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ON CERTAIN RESULTS RELATED TO RAMANUJAN'S q-CONTINUED FRACTIONS

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Abstract. Continued fractions of various orders have been obtained from Ramanujan's general continued fraction identity. We present several identities involving theta functions related to these continued fractions, as well as various color partition identities as an applications. Additionally, we have derived vanishing coefficients and numerous algebraic relations.

1. Introduction

In this article, we give certain q-continued fractions of various orders that are derived from Ramanujan's general theta function identity. One of the significant contributions made by Ramanujan pertains to the field of q continued fractions. The most renowned of these is the Rogers-Ramanujan continued fraction, denoted as $R(q)$. This concept was first introduced by Rogers in 1894. In 1912, Ramanujan revisited the continued fraction, providing numerous explicit values of $R(q)$ in his notebooks [7, 10, 11] and in his initial correspondence with Hardy. These values were subsequently confirmed by Watson and Ramanathan [14, 15]. For further information, one may refer [3, 5, 6] for more details. Surekha [13] in 2017 introduced modular relations associated with continued fractions of order sixteen, highlighting similarities to the Rogers-Ramanujan continued fraction. Subsequently, Saikia and Rajkhowa [8, 9] made noteworthy advancements by creating continued fractions of different orders that bear resemblance to the Rogers-Ramanujan continued fractions and by establishing modular identities for these fractions. Furthermore, they utilized these ideas to formulate color partition identities grounded in partition theory. In this paper, we consider for all complex numbers. z and q, define the q-product $(z;q)_\infty$ as

$$(z;q)_\infty := \prod_{t=0}^{\infty}(1-zq^t), \qquad |q|<1. \tag{1.1}$$

For simplicity, we often write

$$(z_1;q)_\infty(z_2;q)_\infty(z_3;q)_\infty ...(z_m;q)_\infty = (z_1,z_2,z_3,...,z_m;q)_\infty.$$

The theta function $f(x,y)$ as articulated by Ramanujan [4, p.34] is expressed as

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$$f(x,y) = \sum_{t=-\infty}^{\infty} x^{t(t+1)/2} y^{t(t-1)/2}. \tag{1.2}$$

Also, the $f(x,y)$ interms of Jacobi's triple product identity [4, p.35, Entry 19] can be stated as

$$f(x,y) = (-x;xy)_\infty(-y;xy)_\infty(xy;xy)_\infty = (-x,-y,xy;xy)_\infty. \tag{1.3}$$

The special cases of $f(x,y)$ are the theta-functions $\phi(q)$, $\psi(q)$ and $f(-q)$ [4, p.36, Entry 22 (i)-(iii)] are given by,

$$\phi(q) := f(q,q) = \sum_{t=-\infty}^{\infty} q^{t^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.4}$$

$$\psi(q) := f(q,q^3) = \sum_{t=0}^{\infty} q^{t(t+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.5}$$

$$f(-q) := f(-q,-q^2) = \sum_{t=-\infty}^{\infty} (-1)^t q^{t(3t-1)/2} = (q;q)_\infty. \tag{1.6}$$

After Ramanujan, define

$$\chi(q) = (-q;q^2)_\infty. \tag{1.7}$$

Ramanujan documented numerous continued fractions in his notebooks, with the most renowned being the Rogers-Ramanujan continued fraction of order 5 and has recorded various general identities pertaining to continued fractions in his notebook. Among these, he highlighted a specific general continued fraction identity [4, p. 24, Entry 12]. Suppose that w, z and q are complex numbers with $|wz| < 1$ and $|q| < 1$, or that $w = z^{2t+1}$ for some integer t. Then,

$$\frac{(w^2q^3;q^4)_\infty(z^2q^3;q^4)_\infty}{(w^2q;q^4)_\infty(z^2q;q^4)_\infty} = \cfrac{1}{(1-wz) + \cfrac{(w-zq)(z-wq)}{(1-wz)(q^2+1) + \cfrac{(w-zq^3)(z-wq^3)}{(1-wz)(q^4+1)+\dots}}}. \tag{1.8}$$

In this study, we focus on the q-continued fractions of orders fifty and sixty six. By selecting appropriate values for w and z, along with suitable powers of q, it is possible to derive q-continued fractions of specific orders that fulfill theta function identities similar to those associated with $R(q)$.

