is defined. Further, a block- cutvertex graph $BC(G)$ is a tree with vertex set $B(G) \cup C(G)$ and a cutvertex $c \in C(G)$ and a block $b \in B(G)$ are adjacent in $BC(G)$ if and only if c is incident on the block b. A block-vertex tree $b_p(G)$ as defined by V.R.Kulli [11] is a tree with vertex set $B(G) \cup V(G)$ and a vertex $v \in V(G)$ and a block $b \in B(G)$ are adjacent in $b_p(G)$ if and only if v is incident on the block b. It is observed that $B_G(B_G(G)) = C_G(G)$. A graph G is a block-graph of some graph if and only if every block is a clique in G. A similar

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characterization is true for a cutvertex graph also. We have observed that a block behaves like an edge of a graph with multiple vertices. This intuitive notion lead us to define block paths in a graph.

2. Block degree and cutvertex degree

Two vertices $u, w \in V$, are vv-adjacent if they incident on the same block. Then vv-degree of $u = d_{vv}(u)$, is the number of vertices vv-adjacent to u. Similarly, vb-degree (vertex-block degree) $d_{vb}(u)$ of a vertex u, is the number of blocks incident on u. For any noncutvertex $w \in V, d_{vb}(w) = 1$ and for any cutvertex $c \in C(G), d_{vb}(c) \geq 2$. bv-degree (block vertex-degree) of a block $f, d_{bv}(f)$ is the number of vertices in the block f and be-degree (block edge- degree) of a block $f, d_{be}(f)$ is the number of edges in the block f. bb-degree (block block-degree) of a block $h, d_{bb}(h)$ is the number of blocks adjacent to h. Finally, cutvertex degree of a block $b, d_c(b)$ is the number of cutvertices incident on b. Let $\triangle_{vv}(G)$ and $\delta_{vv}(G)$ denote the maximum and minimum vv-degrees of G respectively. Then $\triangle_{vb}(G), \delta_{vb}(G), \triangle_{bv}(G), \delta_{bv}(G), \triangle_{be}(G), \delta_{be}(G), \triangle_{bb}(G), \delta_{bb}(G), \triangle_c(G)$ and $\delta_c(G)$ are similarly defined. A block b is a pendant block if $d_c(b) = 1$. One can also find several new degree concepts in graphs defined and studied in [3, 8, 9]. The following results appear in [5, 7]. Those results in our terminologies read as follows.

Proposition 2.1. *For any graph G with m blocks and n cutvertices,*

$$\sum_{u \in V} (d_{vb}(u) - 1) = m - 1 \qquad \textit{Harary} \ [8] \tag{2.1}$$

$$\sum_{h \in B(G)} (d_c(h) - 1) = n - 1 \qquad \textit{Gallai} \ [6] \tag{2.2}$$

Corollary 2.2. *For any graph G with p vertices,*

$$\sum_{h \in B(G)} (d_{bv}(h) - 1) = p - 1 \tag{2.3}$$

Proof. Infact, $d_{bv}(h) = d_c(h) +$ number of noncutvertices of h. Noting that there are $p - n$ noncutvertices in any graph, we have $\sum_{h \in B(G)} (d_{bv}(h) - 1)$
$= \sum_{h \in B(G)} (d_c(h) - 1) + p - n = n - 1 + p - n = p - 1$ using equation (2.2). □

It is interesting to see that sum of vb-degree of all cutvertices and sum of cutvertex degree of all blocks are equal.

