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INTERVAL VALUED L-FUZZY IDEALS ON SEMINEARRINGS

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Abstract. In this paper, we define a congruence relation on a seminearring and obtain the relationship with the existing congruence relations. In addition, we define an interval valued L-fuzzy ideal of a seminearring. Then we prove that if the level set is strong ideal of a seminearing S, then any L-fuzzy subset of a seminearring is a L-fuzzy ideal and the result is illustrated with the suitable example.

1. Introduction

A right seminearring is an algebraic structure which is a semigroup with respect to both the binary operations $\cdot$ and $+$, satisfies right distributive law. Hoorn and Rootselaar[20] considered an ideal of a seminearring as a kernel of seminearring. Then Ahsan[2, 3] generalized this ideal definition and obtained results on ideals of seminearring. Koppula, Kedukodi and Kuncham[15] defined strong ideal of a seminearring and proved the classical isomorphism theorems. Fuzzy set theory was introduced by Zadeh[21]. Subsequently, various authors studied results on these fuzzy sets. L-fuzzy sets are introduced by Goguen[4], which are the combination of lattices and fuzzy sets. Jifan and Tiejun[9] defined fuzzy ideal in terms of t-norm and t-conorm and studied various properties. By using interval valued idempotent t-conorm and t-norm Li and Wang[17, 18] defined interval valued fuzzy ideals. Davvaz[1] studied interval valued fuzzy ideals on a distributive lattice. Results on L-fuzzy ideal of a ring are studied by Koc and Balkanay[12]. Then results on L-fuzzy left ideals and normal L-fuzzy ideals of semiring are studied by Jun, Neggers and Kim [10, 11]. The results on interval valued L-fuzzy ideals of a nearring using the concepts t-norm and t-conorm are studied by Jagadeesha, Kedukodi and Kuncham[7, 8].
In this paper, we define a congruence relation on a seminearring and obtain the relationship with the existing congruence relations. Later, we define interval valued L-fuzzy ideal of a seminearring. Then we prove that if the level set $\eta_{\hat{t}}$ is a strong ideal of S then the $L-$ fuzzy subset of S is an interval valued $L-$ fuzzy ideal of S.

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2. Preliminaries

Here, basic results and definitions are provided to obtain the results presented in the manuscript.

Definition 2.1. [20] Let S be a non-empty set. Then S is said be a right seminearring with respect to $+$ and $\cdot$, if S is a semigroup with respect to both the operations $+$ and $\cdot$, and satisfies the following conditions.

(1) $(a_2 + a_3)a_1 = a_2a_1 + a_3a_1, \forall a_1, a_2, a_3 \in S$.
(2) $0 + a_1 = a_1 + 0 = a_1, \ \forall \ a_1 \in S$.
(3) For all $a_1 \in S, 0a_1 = 0$.

Definition 2.2. Let T be a nonempty subset of S. Then, for $a_1, a_2 \in S, \ \ a_1 \equiv_T a_2$ if and only if there exist $i_1, i_2 \in T$ such that $i_1 + a_1 = i_2 + a_2$.

Through out this paper, S is considered as a right seminearring.

Definition 2.3. [15] A non-empty subset A of S is a strong ideal of S, if the following mentioned conditions hold.

(1) $a_1 + a_2 \in A$, forall $a_1, a_2 \in A$.
(2) $s_1 + A \subseteq A + s_1, \forall s_1 \in S$.
(3) For $s_1, s_2 \in S$, if $s_1 \equiv_A s_1$ then $s_1 \in A + s_2$.
(4) $s_1(A + s_2) \subseteq A + s_1s_2$ for all $s_1, s_2 \in S$.
(5) $As_1 \subseteq A$ for all $s_1 \in S$.

Definition 2.4. [15] Let A be any non-empty subset of S. For $s_1, s_2 \in S$, $s_1 \ {}_A\!\equiv \ s_2$ implies there exist $a_1, a_2 \in A$ such that $s_1 + a_1 = s_2 + a_2$.

Definition 2.5. An equivalence relation ϱ on S is said to be a congruence relation on S, if $a_1 \ \varrho \ a_2$ and $a_3 \ \varrho \ a_4$

(1) Then $(a_1 + a_3) \ \varrho \ (a_2 + a_4)$
(2) Then $(a_1a_3) \ \varrho \ (a_2a_4)$.

Definition 2.6. [15] Let $\phi : S \to R$ be a seminearring homomorphism. Then

$$ker \ \phi = \{s \in S \mid \phi(s) = \phi(0)\}.$$

Definition 2.7. [15] Let S and R be seminearrings. Then a homomorphism $\phi : S \to R$ is said to be a strong homomorphism if $\phi(x) = \phi(y)$ then $x \in ker \ \phi + y$.

Theorem 2.8. [15] The following statements hold.
1. The projection map $\pi : S \to S/P$ is an onto seminearring strong homomorphism.
2. If $\phi : S \to R$ is an onto seminearring strong homomorphism then $ker \ \phi$ is a strong ideal of S and $S/ker \ \phi \cong R$.