Theorem 1.1. *We have, for* $i = 0, 1, 3, 4, 5, 6, 8, 9, 10$ *and* 11

$$\begin{aligned}\omega_i(q) :&= q^{(2i+1)/4}\frac{f(q^{12-i}, q^{38+i}; q^{50})}{f(q^{13+i}, q^{37-i}; q^{50})} = q^{(2i+1)/4}\frac{f(-q^{12-i}, -q^{38+i})}{f(-q^{13+i}, -q^{37-i})} \\ &= \cfrac{q^{(2i+1)/4}(1-q^{12-i})}{(1-q^{25/2}) + \cfrac{q^{25/2}(1-q^{(2i+1)/2})(1-q^{(49-2i)/2})}{(1-q^{25/2})(q^{25}+1) + \cfrac{q^{25/2}(1-q^{(51+2i)/2})(1-q^{(99-2i)/2})}{(1-q^{25/2})(q^{50}+1)+\cdots}}}.\end{aligned} \tag{1.9}$$

Proof. By replacing q by $q^{25/2}$ in ((1.8), setting the values, $\{s = q^{23/4}, t = q^{27/4}\}$, $\{s = q^{21/4}, t = q^{29/4}\}$, $\{s = q^{19/4}, t = q^{31/4}\}$, $\{s = q^{17/4}, t = q^{33/4}\}$, $\{s = q^{13/4}, t = q^{37/4}\}$, $\{s = q^{11/4}, t = q^{39/4}\}$, $\{s = q^{9/4}, t = q^{41/4}\}$, $\{s = q^{7/4}, t = q^{43/4}\}$, $\{s = q^{3/4}, t = q^{47/4}\}$, $\{s = q^{1/4}, t = q^{49/4}\}$ and simplifying using the results that $\{(q^{51}; q^{50})_\infty = (q; q^{50})_\infty/(1-q)\}$, $\{(q^{52}; q^{50})_\infty = (q^2; q^{50})_\infty/(1-q^2)\}$, $\{(q^{53}; q^{50})_\infty = (q^3; q^{50})_\infty/(1-q^3)\}$, $\{(q^{54}; q^{50})_\infty = (q^4; q^{50})_\infty/(1-q^4)\}$, $\{(q^{56}; q^{50})_\infty = (q^6; q^{50})_\infty/(1-q^6)\}$, $\{(q^{57}; q^{50})_\infty = (q^7; q^{50})_\infty/(1-q^7)\}$, $\{(q^{58}; q^{50})_\infty = (q^8; q^{50})_\infty/(1-q^8)\}$, $\{(q^{59}; q^{50})_\infty = (q^9; q^{50})_\infty/(1-q^9)\}$, $\{(q^{61}; q^{50})_\infty = (q^{11}; q^{50})_\infty/(1-q^{11})\}$, $\{(q^{62}; q^{50})_\infty = (q^{12}; q^{50})_\infty/(1-q^{12})\}$, we obtain the following ten continued fractions of order fifty, given Theorem 1.1. □

Theorem 1.2. *We have, for* $i = 0, 2, 3, 5, 6, 8, 9, 11, 12, 14$ *and* 15

$$\begin{aligned}\zeta_i(q) :&= q^{(2i+1)/4}\frac{f(q^{16-i}, q^{50+i}; q^{66})}{f(q^{17+i}, q^{49-i}; q^{66})} = q^{(2i+1)/4}\frac{f(-q^{16-i}, -q^{50+i})}{f(-q^{17+i}, -q^{49-i})} \\ &= \cfrac{q^{(2i+1)/4}(1-q^{16-i})}{(1-q^{33/2}) + \cfrac{q^{33/2}(1-q^{(2i+1)/2})(1-q^{(65-2i)/2})}{(1-q^{33/2})(q^{33}+1) + \cfrac{q^{33/2}(1-q^{(67+2i)/2})(1-q^{(131-2i)/2})}{(1-q^{33/2})(q^{66}+1)+\cdots}}}.\end{aligned} \tag{1.10}$$