Theorem 2.3. *For any graph G with m blocks and n cutvertices,*

$$\sum_{c \in C(G)} d_{vb}(c) = \sum_{h \in B(G)} d_c(h) = m + n - 1 \tag{2.4}$$

$$\sum_{u \in V(G)} d_{vb}(u) = \sum_{h \in B(G)} d_{bv}(h) = p + m - 1 \tag{2.5}$$

Proof. For each noncutvertex $u, d_{vb}(u) = 1$. Therefore noncutvertices contribute null to the sum in (2.1) and hence equation (2.1) can be written as $m - 1 = \sum_{c\in C(G)}(d_{vb}(c) - 1)$. This yields $\sum_{c\in C(G)} d_{vb}(c) = m + n - 1$. Now from Gallai's result (2.2), we have $n - 1 = \sum_{h\in B(G)} d_c(h) - m$. Hence $\sum_{h\in B(G)} d_c(h) = m + n - 1$. Thus the result (2.4) is proved. Again from (2.1), $\sum_{u\in V}(d_{vb}(u) - 1) = \sum_{u\in V} d_{vb}(u) - p = m - 1$. Then $\sum_{u\in V} d_{vb}(u) = p + m - 1$. Also from (2.3), $\sum_{h\in B(G)}(d_{bv}(h) - 1) = \sum_{h\in B(G)} d_{bv}(h) - m = p - 1$. Therefore $\sum_{h\in B(G)} d_{bv}(h) = p + m - 1$ which yeilds the equation (2.5).

□

Corollary 2.4. *Let $B_P(G)$ and $B_{NP}(G)$ denote the set of all pendant and non-pendant blocks of G respectively. Let $m_p = |B_P(G)|$ and $m_{NP} = |B_{NP}(G)|$, so that $m = m_p + m_{NP}$. Then,*

$$\sum_{h\in B_{NP}(G)} d_c(h) = m_{NP} + n - 1 \tag{2.6}$$

Proof. Since each pendant block has only one cutvertex, each pendant block contributes one to the sum in (2.4). Therefore $\sum_{h\in B(G)} d_c(h) = \sum_{h\in B_{NP}(G)} d_c(h) + m_p = m+n-1$. This yields $\sum_{h\in B_{NP}(G)} d_c(h) = (m-m_p)+n-1 = m_{NP}+n-1$. □

In the next result we get a lower bound for sum of the squares of vb-degree of all cutvertices using well known Cauchy-Schwarz inequality,
$\sum_{i=1}^{n} |a_i b_i| \leq \sqrt{\sum_{i=1}^{n} |a_i|^2}\sqrt{\sum_{i=1}^{n} |b_i|^2}$ where a_i and b_i are integers.

Proposition 2.5. *For any graph G with m blocks and n cutvertices,*

$$\sum_{c\in C(G)} (d_{vb}(c))^2 \geq \frac{(m+n-1)^2}{n} \tag{2.7}$$

$$\sum_{h\in B(G)} (d_{bv}(h))^2 \geq \frac{(p+m-1)^2}{m} \tag{2.8}$$

Further, these bounds are sharp.

Proof. Taking $a_i = d_{vb}(c_i)$ and $b_i = 1$ in the above Cauchy-schwarz inequality, we get $(\sum_{i=1}^{n} d_{vb}(c_i))^2 \leq n\sum_{i=1}^{n}(d_{vb}(c))^2$. The equation (2.7) now follows from equation (2.4). The bound is attained for any B-path and B-complete graph. The equation can (2.8) be proved similarly and hence we omit the proof.

□

The line graph $L(G)$ of a graph G is a graph with vertex set as edges of G and any two vertices in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. It is well known (see [1]) that the sum of edge degrees of all edges of a graph G is given by $\sum_{x\in E(G)} d(x) = 2q_L \sum_{u\in V(G)}([d(u)]^2) - 2q$, where q_L is the number of edges in $L(G)$. Analogusly, we obtain an expression for the sum of bb-degree of all blocks in a graph.

Theorem 2.6. *Let G be any graph and q_b and q_c denote the number of edges in the block graph $B_G(G)$ and cutvertex graph $C_G(G)$. Then,*

$$\sum_{h\in B(G)} d_{bb}(h) = 2q_b = \sum_{c\in C(G)} [d_{vb}(c)]^2 - (m+n-1) \tag{2.9}$$