Definition 2.9. [16] Let η be a fuzzy subset of seminearring S and $\tau_1, \tau_2 \in [0, 1]$ are thresholds of S such that $\tau_1 < \tau_2$. Then η is called a fuzzy ideal of a seminearring S with τ_1 and τ_2, if the below mentioned conditions are satisfied.

(1) $\tau_1 \vee \eta(p + q) \geq \tau_2 \wedge \eta(p) \wedge \eta(q)$
(2) If $x + a = y + b$ then
$\tau_1 \vee \eta(a) \geq \tau_2 \wedge \eta(y + b) \wedge \eta(x), \ \ a, b, x, y \in S$.

(3) $\tau_1 \vee \eta(p+q+r) \geq \tau_2 \wedge \eta(q) \wedge \eta(q+p+r)$
(4) $\tau_1 \vee \eta(p(q+r)+t) \geq \tau_2 \wedge \eta(q) \wedge \eta(pr+t)$
(5) $\tau_1 \vee \eta(pq) \geq \tau_2 \wedge \eta(p)$, for all $p, q, r, t \in S$.

In the above definition, τ_2 and τ_1 are known as upper and lower thresholds of S respectively.
If $\tau_2 = 1$ and $\tau_1 = 0$, then η is an ordinary fuzzy ideal of S.

In this paper, we consider $< L, \wedge_L, \vee_L >$ is a complete bounded lattice with the partial ordering relation $\leq_L$.

Definition 2.10. [6] Let L be a lattice. A t-norm $T : L \times L \to L$ is a function such that if the following conditions are satisfied. Forall $l_1, l_2, l_3 \in L$,

(1) $T(l_1, T(l_2, l_3)) = T(T(l_1, l_2), l_3)$ (Associativity)
(2) $T(l_1, l_2) = T(l_2, l_1)$ (Commutativity)
(3) If $l_2 \leq_L l_3$ then $T(l_1, l_2) \leq_L T(l_1, l_3)$ (Monotonicity)
(4) $T(l_1, M) = l_1$, where M is the greatest element of L (Boundary condition).

In the following, T denotes a t-norm on L.
If $T(l_1, l_1) = l_1, \forall l_1 \in L$ then T is said be idempotent $t-$ norm.

Definition 2.11. [22] A t-conorm $C : L \times L \to L$ is a function such that if the following mentioned conditions are satisfied. Forall $l_1, l_2, l_3 \in L$,

(1) $C(l_1, C(l_2, l_3)) = C(C(l_1, l_2), l_3)$ (Associativity)
(2) $C(l_1, l_2) = C(l_2, l_1)$ (Commutativity)
(3) If $l_2 \leq_L l_3$ then $C(l_1, l_2) \leq_L C(l_1, l_3)$ (Monotonicity)
(4) $C(l_1, m) = l_1$, where m is the least element of L (Boundary condition).

In the following, C denotes a t-conorm on L.

Remark 2.12. $T(l_1, l_2) \leq_L l_1 \wedge_L l_2 \leq_L l_1 \vee_L l_2 \leq_L C(l_1, l_2)$.

The set $D(L) = \{[x, y] \mid x, y \in L\}$. Let $I_1 = [x_1, y_1]$ and $I_2 = [x_2, y_2]$ be two elements of $D(L)$. Then $I_1 \leq I_2 \Leftrightarrow x_1 \leq_L x_2$ and $y_1 \leq_L y_2$.

Definition 2.13. Let A be any non-empty subset of X. A mapping $\eta : A \to L$ is called a L-fuzzy subset of A. Now, consider $\underline{\eta}, \overline{\eta}$ are any two L-fuzzy subsets of A such that $\underline{\eta}(a) \leq \overline{\eta}(a),\ \forall\ a \in A$.
Define a mapping $\hat{\eta} : A \to D(L)$ as $\hat{\eta} = [\underline{\eta}(a), \overline{\eta}(a)],\ \forall\ a \in A$. Then $\hat{\eta}$ is called an interval valued $L-$ fuzzy subset of A.

For more results on nearrings, semirings and seminearrings, we refer Pilz[19], Golan[5], Koppula, Kedukodi and Kuncham[13, 14].

3. Congruence relations on seminearring

In the present section, we provide a congruence relation and obtained the relationship with the existing congruence relations. Then we define interval valued L-fuzzy ideal using t-norm and t-conorm and proved the related results.

Definition 3.1. Let A be a strong ideal of seminearirng S and $s_1, s_2 \in S$. Then $s_1 \overset{A}{=} s_2$ if and only if there exist $a_1, a_2, a_3, a_4 \in A$ such that $a_1 + s_1 + a_2 = a_3 + s_2 + a_4$.

Proposition 3.2. If A is a strong ideal of seminearring S and $s_1, s_2 \in S$ then $s_1 \equiv_A s_2$ implies $s_1 \;{}_A\!\equiv s_2$.