Proof. we derive eleven continued fractions of order sixty-six, by replacing q by $q^{33/2}$ in (1.8), setting the values, $\{s = q^{31/4}, t = q^{35/4}\}$, $\{s = q^{29/4}, t = q^{37/4}\}$, $\{s = q^{25/4}, t = q^{41/4}\}$, $\{s = q^{23/4}, t = q^{43/4}\}$, $\{s = q^{19/4}, t = q^{47/4}\}$, $\{s = q^{17/4}, t = q^{49/4}\}$, $\{s = q^{13/4}, t = q^{53/4}\}$, $\{s = q^{11/4}, t = q^{55/4}\}$, $\{s = q^{7/4}, t = q^{59/4}\}$, $\{s = q^{5/4}, t = q^{61/4}\}$ and $\{s = q^{1/4}, t = q^{65/4}\}$, and simplifying using the results $\{(q^{67}; q^{66})_\infty = (q; q^{66})_\infty/(1-q)\}$, $\{(q^{68}; q^{66})_\infty = (q^2; q^{66})_\infty/(1-q^2)\}$, $\{(q^{70}; q^{66})_\infty = (q^4; q^{66})_\infty/(1-q^4)\}$, $\{(q^{71}; q^{66})_\infty = (q^5; q^{66})_\infty/(1-q^5)\}$, $\{(q^{73}; q^{66})_\infty = (q^7; q^{66})_\infty/(1-q^7)\}$, $\{(q^{74}; q^{66})_\infty = (q^9; q^{66})_\infty/(1-q^9)\}$, $\{(q^{76}; q^{66})_\infty = (q^{10}; q^{66})_\infty/(1-q^{10})\}$, $\{(q^{77}; q^{66})_\infty = (q^{11}; q^{66})_\infty/(1-q^{11})\}$, $\{(q^{79}; q^{66})_\infty = (q^{13}; q^{66})_\infty/(1-q^{13})\}$, $\{(q^{80}; q^{66})_\infty = (q^{14}; q^{66})_\infty/(1-q^{14})\}$, $\{(q^{82}; q^{66})_\infty = (q^{16}; q^{66})_\infty/(1-q^{16})\}$, we obtain the following eleven continued fractions of order sixty-six, respectively: □

The main objective of this paper is to establish theta-function identities for continued fractions of orders fifty and sixty-six. In Section 2, we define some theta function identities for continued fraction. In Section 3, we presents vanishing coefficient results associated with both orders. In section 4, we demonstrate how theta-function identities can be utilized to derive colored partition identities, providing suitable examples.

2. Theta-function identities for $\omega_i(q)$ and $\zeta_i(q)$

This section is dedicated to, establishing theta-function identities for the continued fractions of order 50 and order 66.

Theorem 2.1. *For* $i = 0, 1, 3, 4, 5, 6, 8, 9, 10,$ *and* 11, *we have*

$$\frac{1}{\omega_i(q)} \mp \omega_i(q) = \frac{\phi(\pm q^{25/2}) f(\mp q^{(2i+1)/2}, \mp q^{(51-2i)/2})}{q^{(2i+1)/4}\psi(q^{25}) f(-q^{12-i}, -q^{13+i})}$$

Proof. From (1.9), we obtain

$$\frac{1}{\sqrt{\omega_1(q)}} - \sqrt{\omega_1(q)} = \frac{f(-q^{13}, -q^{37}) - q^{1/4} f(-q^{12}, -q^{38})}{\sqrt{q^{1/4} f(-q^{12}, -q^{38}) f(-q^{13}, -q^{37})}}. \tag{2.1}$$

From [4, pp.46, Entry 30 (ii) and (iii)], we note that

$$f(x, y) = f(x^3 y, xy^3) + x f(y/x, x^5 y^3). \tag{2.2}$$

Putting $\{x = -q^{1/4},\ y = q^{49/4}\}$ and $\{x = q^{1/4}, y = -q^{49/4}\}$ in (2.2), we get

$$f(-q^{1/4}, q^{49/4}) = f(-q^{13}, -q^{37}) + (-q^{1/4}) f(-q^{12}, -q^{38}) \tag{2.3}$$

and

$$f(q^{1/4}, -q^{49/4}) = f(-q^{13}, -q^{37}) + q^{1/4} f(-q^{12}, -q^{38}). \tag{2.4}$$

Using (2.3) in (2.1), we conclude that

$$\frac{1}{\sqrt{\omega_1(q)}} - \sqrt{\omega_1(q)} = \frac{f(-q^{1/4}, q^{49/4})}{\sqrt{q^{1/4} f(-q^{12}, -q^{38}) f(-q^{13}, -q^{37})}}. \tag{2.5}$$