$$2q_c = \sum_{h\in B(G)} [d_c(h)]^2 - (m+n-1) \tag{2.10}$$

Proof. Since vertices of $B_G(G)$ are blocks of G, bb-degree of a block is the degree of the corresponding vertex in $B_G(G)$. Hence $2q_b = \sum_{u\in V(B_G(G))} d(u) = \sum_{h\in B(G)} d_{bb}(h)$. As every block in $B_G(G)$ is a clique, each cutvertex in G yield $\binom{d_{vb}(c)}{2}$ edges in $B_G(G)$. Then $q_b = \sum_{c\in C(G)} \binom{d_{vb}(c)}{2} = \frac{1}{2}\sum_{c\in C(G)}[(d_{vb}(c))^2 - d_{vb}(c)] = \frac{1}{2}[\sum_{c\in C(G)}[d_{vb}(c)]^2 - (m+n-1)]$ using equation (2.4). Then the result (2.9) follows. The proof of (2.10) is similar and hence we omit the proof. ☐

Remark 2.7. *As the edges of G can be partitioned in to blocks of G, it is immediate that* $\sum_{h\in B(G)} d_{be}(h) = q$.

2.1. Vertex graph and sum of vv-degree. Similar to cutvertex graph, we define *vertex graph* $P_G(G)$ of a graph, whose vertex set is $V(G)$ and any two vertices in $P_G(G)$ are adjacent if and only if they are vv-adjacent in G. We observe that $P_G(P_G(G)) = P_G(G)$. Interestingly, a vertex graph also admits same characterization as that of cutvertex graph. A graph G is a vertex graph of some graph if and only if every block of G is a clique in G. Also $P_G(G) \cong G$ if and only if every block of G is a clique in G.

Theorem 2.8. *Let G be any graph and q_p denote the number of edges in the vertex graph $P_G(G)$. Then,*

$$\sum_{u\in V(G)} d_{vv}(u) = 2q_p = \sum_{h\in B(G)} [d_{bv}(h)]^2 - (p+m-1) \tag{2.11}$$

Proof. Since vv-degree of a vertex is the degree of the corresponding vertex in $P_G(G)$, we have $\sum_{u\in V(G)} d_{vv}(u) = \sum_{u\in V(P_G(G))} d(u) = 2q_p$. As every block in $P_G(G)$ is a clique, each block in G yield $\binom{d_{bv}(h)}{2}$ edges in $P_G(G)$. Then $q_p = \sum_{h\in B(G)} \binom{d_{bv}(h)}{2} = \frac{1}{2}\sum_{h\in B(G)}[(d_{bv}(h))^2 - d_{bv}(h)] = \frac{1}{2}[\sum_{h\in B(G)}[d_{bv}(h)]^2 - (p+m-1)]$ using equation (2.5). Thus the equation (2.11) follows. ☐

The following result provides a lower bound for sum of bb-degree of all blocks when number of blocks and cutvertices are known.

Proposition 2.9. *For any graph G with m blocks and n cutvertices,*

$$\sum_{h\in B(G)} d_{bb}(h) \geq \frac{(m-1)(m+n-1)}{n} \tag{2.12}$$

$$\sum_{u\in V(G)} d_{vv}(u) \geq \frac{(p-1)(p+m-1)}{m} \tag{2.13}$$

Further, these bounds are sharp.

Proof. From Proposition 2.5 and Theorem 2.6, we have

$$\begin{aligned}\sum_{h\in B(G)} d_{bb}(h) \geq & \frac{(m+n-1)^2}{n} - (m+n-1)\\ &= \frac{m^2+n^2+1+2mn-2n-2m-mn-n^2+n}{n}\\ &= \frac{(m-1)^2+n(m-1)}{n} = \frac{(m-1)(m+n-1)}{n}\end{aligned}$$

The bound is sharp is evident from the fact that any B-path and B-complete graph attain the bound. The result (2.13) can be proved similarly. □

2.2. Bounds on number of blocks and cutvertices in a graph. If G has atleast one cutvertex then there exist at least two pendant blocks and hence $\delta_c(G)$ is always equal to 1. For such a graph, if u is a noncutvertex, then $d_{vb}(u) = 1$. Therefore $\delta_{vb}(G)$ is always equal to 1. Hence we introduce two parameters defined as $\delta_{cvb}(G) = min_{c\in C(G)}\{d_{vb}(c)\}$ and $\delta_{NPc}(G) = min_{h\in B_{NP}(G)}\{d_c(h)\}$.