Proof. Suppose $s_1 \;{}_A\!\equiv s_2$. Then there exist $a_1, a_2 \in A$ such that $s_1 + a_1 = s_2 + a_2$. As A is a strong ideal of S, there exist $a_1^{'}, a_2^{'} \in A$ such that $a_1^{'} + s_1 = a_2^{'} + s_2$. This implies $s_1 \equiv_A s_2$. ☐

Proposition 3.3. If A is a strong ideal of seminearring S then $\overset{A}{=}$ is a congruence relation on S.

Proof. Clearly, $\overset{A}{=}$ is reflexive and symmetric. Now, consider $x \overset{A}{=} y$ and $y \overset{A}{=} z$. Then there exist $i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8 \in A$ such that $i_1 + x + i_2 = i_3 + y + i_4$ and $i_5 + y + i_6 = i_7 + z + i_8$.

Now, $i_5 + (i_1 + x + i_2) + i_6 = i_5 + (i_3 + y + i_4) + i_6 = i_5 + i_3 + (y + i_4) + i_6$

$\implies (i_5 + i_1) + x + (i_2 + i_6) = i_5 + i_3 + (i_4^{'} + y) + i_6$, for some $i_4^{'} \in A$.

$\implies i_1^{'} + x + i_2^{'} = (i_5 + i_5^{'}) + y + i_6$

$(i_5 + i_1 = i_1^{'} \in A, \;\; i_2 + i_6 = i_2^{'} \in A \quad \text{and} \quad i_3 + i_4^{'} = i_5^{'} \in A)$

$\implies i_1^{'} + x + i_2^{'} = (i_6^{'} + i_5) + y + i_6$, for some $i_6^{'} \in A$.

$\implies i_1^{'} + x + i_2^{'} = i_6^{'} + (i_5 + y + i_6)$

$\implies i_1^{'} + x + i_2^{'} = i_6^{'} + (i_7 + z + i_8)$

$\implies i_1^{'} + x + i_2^{'} = i_7^{'} + z + i_8 \; (i_6^{'} + i_7 = i_7^{'} \in A)$

This implies $x \overset{A}{=} z$.

Thus, the relation $\overset{A}{=}$ is an equivalence relation on S.

Now, we show that the relation $\overset{A}{=}$ is congruence relation on S.

Suppose $x \overset{A}{=} y$ and $x^{'} \overset{A}{=} y^{'}$.

Then there exist $i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8 \in A$ such that $i_1 + x + i_2 = i_3 + y + i_4$ and $i_5 + x^{'} + i_6 = i_7 + y^{'} + i_8$.

Now, we have $(i_1 + x + i_2) + (i_5 + x^{'} + i_6) = (i_3 + y + i_4) + (i_7 + y^{'} + i_8)$

$\Rightarrow i_1 + (x + i_2) + (i_5 + x^{'} + i_6) = i_3 + (y + i_4) + (i_7 + y^{'} + i_8)$

$\Rightarrow i_1 + (x + i_2) + (i_5 + x^{'} + i_6) = i_3 + (y + i_4) + (i_7 + y^{'} + i_8)$

$\Rightarrow i_1 + (i_2^{'} + x) + (i_5 + x^{'} + i_6) = i_3 + (i_4^{'} + y) + (i_7 + y^{'} + i_8)$, for some $i_2^{'}, i_4^{'} \in A$.

$\Rightarrow i_1 + i_2^{'} + (x + i_5) + x^{'} + i_6 = i_3 + i_4^{'} + (y + i_7) + y^{'} + i_8$

$\Rightarrow i_1 + i_2^{'} + (i_5^{'} + x) + x^{'} + i_6 = i_3 + +i_4^{'} + (i_7^{'} + y) + y^{'} + i_8$ for some $i_5^{'}, i_7^{'} \in A$.

$\Rightarrow i_9 + x + x^{'} + i_6 = i_{10} + y + y^{'} + i_8$

$(i_1 + i_2^{'} + i_5^{'} = i_9 \in A, \;\; i_3 + +i_4^{'} + i_7^{'} = i_{10} \in A)$

This implies $x + x^{'} \overset{A}{=} y + y^{'}$.

Now, consider $(i_1 + x + i_2)(i_5 + x^{'} + i_6) = (i_3 + y + i_4)(i_7 + y^{'} + i_8)$

$\Rightarrow (i_1 + x)(i_5 + x^{'} + i_6) + i_2(i_5 + x^{'} + i_6) = (i_3 + y)(i_7 + y^{'} + i_8) + i_4(i_7 + y^{'} + i_8)$

$\Rightarrow i_1(i_5 + x^{'} + i_6) + x(i_5 + (x^{'} + i_6)) + i_2^{''} = i_3(i_7 + y^{'} + i_8) + y(i_7 + (y^{'} + i_8)) + i_4^{''}$

$[i_2(i_5 + x^{'} + i_6) = i_2^{''} \in A \text{ and } i_4(i_7 + y^{'} + i_8) = i_4^{''} \in A]$