Similarly, from (1.9) and applying (2.4), we find that

$$\frac{1}{\sqrt{\omega_1(q)}} + \sqrt{\omega_1(q)} = \frac{f(q^{1/4}, -q^{49/4})}{\sqrt{q^{1/4} f(-q^{12}, -q^{38}) f(-q^{13}, -q^{37})}}. \tag{2.6}$$

Combining (2.5) and (2.6) we arrive at

$$\frac{1}{\omega_1(q)} - \omega_1(q) = \frac{f(-q^{1/4}, q^{49/4}) f(q^{1/4}, -q^{49/4})}{q^{1/4} f(-q^{12}, -q^{38}) f(-q^{13}, -q^{37})}. \tag{2.7}$$

Again, from [4, pp.46, Entry 30 (i)(iv)] we note

$$f(x, xy^2) f(y, x^2 y) = f(x, y)\psi(xy) \tag{2.8}$$

and

$$f(x, y) f(-x, -y) = f(-x^2, -y^2)\phi(-xy). \tag{2.9}$$

Putting $\{x = -q^{12},\ y = -q^{13}\}$ in (2.8) and $\{x = -q^{1/4}, y = q^{49/4}\}$ in (2.9), we get

$$f(-q^{12}, -q^{38}) f(-q^{13}, -q^{37}) = f(-q^{12}, -q^{13})\psi(q^{25}) \tag{2.10}$$

and

$$f(-q^{1/4}, q^{49/4}) f(q^{1/4}, -q^{49/4}) = f(-q^{1/2}, -q^{49/2})\phi(q^{25/2}). \tag{2.11}$$

respectively. Using (2.10) and (2.11) in (2.7), we complete the proof first identity. Squaring on both sides (2.6), we obtain

$$\frac{1}{\omega_1(q)} + \omega_1(q) = \frac{f^2(q^{1/4}, -q^{49/4})}{q^{1/4} f(-q^{12}, -q^{13})\psi(q^{25})} - 2. \tag{2.12}$$

From[4, pp.46, Entry 30 (v), (vi)], we note that

$$f^2(x, y) = f(x^2, y^2)\phi(xy) + 2xf(y/x, x^3y)\psi(x^2y^2). \tag{2.13}$$

Setting $\{x = q^{1/4},\ y = -q^{49/4}\}$ we get

$$f^2(q^{1/4}, -q^{49/4}) = f(q^{1/2}, q^{49/2})\phi(-q^{25/2}) + 2q^{1/4} f(-q^{12}, -q^{13})\psi(q^{25}). \tag{2.14}$$

Using (2.14) and (2.10) in (2.12) and simplifying, we arrive at (i). Proofs of other theta function identities are similar, so we omit the proof. □

Theorem 2.2. *We have*

$$\frac{1}{\zeta_i(q)} \mp \zeta_i(q) = \frac{\phi(\pm q^{33/2}) f(\mp q^{(2i+1)/2}, \mp q^{(65-2i)/2})}{q^{(2i+1)/4}\psi(q^{33}) f(-q^{16-i}, -q^{17+i})}$$

for $i = 0, 2, 3, 5, 6, 8, 9, 11, 12, , 14$ *and* 15

Proof. From (1.10), we obtain

$$\frac{1}{\sqrt{\zeta_1(q)}} - \sqrt{\zeta_1(q)} = \frac{f(-q^{17}, -q^{49}) - q^{1/4} f(-q^{16}, -q^{50})}{\sqrt{q^{1/4} f(-q^{16}, -q^{50}) f(-q^{17}, -q^{49})}}. \tag{2.15}$$

From [4, pp. 46, Entry 30 (ii) and (iii)], we note that

$$f(x, y) = f(x^3y, xy^3) + xf(y/x, x^5y^3). \tag{2.16}$$

Setting $x = -q^{1/4}$, $y = q^{65/4}$ and $x = q^{1/4}$, $y = -q^{65/4}$ in (2.16), we get

$$f(-q^{1/4}, q^{65/4}) = f(-q^{17}, -q^{49}) + (-q^{1/4}) f(-q^{16}, -q^{50}), \tag{2.17}$$

and

$$f(q^{1/4}, -q^{65/4}) = f(-q^{17}, -q^{49}) + q^{1/4} f(-q^{16}, -q^{50}). \tag{2.18}$$