It is well known that $\frac{p\delta}{2} \leq q \leq \frac{p\Delta}{2}$. Along these lines, the next results provide bounds for number of blocks and cutvertices.

Theorem 2.10. *For any graph G, with m blocks, m_{NP} nonpendant blocks and n cutvertices,*

$$\frac{q}{\Delta_{be}} \leq m \leq \frac{q}{\delta_{be}} \tag{2.14}$$

$$\frac{p-1}{\Delta_{bv}-1} \leq m \leq \frac{p-1}{\delta_{bv}-1} \tag{2.15}$$

$$\frac{n-1}{\Delta_c-1} \leq m_{NP} \leq \frac{n-1}{\delta_{NPc}-1} \tag{2.16}$$

$$\frac{m-1}{\Delta_{vb}-1} \leq n \leq \frac{m-1}{\delta_{cvb-1}} \tag{2.17}$$

Further, these bounds are sharp.

Proof. By Remark 2.7, $\sum_{h\in B(G)} d_{be}(h) = q$. Then $m\delta_{be} \leq \sum_{h\in B(G)} d_{be}(h) = q \leq m\Delta_{be}$. Hence
$\frac{q}{\Delta_{be}} \leq m \leq \frac{q}{\delta_{be}}$ follows. Again from Theorem 2.3, we have $m\delta_{bv} \leq \sum_{h\in B(G)} d_{bv}(h) = p+m-1 \leq m\Delta_{bv}$. This yields equation (2.15). Similarly from Corollary 2.4, we have $(m_{NP})(\delta_{NPc}) \leq \sum_{b\in B_{NP}(G)} d_c(b) = m_{NP}+n-1 \leq (m_{NP})\Delta_c$ which yields equation (2.16) .
Finally, from Theorem 2.3, we have $m\delta_{cvb} \leq \sum_{c\in C(G)} d_{vb}(c) = m+n-1 \leq n\Delta_{vb}$. This yields equation (2.17).

For any tree T with p vertices $m = q = p-1, \delta_{bv} = \Delta_{bv} = 2$ and $\delta_{be} = \Delta_{be} = 1$. Hence any tree attain both upper and lower bounds in (2.14) and (2.15). For any B-path BP_m with m blocks $\Delta_c = \delta_{NPc} = 2, \Delta_{vb} = \delta_{cvb} = 2, m_{NP} = m-2$ and $n = m-1$. Now one can verify that any B-path attains upper and lower bounds in (2.16) and (2.17). Thus the bounds are sharp.

□

3. Semitotal-block-cutvertex graph and Total-block-cutvertex graph

Semitotal graphs and Total graphs are well studied in [2, 12]. Further, semitotal block graph and total block graph are defined by V.R.Kulli [10]. On the similar lines, we define two new graphs arising from the given graph. The blocks and cutvertices of a graph are called its members. *Semitotal block-cutvertex graph* $T_{bc}(G)$ of a graph G is a graph with vertex set $B(G) \cup C(G)$ and any two vertices in $T_{bc}(G)$ are adjacent if and only if the corresponding cutvertices are adjacent or the corresponding members are incident. It is easy to note that $T_{bc}(G) = BC(G) \cup C_G(G)$. The *total block cutvertex graph* $T_{BC}(G)$ of a graph G is a graph with vertex set $B(G) \cup C(G)$ and any two vertices in $T_{BC}(G)$ are adjacent if and only if the corresponding members are adjacent or incident. Again we note that $T_{BC}(G) = BC(G) \cup C_G(G) \cup B_G(G)$.

We are now ready to detrmine the number of edges in the above newly defined graphs. In what follows by $q(G)$ we mean the number of edges in the corresponding graph G.