As A is a strong ideal of S, we get $i_1^{''}+(i_5^{''}+x(x^{'}+i_6))+i_2^{''}=i_3^{''}+(i_7^{''}+y(y^{'}+i_8))+i_4^{''}$, for some $i_5^{''},i_7^{''}\in A$ $[i_1(i_5+x^{'}+i_6)=i_1^{''}\in A$ and $i_3(i_7+y^{'}+i_8)=i_3^{''}\in A]$.
$\Rightarrow i_1^{''}+i_5^{''}+x(i_6^{'}+x^{'})+i_2^{''}=i_3^{''}+i_7^{''}+y(i_8^{'}+y^{'})+i_4^{''}$ for some $i_6^{'},i_8^{'}\in A$.
$\Rightarrow i_1^{''}+i_5^{''}+i_6^{''}+xx^{'}+i_2^{''}=i_3^{''}+i_7^{''}+i_8^{''}+yy^{'}+i_4^{''}$ for some $i_6^{''},i_8^{''}\in A$.
$\Longrightarrow i_9^{''}+xx^{'}+i_2^{''}=i_{10}^{''}+yy^{'}+i_4^{''}$ $[i_1^{''}+i_5^{''}+i_6^{''}=i_9^{''}\in A$ and $i_3^{''}+i_7^{''}+i_8^{''}=i_{10}^{''}\in A]$
This implies $xx^{'}\overset{A}{=}yy^{'}$. Thus, the relation $\overset{A}{=}$ is a congruence relation on S. □

Proposition 3.4. If A is a strong ideal of seminearring S, and $x,y\in S$, then $x\overset{A}{=}y$ if and only if $x\equiv_A y$.

Proof. First, we assume that $x\overset{A}{=}y$. This implies there exist $i_1,i_2,i_3,i_4\in A$ such that $i_1+(x+i_2)=i_3+(y+i_4)$.
Then there exist $i_2^{'},i_4^{'}\in A$ such that $i_1+(i_2^{'}+x)=i_3+(i_4^{'}+y)$.
This implies $(i_1+i_2^{'})+x=(i_3+i_4^{'})+y$.
Then $i_1^{'}+x=i_3^{'}+y$
$(i_1+i_2^{'}=i_1^{'}\in A$ and $i_3+i_4^{'}=i_3^{'}\in A)$.
This implies $x\equiv_A y$.
Conversely, $x\equiv_A y$ implies there exist $a_1,a_2\in A$ such that $a_1+x=a_2+y$.
This also can be written as $a_1+x+0=a_2+y+0$. This implies $x\overset{A}{=}y$. □

Proposition 3.5. If $\varphi:S\to S^{'}$ is a seminearring homomorphism and η is a fuzzy ideal of $S^{'}$ then $\varphi^{-1}(\eta)$ is a fuzzy ideal of S.

Proof. Let $s_1,s_2\in S$. Then consider $\tau_1\vee\varphi^{-1}(\eta)(s_1+s_2)=\tau_1\vee\eta(\varphi(s_1+s_2))$
$=\tau_1\vee\eta(\varphi(s_1)+\varphi(s_2))$
$\geq\tau_2\wedge\eta(\varphi(s_1))\wedge\eta(\varphi(s_2))$ (Because η is a fuzzy ideal of $S^{'}$.)
$=\tau_2\wedge\varphi^{-1}(\eta)(s_1)\wedge\varphi^{-1}(\eta)(s_2)$.
Let $x,y,s_1,s_2\in S$ be such that $x+s_1=y+s_2$.
Then $\varphi(x+s_1)=\varphi(y+s_2)$. As φ is a homomorphism, then we have $\varphi(x)+\varphi(s_1)=\varphi(y)+\varphi(s_2)$.
Now, $\tau_1\vee\varphi^{-1}(\eta)(s_1)=\tau_1\vee\eta(\varphi(s_1))$
$\geq\tau_2\wedge\eta(\varphi(y)+\varphi(s_2))\wedge\eta(\varphi(x))$ (Because η is a fuzzy ideal of $S^{'}$.)
$=\tau_2\wedge\eta(\varphi(y+s_2))\wedge\eta(\varphi(x))$
$=\tau_2\wedge\varphi^{-1}(\eta)(y+s_2)\wedge\varphi^{-1}(\eta)(x)$.
Let $a,b,c\in S$ be such that $\tau_1\vee\varphi^{-1}(\eta)(a+b+c)=\tau_1\vee\eta(\varphi(a+b+c))$
$=\tau_1\vee\eta(\varphi(a)+\varphi(b)+\varphi(c))$
$\geq\tau_2\wedge\eta(\varphi(b))\wedge\eta(\varphi(b)+\varphi(a)+\varphi(c))$
$=\tau_2\wedge\eta(\varphi(b))\wedge\eta(\varphi(b+a+c))$
$=\tau_2\wedge\varphi^{-1}(\eta)(b)\varphi^{-1}(\eta)(b+a+c)$.
Now, take $a,b,c,t\in S$ such that $\tau_1\vee\varphi^{-1}(\eta)(a(b+c)+t)=\tau_1\vee\eta(\varphi(a(b+c)+t))$
$=\tau_1\vee\eta(\varphi(a(b+c))+\varphi(t))=\tau_1\vee\eta(\varphi(a)\varphi(b+c))+\varphi(t))$
$=\tau_1\vee\eta(\varphi(a)(\varphi(b)+\varphi(c))+\varphi(t))$
$\geq\tau_2\wedge\eta(\varphi(b))\wedge\eta(\varphi(a)\varphi(c)+\varphi(t))$
$=\tau_2\wedge\eta(\varphi(b))\wedge\eta(\varphi(ac)+\varphi(t))$
$=\tau_2\wedge\eta(\varphi(b))\wedge\eta((\varphi(ac+t))$
$=\tau_2\wedge\varphi^{-1}(\eta)(b)\wedge\varphi^{-1}(\eta)(ac+t)$.