Using (2.17) in (2.15), we obtain

$$\frac{1}{\sqrt{\zeta_1(q)}} - \sqrt{\zeta_1(q)} = \frac{f(-q^{1/4}, q^{65/4})}{\sqrt{q^{1/4} f(-q^{16}, -q^{50}) f(-q^{17}, -q^{49})}}. \tag{2.19}$$

Similarly, from (1.10) and applying (2.18), we find that

$$\frac{1}{\sqrt{\zeta_1(q)}} + \sqrt{\zeta_1(q)} = \frac{f(q^{1/4}, -q^{65/4})}{\sqrt{q^{1/4} f(-q^{16}, -q^{50}) f(-q^{17}, -q^{49})}}. \tag{2.20}$$

Combining (2.19) and (2.20), we arrive at

$$\frac{1}{\zeta_1(q)} - \zeta_1(q) = \frac{f(-q^{1/4}, q^{65/4}) f(q^{1/4}, -q^{65/4})}{q^{1/4} f(-q^{16}, -q^{50}) f(-q^{17}, -q^{49})}. \tag{2.21}$$

From [4, pp. 46, Entry 30 (i) and (iv)], we note

$$f(x, xy^2) f(y, x^2y) = f(x, y)\psi(xy), \tag{2.22}$$

and

$$f(x, y) f(-x, -y) = f(-x^2, -y^2)\phi(-xy). \tag{2.23}$$

Setting $x=-q^{16}$, $y=-q^{17}$ in (2.22) and $x=-q^{1/4}$, $y=q^{65/4}$ in (2.23), we get

$$f(-q^{16},-q^{50})f(-q^{17},-q^{49})=f(-q^{16},-q^{17})\psi(q^{33}), \tag{2.24}$$

and

$$f(-q^{1/4},q^{65/4})f(q^{1/4},-q^{65/4})=f(-q^{1/2},-q^{65/2})\phi(q^{33/2}). \tag{2.25}$$

Using (2.24) and (2.25) in (2.21), we obtain the first identity.
Squaring both sides of (2.20), we obtain

$$\frac{1}{\zeta_1(q)}+\zeta_1(q)=\frac{f^2(q^{1/4},-q^{65/4})}{q^{1/4}f(-q^{16},-q^{17})\psi(q^{33})}-2. \tag{2.26}$$

From [4, pp. 46, Entry 30 (v) and (vi)], we note that

$$f^2(x,y)=f(x^2,y^2)\phi(xy)+2xf(y/x,x^3y)\psi(x^2y^2). \tag{2.27}$$

Setting $x=q^{1/4}$, $y=-q^{65/4}$, we get

$$f^2(q^{1/4},-q^{65/4})=f(q^{1/2},q^{65/2})\phi(-q^{33/2})+2q^{1/4}f(-q^{16},-q^{17})\psi(q^{33}). \tag{2.28}$$

Using (2.28) and (2.24) in (2.26) and simplifying, we arrive at the first result. Proofs of other theta function identities are similar, so we omit them. □

3. Vanishing Coefficients

In this section we obtain vanishing coefficients for continued fraction defined earlier in section 1. For further details one can refer [2, 12].

Theorem 3.1.

$$\omega_1^*(q)=\frac{(q^{12},q^{38};q^{50})_\infty}{(q^{13},q^{37};q^{50})_\infty}=\sum_{n=0}^{\infty}\alpha_n q^n,$$

then, $\alpha'_{25n+9}=0$.

Proof. The following p-dissection formula stated by Andrews and Bressoud [1]

$$\frac{(q^m,q^m,q^{k+l},q^{m-k-l};q^m)_\infty}{(q^l,q^{m-l},q^k,q^{m-k};q^m)_\infty}=\sum_{j=0}^{p-1}q^{jk}\frac{(q^{pm},q^{pm},q^{pk+l+jm},q^{(p-j)m-pk-l};q^{pm})_\infty}{(q^{jm+l},q^{(p-j)m-l},q^{pk},q^{(m-k)p};q^{pm})_\infty} \tag{3.1}$$