Theorem 3.1. *Let G be a graph with m blocks and n cutvertices. Let q_{bc} and q_{BC} denote the number of edges in $T_{bc}(G)$ and $T_{BC}(G)$ respectively. Then,*

$$q_{bc} = \frac{1}{2}[(m + n - 1) + \sum_{h \in B(G)} (d_c(h))^2] \tag{3.1}$$

$$q_{BC} = \frac{1}{2}[\sum_{h \in B(G)} (d_c(h))^2 + \sum_{c \in C(G)} (d_{vb}(c))^2] \tag{3.2}$$

Proof. We first prove the following
Claim : For any graph G,

$$\sum_{h \in B_{NP}(G)} [(d_c(h))^2 - d_c(h)] = \sum_{h \in B(G)} [(d_c(h))^2 - d_c(h)] \tag{3.3}$$

The proof of the claim follows from the fact that, for any pendant block $f, d_c(f) = 1$ and hence all pendant blocks contribute null to the sum on the right hand side of equation (3.3).

To prove (3.1)**.** Since every cutvertex yields $d_{vb}(c)$ edges in $BC(G)$, the number of edges in $BC(G) = \sum_{c \in C(G)} d_{vb}(c) = m+n-1$. Further, all the cutvertices incident to a nonpendant block are mutually adjacent, every nonpendant block h yields $\binom{d_c(h)}{2}$ edges in $C_G(G)$. Hence the number of edges in $C_G(G) = \sum_{h \in B_{NP}(G)} \binom{d_c(h)}{2}$. Then,

$$
\begin{aligned}
q_{bc} &= q(BC(G)) + q(C_G(G)) \\
&\quad (Since\ T_{bc}(G) = BC(G) \cup C_G(G)) \\
&= m + n - 1 + \sum_{h \in B_{NP}(G)} \binom{d_c(h)}{2} \\
&= m + n - 1 + \frac{1}{2} \sum_{h \in B_{NP}(G)} [(d_c(h))^2 - d_c(h)] \\
&= m + n - 1 + \frac{1}{2} \sum_{h \in B_{NP}(G)} (d_c(h))^2 - \frac{1}{2} \sum_{h \in B_{NP}(G)} d_c(h) \\
&= m + n - 1 - \frac{1}{2}(m_{NP} + n - 1) + \frac{1}{2} \sum_{h \in B_{NP}(G)} (d_c(h))^2 \\
&\quad (using\ Corollary\ 2.4) \\
&= \frac{1}{2}(2m - m_{NP} + n - 1) + \frac{1}{2} \sum_{h \in B_{NP}(G)} (d_c(h))^2 \\
&= \frac{1}{2}[(m + m_P + n - 1) + \sum_{h \in B_{NP}(G)} (d_c(h))^2] \\
&\quad (since\ m - m_{NP} = m_p). \\
&= \frac{1}{2}[(m + n - 1) + \sum_{h \in B_{NP}(G)} [(d_c(h))^2] + m_P] \\
&= \frac{1}{2}[(m + n - 1) + \sum_{h \in B(G)} (d_c(h))^2].
\end{aligned}
$$

To prove (3.2)**.**

$$
\begin{aligned}
q_{BC} =& q(BC(G)) + q(C_G(G)) + q(B_G(G)) \\
&= m + n - 1 + \sum_{h \in B_{NP}(G)} \tbinom{d_c(h)}{2} + \frac{1}{2}[\sum_{c \in C(G)} [d_{vb}(c)]^2] - (m + n - 1)] \\
&\quad (using\ Theorem\ 2.6) \\
&= m + n - 1 + \frac{1}{2}[\sum_{h \in B(G)} [(d_c(h))^2 - d_c(h)] + \sum_{c \in C(G)} [d_{vb}(c)]^2 - (m + n - 1)] \\
&\quad (using\ eqation\ (3.3)) \\
&= m + n - 1 + \frac{1}{2}[\sum_{h \in B(G)} [d_c(h)]^2 - (m + n - 1) + \sum_{c \in C(G)} [d_{vb}(c)]^2 - (m + n - 1)] \\
&\quad (using\ Theorem\ 2.3) \\
&= \frac{1}{2}[\sum_{h \in B(G)} [d_c(h)]^2 + \sum_{c \in C(G)} [d_{vb}(c)]^2]
\end{aligned}
$$