Now, take $a,b \in S$ such that $\tau_1 \vee \varphi^{-1}(\eta)(ab) = \tau_1 \vee \eta(\varphi(ab))$
$\geq \tau_2 \wedge \eta(\varphi(a)) = \tau_2 \wedge \varphi^{-1}(\eta)(a)$.
Thus $\varphi^{-1}(\eta)$ is a fuzzy ideal of S. □

Definition 3.6. Let $(S,+,\cdot)$ be a seminearring and T_{IL}, C_{IL} be interval valued t-norm and t-conorm on $D(L)$. Let $\hat{\tau}_1, \hat{\tau}_2 \in D(L)$ be the thresholds such that $\hat{\tau}_1 < \hat{\tau}_2$. An interval valued $L-$fuzzy subset $\hat{\eta}$ on S is said to be an interval valued $L-$ fuzzy ideal with $\hat{\tau}_1, \hat{\tau}_2$ such that if the below mentioned conditions hold. Forall $s_1, s_2, s_3, t \in S$,

(1) $C_{IL}(\hat{\tau}_1, \hat{\eta}(s_1+s_2)) \geq T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(s_1)), C_{IL}(\hat{\tau}_1, \hat{\eta}(s_2))))$
(2) If $x_1 + a = x_2 + b$ then
$C_{IL}(\hat{\tau}_1, \hat{\eta}(a)) \geq T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(x_2+b)), C_{IL}(\hat{\tau}_1, \hat{\eta}(x_1))))$,
$a, b, x_1, x_2 \in S$.
(3) $C_{IL}(\hat{\tau}_1, \hat{\eta}(s_1+s_2+s_3)) \geq T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(s_2)), C_{IL}(\hat{\tau}_1, \hat{\eta}(s_2+s_1+s_3))))$
(4) $C_{IL}(\hat{\tau}_1, \hat{\eta}(s_1(s_2+s_3)+t)) \geq T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(s_2)), C_{IL}(\hat{\tau}_1, \hat{\eta}(s_1 s_3+t))))$
(5) $C_{IL}(\hat{\tau}_1, \hat{\eta}(s_1 s_2)) \geq T_{IL}(\hat{\tau}_2, C_{IL}(\hat{\tau}_1, \hat{\eta}(s_1)))$.

Proposition 3.7. If $\hat{\eta}$ is an interval valued $L-$ fuzzy subset of S and the level set $\eta_{\hat{t}}, \forall\ \hat{t} \in (\hat{\tau}_1, \hat{\tau}_2]$ is a strong ideal of S then $\hat{\eta}$ is an interval valued $L-$ fuzzy ideal of S.