Where all of the powers of q in each of the infinite products on the right hand side must be multiple of p and the integer k must satisfy $gcd(k,p)=1$ i.e., $gcd(13,25)=1$.
Now, setting $m=50$, $k=13$, $p=25$ and $l=25$ in (3.1), we get

$$\frac{(q^{50},q^{50},q^{38},q^{12};q^{50})_\infty}{(q^{25},q^{25},q^{13},q^{37};q^{50})_\infty}=\sum_{j=0}^{24}q^{13j}\frac{(q^{1250},q^{1250},q^{350+50j},q^{900-50j};q^{1250})_\infty}{(q^{25+50j},q^{1225-50j},q^{325},q^{925};q^{1250})_\infty}$$

$$=q^0\frac{(q^{1250},q^{1250},q^{350},q^{900};q^{1250})_\infty}{(q^{25},q^{1225},q^{325},q^{925};q^{1250})_\infty}+q^{13}\frac{(q^{1250},q^{1250},q^{400},q^{850};q^{1250})_\infty}{(q^{75},q^{1175},q^{325},q^{925};q^{1250})_\infty}$$

$$+q^{26}\frac{(q^{1250},q^{1250},q^{450},q^{800};q^{1250})_\infty}{(q^{125},q^{1125},q^{325},q^{925};q^{1250})_\infty}+q^{39}\frac{(q^{1250},q^{1250},q^{500},q^{750};q^{1250})_\infty}{(q^{175},q^{1075},q^{325},q^{925};q^{1250})_\infty}$$

$$+q^{52}\frac{(q^{1250},q^{1250},q^{550},q^{700};q^{1250})_\infty}{(q^{225},q^{1025},q^{325},q^{925};q^{1250})_\infty}+q^{65}\frac{(q^{1250},q^{1250},q^{600},q^{650};q^{1250})_\infty}{(q^{275},q^{975},q^{325},q^{925};q^{1250})_\infty}$$
$$+q^{78}\frac{(q^{1250},q^{1250},q^{650},q^{600};q^{1250})_\infty}{(q^{325},q^{925},q^{325},q^{925};q^{1250})_\infty}+q^{91}\frac{(q^{1250},q^{1250},q^{700},q^{550};q^{1250})_\infty}{(q^{375},q^{875},q^{325},q^{925};q^{1250})_\infty}$$
$$+q^{104}\frac{(q^{1250},q^{1250},q^{750},q^{500};q^{1250})_\infty}{(q^{425},q^{825},q^{325},q^{925};q^{1250})_\infty}+q^{117}\frac{(q^{1250},q^{1250},q^{800},q^{450};q^{1250})_\infty}{(q^{475},q^{775},q^{325},q^{925};q^{1250})_\infty}$$
$$+q^{130}\frac{(q^{1250},q^{1250},q^{850},q^{400};q^{1250})_\infty}{(q^{525},q^{725},q^{325},q^{925};q^{1250})_\infty}+q^{143}\frac{(q^{1250},q^{1250},q^{900},q^{350};q^{1250})_\infty}{(q^{575},q^{675},q^{325},q^{925};q^{1250})_\infty}$$
$$+q^{156}\frac{(q^{1250},q^{1250},q^{950},q^{300};q^{1250})_\infty}{(q^{625},q^{625},q^{325},q^{925};q^{1250})_\infty}+q^{169}\frac{(q^{1250},q^{1250},q^{1000},q^{250};q^{1250})_\infty}{(q^{675},q^{575},q^{325},q^{925};q^{1250})_\infty}$$
$$+q^{182}\frac{(q^{1250},q^{1250},q^{1050},q^{200};q^{1250})_\infty}{(q^{725},q^{525},q^{325},q^{925};q^{1250})_\infty}+q^{195}\frac{(q^{1250},q^{1250},q^{1100},q^{150};q^{1250})_\infty}{(q^{775},q^{475},q^{325},q^{925};q^{1250})_\infty}$$
$$+q^{208}\frac{(q^{1250},q^{1250},q^{1150},q^{100};q^{1250})_\infty}{(q^{825},q^{425},q^{325},q^{925};q^{1250})_\infty}+q^{221}\frac{(q^{1250},q^{1250},q^{1200},q^{50};q^{1250})_\infty}{(q^{875},q^{375},q^{325},q^{925};q^{1250})_\infty}$$
$$+q^{234}\frac{(q^{1250},q^{1250},q^{1250},q^{0};q^{1250})_\infty}{(q^{925},q^{325},q^{325},q^{925};q^{1250})_\infty}+q^{247}\frac{(q^{1250},q^{1250},q^{1300},q^{-50};q^{1250})_\infty}{(q^{975},q^{275},q^{325},q^{925};q^{1250})_\infty}$$
$$+q^{260}\frac{(q^{1250},q^{1250},q^{1350},q^{-100};q^{1250})_\infty}{(q^{1025},q^{225},q^{325},q^{925};q^{1250})_\infty}+q^{273}\frac{(q^{1250},q^{1250},q^{1400},q^{-150};q^{1250})_\infty}{(q^{1075},q^{175},q^{325},q^{925};q^{1250})_\infty}$$
$$+q^{286}\frac{(q^{1250},q^{1250},q^{1450},q^{-200};q^{1250})_\infty}{(q^{1125},q^{125},q^{325},q^{925};q^{1250})_\infty}+q^{299}\frac{(q^{1250},q^{1250},q^{1500},q^{-250};q^{1250})_\infty}{(q^{1175},q^{75},q^{325},q^{925};q^{1250})_\infty}$$
$$+q^{312}\frac{(q^{1250},q^{1250},q^{1550},q^{-300};q^{1250})_\infty}{(q^{1225},q^{25},q^{325},q^{925};q^{1250})_\infty}. \tag{3.2}$$