□

3.1. Semitotal-block-vertex graph and total block-vertex graph.
In the above graphs if we replace $C(G)$ by $V(G)$ we get two different graphs. The blocks and vertices of a graph are called its members. *Semitotal block-vertex graph* $T_{bp}(G)$ is a graph with vertex set $V(G) \cup B(G)$ and any two vertices in $T_{bp}(G)$ are adjacent if and only if corresponding vertices are vv-adjacent or the corresponding members are incident. In this case $T_{bp}(G) = b_p(G) \cup P_G(G)$. The total block-vertex graph $T_{BP}(G)$ is a graph with vertex set $V(G) \cup B(G)$ and any two vertices in $T_{BP}(G)$ are adjacent if and only if the corresponding members are vv-adjacent or adjacent or incident to each other. Further, $T_{BP}(G) = b_p(G) \cup P_G(G) \cup B_G(G)$. Expressions for the number of edges in these graphs are derived in the next theorem.

Theorem 3.2. *Let G be a (p,q) graph with m blocks and n cutvertices. Let q_{bp} and q_{BP} denote the number of edges in $T_{bp}(G)$ and $T_{BP}(G)$ respectively. Then,*

$$q_{bp} = \frac{1}{2}[(p+m-1) + \sum_{h \in B(G)} [d_{bv}(h)]^2] \tag{3.4}$$

$$q_{BC} = \frac{1}{2}[p-n + \sum_{h \in B(G)} [d_{bv}(h)]^2 + \sum_{c \in C(G)} [d_{vb}(c)]^2] \tag{3.5}$$

Proof. Result (3.4) can be proved in a similar way as in Theorem 3.1. Therefore we prove (3.5) only. Since every block h yields $d_{bv}(h)$ edges in $b_p(G)$, we have the number of edges in

$$b_p(G) = \sum_{h \in B(G)} d_{bv}(h) = p+m-1$$

Therefore

$$\begin{aligned} q_{BC} =& q(b_p(G)) + q(P_G(G)) + q(B_G(G)) \\ &= p+m-1+\frac{1}{2}\Bigg[\sum_{h \in B(G)} [d_{bv}(h)]^2 - (p+m-1)\Bigg] \\ &+ \frac{1}{2}\Bigg[\sum_{c \in C(G)} [d_{vb}(c)]^2 - (m+n-1)\Bigg] (using\ Theorem\ 2.4\ and\ Theorem\ 2.6) \\ &= \frac{1}{2}[p-n+\sum_{h \in B(G)} [d_{bv}(h)]^2 + \sum_{c \in C(G)} [d_{vb}(c)]^2] \end{aligned}$$

□

3.2. Semitotal-block-vertex-edge graph and total block-vertex- edge graph.
We now define the most generalized total graphs. The blocks, vertices and edges of a graph are called its members. *Semitotal block-vertex-edge graph* $T_{bpe}(G)$ is a graph with vertex set $V(G) \cup B(G) \cup E(G)$ and any two vertices in $T_{bpe}(G)$ are adjacent if and only if corresponding vertices are vv-adjacent or the corresponding members are incident. A *vertex edge graph* $V_e(G)$ is a bigraph with vertex set as

$V(G) \cup E(G)$ and a vertex $v \in V$ and an edge $x \in E(G)$ are adjacent in $V_e(G)$ if and only if v is incident on the edge x. A *block edge graph* $b_e(G)$ is a bigraph with vertex set $B(G) \cup E(G)$ and a block $b \in B(G)$ are adjacent if and only if the edge x is incident on the block b. we observe that $T_{bpe} = P_G(G) \cup b_e(G) \cup V_e(G) \cup b_p(G)$. The total block-vertex-edge graph $T_{BPE}(G)$ is a graph with vertex set $V(G) \cup B(G) \cup E(G)$ and any two vertices in $T_{BPE}(G)$ are adjacent if and only if the corresponding members are vv-adjacent or adjacent or incident to each other. It is not hard to see that $T_{BPE}(G) = T_bpe(G) \cup B_G(G) \cup L(G)$. We now aim at getting expressions for the number of edges in these generalized total graphs.