Proof. First, we will show that
$C_{IL}(\hat{\tau}_1, \hat{\eta}(x+y)) \geq T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(x))), C_{IL}(\hat{\tau}_1, \hat{\eta}(y)))), \forall\ x,y \in S$. Suppose, we assume that $C_{IL}(\hat{\tau}_1, \hat{\eta}(x+y)) \leq T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(x)), C_{IL}(\hat{\tau}_1, \hat{\eta}(y))))$, for some x and y.
Let $\hat{t} = T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(x)), C_{IL}(\hat{\tau}_1, \hat{\eta}(y))))$.
Then $\hat{t} \leq \hat{\tau}_2 \wedge T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(x)), C_{IL}(\hat{\tau}_1, \hat{\eta}(y)))$.
$\leq \hat{\tau}_2 \wedge C_{IL}(\hat{\tau}_1, \hat{\eta}(x)) \wedge C_{IL}(\hat{\tau}_1, \hat{\eta}(y))$
$\Rightarrow \hat{t} \leq \hat{\tau}_2$, $\hat{t} \leq C_{IL}(\hat{\tau}_1, \hat{\eta}(x))$ and $\hat{t} \leq C_{IL}(\hat{\tau}_1, \hat{\eta}(y))$.
$\Rightarrow x \in \hat{\eta}_k, y \in \hat{\eta}_k$.
As $C_{IL}(\hat{\tau}_1, \hat{\eta}(x+y)) < \hat{t}$, we get $x+y \notin \eta_{\hat{t}}$.
Now, $x \in \eta_{\hat{t}}$, $y \in \eta_{\hat{t}}$ and $x+y \notin \eta_{\hat{t}}$. This is a contradiction to $\eta_{\hat{t}}$ is a strong ideal of S.
Let $x,a,y,b \in S$ such that $x+a = y+b$.
Then we show that $C_{IL}(\hat{\tau}_1, \hat{\eta}(a)) \geq T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(y+b)), C_{IL}(\hat{\tau}_1, \hat{\eta}(x))))$.
Suppose $C_{IL}(\hat{\tau}_1, \hat{\eta}(a)) < T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(y+b)), C_{IL}(\hat{\tau}_1, \hat{\eta}(x))))$,
for some $x,y,a,b \in S$. Let $\hat{t} \in (\hat{\tau}_1, \hat{\tau}_2]$ such that
$C_{IL}(\hat{\tau}_1, \hat{\eta}(a)) < \hat{t} < T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(y+b)), C_{IL}(\hat{\tau}_1, \hat{\eta}(x))))$. This implies $C_{IL}(\hat{\tau}_1, \hat{\eta}(a)) < \hat{t}$ and $\hat{t} < T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(y+b)), C_{IL}(\hat{\tau}_1, \hat{\eta}(x)))) \leq \hat{\tau}_2 \wedge T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(y+b)), C_{IL}(\hat{\tau}_1, \hat{\eta}(x)))$
$\Rightarrow a \notin \eta_{\hat{t}}$ and $\hat{t} \leq \hat{\tau}_2 \wedge C_{IL}(\hat{\tau}_1, \hat{\eta}(x)) \wedge C_{IL}(\hat{\tau}_1, \hat{\eta}(y+b))$
$\Rightarrow \hat{t} \leq \hat{\tau}_2$, $\hat{t} \leq C_{IL}(\hat{\tau}_1, \hat{\eta}(x))$ and $\hat{t} \leq (C_{IL}(\hat{\tau}_1, \hat{\eta}(y+b))$ and $a \notin \eta_{\hat{t}}$.
This implies $x \in \eta_{\hat{t}}$ and $y+b \in \eta_{\hat{t}}$ and $a \notin \eta_{\hat{t}}$.
As $x \in \eta_{\hat{t}}$, $y+b \in \eta_{\hat{t}}$ and $x+a = y+b$, we get $a \equiv_{\eta_{\hat{t}}} 0$. This implies $a \in \eta_{\hat{t}}$, (Because $\eta_{\hat{t}}$ is a strong ideal of S), which is a contradiction.

Now, we have to show that
$C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1 + s_2 + r)) \geq T_{IL}(\hat{\tau_2}, T_{IL}(C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2)), C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2 + s_1 + r))))$,
$\forall\ s_1, s_2, r \in S$. Suppose there exist $s_1, s_2, r \in S$ such that
$C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1 + s_2 + r)) < T_{IL}(\hat{\tau_2}, T_{IL}(C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2)), C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2 + s_1 + r))))$.
Let $\hat{t} \in (\hat{\tau_1}, \hat{\tau_2}]$ such that
$C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1 + s_2 + r)) < \hat{t} < T_{IL}(\hat{\tau_2}, T_{IL}(C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2)), C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2 + s_1 + r))))$.
Then $C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1 + s_2 + r)) < \hat{t}$ and $\hat{t} \leq \hat{\tau_2} \wedge T_{IL}(C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2)), C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2 + s_1 + r))))$.
$\Rightarrow s_1 + s_2 + r \notin \hat{\eta}_t$ and $\hat{t} \leq \hat{\tau_2} \wedge C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2)) \wedge C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2 + s_1 + r))$.
This implies $s_2 \in \eta_{\hat{t}}$, $s_2 + s_1 + r \in \eta_{\hat{t}}$ and $s_1 + s_2 + r \notin \eta_{\hat{t}}$.
As $\eta_{\hat{t}}$ is a strong ideal of S, $s_2 \in \eta_{\hat{t}}$ and $s_2 + s_1 + r \in \eta_{\hat{t}}$ implies $s_1 + r \in \eta_{\hat{t}}$. Because $s_2 \in \eta_{\hat{t}}$, there exists $q_1 \in \eta_{\hat{t}}$ such that $(s_1 + s_2) + r = (q_1 + s_1) + r \in \eta_{\hat{t}}$, which is a contradiction to $s_1 + s_2 + r \notin \eta_{\hat{t}}$.
Now, suppose there exist $s_1, s_2, r, t \in S$ such that $C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1(s_2 + r) + t)) < T_{IL}(\hat{\tau_2}, T_{IL}(C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2)), C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1 r + t))))$. Let $\hat{t} \in (\hat{\tau_1}, \hat{\tau_2}]$ such that
$C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1(s_2 + r) + t)) < \hat{t} < T_{IL}(\hat{\tau_2}, T_{IL}(C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2)), C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1 r + t))))$.
This implies $C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1(s_2 + r) + t)) < \hat{t}$ and
$\hat{t} \leq \hat{\tau_2} \wedge T_{IL}(C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2)), C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1 r + t)))$.
This gives $t \leq \hat{\tau_2} \wedge C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2)) \wedge C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1 r + t))$ and $s_1(s_2 + r) + t \in \eta_{\hat{t}}$.
This implies $s_2 \in \hat{\eta}_t$, $s_1 r + t \in \eta_{\hat{t}}$ and $s_1(s_2 + r) + t \notin \eta_{\hat{t}}$.
As $\eta_{\hat{t}}$ is a strong ideal of S and $s_2 \in \eta_{\hat{t}}$ then $s_1(s_2 + r) + t = q_1 + s_1 r + t \in \eta_{\hat{t}}$ for some $q_1 \in \eta_{\hat{t}}$.
This implies $s_1(s_2 + r) + t \in \eta_{\hat{t}}$.
This is a contradiction.
Hence $C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1(s_2+r)+t)) \geq T_{IL}(\hat{\tau_2}, T_{IL}(C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2)), C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1 r+t))))$.
Now, we assume that $C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1 s_2)) < T_{IL}(\hat{\tau_2}, C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1)))$, for some $s_1, s_2 \in S$. Then there exists $\hat{t} \in (\hat{\tau_1}, \hat{\tau_2}]$ such that
$C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1 s_2)) < \hat{t} < T_{IL}(\hat{\tau_2}, C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1)))$. This implies $\hat{t} \leq \hat{\tau_2} \wedge C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1))$ and $s_1 s_2 \notin \hat{\eta}_t$. This gives $s_1 \in \hat{\eta}_t$ and $s_1 s_2 \notin \hat{\eta}_t$. This is a contradiction. Hence $\hat{\eta}$ is an interval valued $L-$ fuzzy ideal of S. □