Theorem 3.3. *Let G be a (p,q) graph with m blocks and n cutvertices. Let q_{bpe} and q_{BPE} denote the number of edges in $T_{bpe}(G)$ and $T_{BPE}(G)$ respectively. Then,*

$$q_{bpe} = \frac{1}{2}[6q + (p + m - 1) + \sum_{h \in B(G)} [d_{bv}(h)]^2] \tag{3.6}$$

$$q_{BC} = \frac{1}{2}[4q + (p - n) + \sum_{u \in V(G)} [d(u)]^2 + \sum_{c \in C(G)} [d_{vb}(c)]^2 + \sum_{h \in B(G)} [d_{bv}(h)]^2] \tag{3.7}$$

Proof. **To prove** (3.6)**.** Since each edge is incident on two vertices, there are $2q$ edges in $V_e(G)$. As every edge is incident on a unique block, there are q edges in $b_e(G)$. Then

$$\begin{aligned} q_{bpe} &= q(P_G(G)) + q(b_e(G)) + q(V_e(G)) + q(b_p(G)) \\ &= \frac{1}{2}\left[\sum_{h \in B(G)} [(d_{bv}(h))^2] - (p + m - 1)\right] + q + 2q + (p + m - 1) \\ &= \frac{1}{2}\left[6q + (p + m - 1) + \sum_{h \in B(G)} [(d_{bv}(h))^2]\right] \end{aligned}$$

To prove (3.7)**.**

$$\begin{aligned} q_{BPE} &= q_{bpe} + q(B_G(G)) + q(L(G)) \\ &= \frac{1}{2}[6q + (p + m - 1) + \sum_{h \in B(G)} (d_{bv}(h))^2] + \frac{1}{2}\Big[\sum_{c \in C(G)} (d_{vb}(c))^2 \\ &- (m + n - 1)\Big] + \frac{1}{2}\left[\sum_{u \in V(G)} (d(u))^2 - 2q\right] \\ &= \frac{1}{2}\left[4q + (p - n) + \sum_{u \in V(G)} (d(u))^2 + \sum_{c \in C(G)} (d_{vb}(c))^2 + \sum_{h \in B(G)} (d_{bv}(h))^2\right] \end{aligned}$$

□

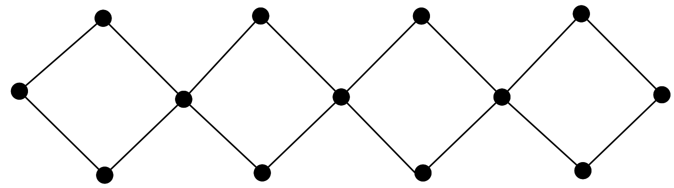

FIGURE 1. The Cactus $K(4)$

4. B-regular graphs

The new degree concepts paved the way to define varieties of block regular graphs. A graph G is said to be BVk- regular if $d_{bv}(h) = k$ for every $h \in B(G)$. A graph G is BEk- regular if $d_{be}(h) = k$ for every $h \in B(G)$. Similarly if $d_{vb}(c) = k$ for every cutvertex $c \in C(G)$, then G is a VBk- regular graph. If $d_{bb}(h) = k$ for every block $h \in B(G)$ then G is a BBk- regular graph. On the other hand, if $d_c(f) = k$ for every nonpendant block f, then G is a Ck - regular (cutvertex k regular) graph. Finally, if $d_{vv}(c) = k$ for every cutvertex $c \in C(G)$, then G is a VVk-regular graph. Any path P_n is a $BE1, BV2, C2, VB2$ and $VV2$-regular graph. Any B-path BP_m in which every block has k vertices and r edges is a $BEr, BVk, C2$ and $VB2$ and $VV(2k-2)$ regular graph. Any tree T is a BV2 and BE1 regular graph.The cactus $K(i)$ for $i \in N$, has vertex set $V(K(i)) = \{a_k | k = 1, 2, ..., i+1\}; \{b_j | j = 1, 2, ..., 2i\}$ and the edge set $E(K(i)) = \{a_k b_{2k-1}, a_{k+1} b_{2k-1} | k = 1, 2, ..., i\}; \{a_k b_{2k}, a_{k+1} b_{2k} | k = 1, 2, ..., i\}$. The *Cactus* $K(4)$ shown in Figure 1 is a BE4, BV4, C2,VB2 and VV6 regular graph. We observe that, in general any *Cactus* $K(i)$ is a B-path with i blocks and all the blocks being the cycle C_i. Hence $K(i)$ is a BEi, BV2, C2,VB2 and VV(2i-2) regular graph.

Proposition 4.1. *If G is a connected BVk- regular graph with $k \geq 3$ and m blocks, then*

$$p = m(k-1) + 1 \tag{4.1}$$

$$mk \leq q \leq \frac{mk(k-1)}{2} \tag{4.2}$$

Proof. If G is a connected BVk- regular graph with $k \geq 3$, then every block of G has exactly k vertices. Then $mk = \sum_{h \in B(G)} d_{bv}(h) = p + m - 1$ from Theorem 2.3. Therefore $p = m(k-1) + 1$ and the result (4.1) follows. A block with k vertices with minimum number of edges is the cycle C_k, and every cycle C_k has k edges and hence $mk \leq q$. Similarly, a block with k vertices with maximum number of edges is a complete graph K_k, which has $\binom{k}{2}$ edges, and thus we have $q \leq m\binom{k}{2}$. Then the result (4.2) follows. □

In the next proposition we characterize the BVk - regular graphs attaining $q = mk$ and $q = \frac{mk(k-1)}{2}$. The result being straight forward from the above proof.

Proposition 4.2. *If G is a connected BVk- regular graph with $k \geq 3$ and m blocks, then*

(i) $q = mk$ *if and only if every block of* G *is a cycle* C_k.
(ii) $q = \frac{mk(k-1)}{2}$ *if and only if every block of* G *is a clique* Q_k

Proposition 4.3. *Let* G *be a connected BEk- regular graph with* m *blocks, then*
(i) $q = mk$
(ii) $m(r-1) + 1 \leq p \leq m(k-1) + 1$
where r *is a least positive integer such that the clique* Q_r *has* k *edges.*

Proof. Since G is BEk -regular graph, every block of G contains k edges. A block containing k edges with minimum number of vertices must be a clique Q_r in G. If every block of G contains atleast r vertices then $mr \leq \sum_{h \in B(G)} d_{bv} = p + m - 1$ which yields lower bound in (ii). Simlarly, a block containing k edges with maximum number of vertices is the cycle C_k in G. If every block of G contains atmost k vertices, then $mk \geq \sum_{h \in B(G)} d_{bv} = p + m - 1$ which yields upper bound in (ii). □

Proposition 4.4. *Let* G *be a BEk- regular graph with* m *blocks, then*
(i) $p = m(r-1) + 1$ *if and only if every block of* G *is a clique* Q_r *where* r *is a least positive integer such that the clique* Q_r *has* k *edges.*
(ii) $p = m(k-1) + 1$ *if and only if every block of* G *is a cycle* C_k

Proof. The result follows from the fact that a block containing k edges with minimum number of vertices must be a clique Q_r and a block containing k edges with maximum number of vertices is a cycle C_k. □

Conclusion: B-regular graph structures are important in the study of biotechnology to get unique type of group cell structures. It finds its application in the study of regular structure of hydrocarbons in organic chemistry and crystal structure in crystallography. Recently, mixed block domination parameters are studied and using the maximum block degree, several bounds for mixed block domination are obtained in [4, 13]. The following problems are kept open.

Open Problem 1. *Characterize all VBk regular graphs*

Open Problem 2. *Characterize all VVk regular graphs*

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