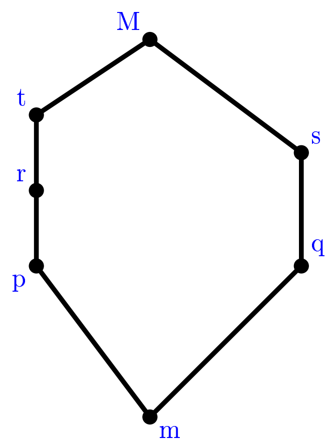
The following mentioned example shows that, if $\hat{\eta}$ is an interval valued $L-$ fuzzy ideal of S then $\eta_{\hat{t}},\ \hat{t} \in (\hat{\tau_1}, \hat{\tau_2}]$ need not be a strong ideal of S.

Example 3.8. Let $S = \{0, p, q, r\}$ be a set with respect to $+$ and $\cdot$ defined as follows.

$+$	0	p	q	r
0	0	p	q	r
p	p	p	q	r
q	q	r	q	r
r	r	r	q	r

.	0	p	q	r
0	0	0	0	0
p	0	p	0	0
q	q	q	q	q
r	q	r	q	q

Then $(S, +, \cdot)$ is a right seminearring. Now, we consider the lattice L as shown in the following figure.



Define $\hat{\eta} : S \to D(L)$ as shown in the below.

$$\hat{\eta}(s) = \begin{cases} [t, M] & \text{if} \quad s = 0 \\ [r, s] & \text{if} \quad s = p \\ [p, s] & \text{if} \quad s \in \{q, r\}. \end{cases}$$

$C_{fL}(s_1, s_2) = s_1 \vee s_2$

$$C_{sL}(s_1, s_2) = \begin{cases} s_1 & \text{if} \quad s_2 = m \\ s_2 & \text{if} \quad s_1 = m \\ M & \text{otherwise.} \end{cases}$$

Let $\hat{\tau_1} = [m, q]$ and $\hat{\tau_2} = [r, s]$.
Then $C_{IL}(\hat{\alpha}, \hat{\eta}(0)) = [C_{fL}(m, r), C_{sL}(q, s)] = [t, M]$
Similarly, $T_{IL}(\hat{\alpha}, \hat{\eta}(0)) = [T_{fL}(m, r), T_{sL}(q, s)]$.
$C_{IL}(\hat{\alpha}, \hat{\eta}(a)) = [r, M]$
$C_{IL}(\hat{\alpha}, \hat{\eta}(b)) = C_I(\hat{\alpha}, \hat{\eta}(c)) = [p, M]$.
Then $\hat{\eta}$ is an interval valued L-fuzzy ideal of S.
Let $\hat{t} = [r, s]$. Then $\alpha < \hat{t} \leq \beta$.
Then $\eta_{\hat{t}} = \{0, p\}$ is not a strong ideal of S.
Because $q + \{0, p\} = \{q, r\} \nsubseteq \{0, p\} + q = q$.

F1: For $a, b \in S$, if $C_{IL}(\hat{\tau_1}, \hat{\eta}(a)) \geq T_{IL}(\hat{\tau_2}, T_{IL}(C_{IL}(\hat{\tau_1}, \hat{\eta}(b)), \hat{t}))$, then we assume that $a \in \eta_{\hat{t}} + b, \ \forall \ \hat{t} \in (\hat{\tau_1}, \hat{\tau_2}]$.

Proposition 3.9. If $\hat{\eta}$ is an interval L-fuzzy ideal of a seminearring S, T_{IL} is an idempotent interval valued t-norm on $D(L)$ and satifies condition F1, then $\eta_{\hat{t}}, \ \forall \hat{t} \in (\hat{\tau_1}, \hat{\tau_2}]$ is a strong ideal of seminearring S.

Proof. Let $s_1, s_2 \in \eta_{\hat{t}}$. Then $C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1)) \geq \hat{t}$ and $C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2)) \geq \hat{t}$.
As $\hat{\eta}$ is an interval L-fuzzy ideal of a seminearring S, we have
$C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1 + s_2)) \geq T_{IL}(\hat{\tau_2}, T_{IL}(C_{IL}(\hat{\tau_1}, \hat{\eta}(s_1)), C_{IL}(\hat{\tau_1}, \hat{\eta}(s_2))))$.

This implies $C_{IL}(\hat{\tau}_1, \hat{\eta}(s_1 + s_2)) \geq T_{IL}(\hat{\tau}_2, T_{IL}(\hat{t}, (C_{IL}(\hat{\tau}_1, \hat{\eta}(s_2))))$
$\Rightarrow C_{IL}(\hat{\tau}_1, \hat{\eta}(s_1 + s_2)) \geq T_{IL}(\hat{\tau}_2, T_{IL}(\hat{t}, \hat{t}))$
$\Rightarrow C_{IL}(\hat{\tau}_1, \hat{\eta}(s_1 + s_2)) \geq T_{IL}(\hat{\tau}_2, \hat{t}) \geq T_{IL}(\hat{t}, \hat{t}) = \hat{t}$.
This implies $s_1 + s_2 \in \eta_{\hat{t}}$.
Now, take $s \in S$ such that $a \in s + \eta_{\hat{t}}$. Then there exists $i_1 \in \eta_{\hat{t}}$ such that $a = s + i_1$.
As $\hat{\eta}$ is an interval L-fuzzy ideal of a seminearring S, we have $C_{IL}(\hat{\tau}_1, \hat{\eta}(a)) = C_{IL}(\hat{\tau}_1, \hat{\eta}(s + i_1)) \geq T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(s)), C_{IL}(\hat{\tau}_1, \hat{\eta}(i_1))))$
$\geq T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(s)), \hat{t}))$.
Then by condition F1, we get $a \in \eta_{\hat{t}} + s$. Therefore $s + \eta_{\hat{t}} \subseteq \eta_{\hat{t}} + s, \ \forall\ s \in S$.
For $s_1, s_2 \in S$, consider $s_1 \equiv_{\eta_{\hat{t}}} s_2$. This implies there exist $i_1, i_2 \in \eta_{\hat{t}}$ such that $i_1 + s_1 = i_2 + s_2$.
Then $C_{IL}(\hat{\tau}_1, \hat{\eta}(s_1)) \geq T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(i_2 + s_2)), C_{IL}(\hat{\tau}_1, \hat{\eta}(i_1))))$
$\geq T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(i_2 + s_2)), \hat{t}))$.
This implies $s_1 \in \eta_{\hat{t}} + i_2 + s_2 \subseteq \eta_{\hat{t}} + s_2$.
Now, take $x \in s_1(\eta_{\hat{t}} + s_2)$. Then there exists $k_1 \in \eta_{\hat{t}}$ such that $x = s_1(k_1 + s_2)$.
Then $C_{IL}(\hat{\tau}_1, s_1(k_1 + s_2)) \geq T_{IL}(\hat{\tau}_2, T_{IL}(C_{IL}(\hat{\tau}_1, \hat{\eta}(k_1)), C_{IL}(\hat{\tau}_1, \hat{\eta}(s_1 s_2))))$
$\geq T_{IL}(\hat{\tau}_2, T_{IL}(\hat{t}, C_{IL}(\hat{\tau}_1, \hat{\eta}(s_1 s_2))))$.
This implies $x \in \eta_{\hat{t}} + s_1 s_2$.
Let $x \in \eta_{\hat{t}} s$. Then there exists $i \in \eta_{\hat{t}}$ such that $x = is$. Then $C_{IL}(\hat{\tau}_1, \hat{\eta}(is)) \geq T_{IL}(\hat{\tau}_2, C_{IL}(\hat{\tau}_1, \hat{\eta}(i))) \geq T_{IL}(\hat{\tau}_2, C_{IL}(\hat{\tau}_1, \hat{t})) \geq T_{IL}(\hat{\tau}_2, \hat{\tau}_2) = \hat{\tau}_2$. Thus $\eta_{\hat{t}}, \ \forall\ \hat{t} \in (\hat{\tau}_1, \hat{\tau}_2]$ is a strong ideal of seminearring S.

□

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