□

Since the right-hand side of (3.2) involves extracting the terms containing q^{25n+9}, we arrive at the result.
In the following table, we present the vanishing coefficients of the remaining q-series expressions related to the continued fractions of order fifty. Proofs are identical to the above Theorem 3.1, therefore, we omit them.

TABLE 1. Vanishing Coefficients for Continued Fractions

q-series/continued fractions	**vanishing coefficients**
$\frac{1}{\omega_2^*(q)}=\frac{(q^{14},q^{36};q^{50})_\infty}{(q^{11},q^{39};q^{50})_\infty}=\sum_{n=0}^\infty\beta_n^{'}q^n$	$\beta'_{25n+9}=0$
$\frac{1}{\omega_4^*(q)}=\frac{(q^{16},q^{34};q^{50})_\infty}{(q^{9},q^{41};q^{50})_\infty}=\sum_{n=0}^\infty\gamma_n^{'}q^n$	$\gamma'_{25n+5}=0$
$\omega_5^*(q)=\frac{(q^{8},q^{42};q^{50})_\infty}{(q^{17},q^{33};q^{50})_\infty}=\sum_{n=0}^\infty\delta_nq^n$	$\delta_{25n+5}=0$
$\frac{1}{\omega_6^*(q)}=\frac{(q^{18},q^{32};q^{50})_\infty}{(q^{7},q^{43};q^{50})_\infty}=\sum_{n=0}^\infty\epsilon_n^{'}q^n$	$\epsilon'_{25n+22}=0$

$\omega_7^*(q) = \frac{(q^6,q^{44};q^{50})_\infty}{(q^{19},q^{31};q^{50})_\infty} = \sum_{n=0}^{\infty} \varepsilon_n q^n$	$\varepsilon_{25n+10} = 0$
$\omega_9^*(q) = \frac{(q^4,q^{46};q^{50})_\infty}{(q^{21},q^{29};q^{50})_\infty} = \sum_{n=0}^{\infty} \zeta_n q^n$	$\zeta_{25n+19} = 0$
$\frac{1}{\omega_{10}^*(q)} = \frac{(q^{22},q^{28};q^{50})_\infty}{(q^3,q^{47};q^{50})_\infty} = \sum_{n=0}^{\infty} \eta_n' q^n$	$\eta'_{25n+19} = 0$
$\omega_{11}^*(q) = \frac{(q^2,q^{48};q^{50})_\infty}{(q^{23},q^{27};q^{50})_\infty} = \sum_{n=0}^{\infty} \lambda_n q^n$	$\lambda_{25n+24} = 0$
$\frac{1}{\omega_{12}^*(q)} = \frac{(q^{24},q^{26};q^{50})_\infty}{(q^1,q^{49};q^{50})_\infty} = \sum_{n=0}^{\infty} \mu_n' q^n$	$\mu'_{25n+24} = 0$

Theorem 3.2.

$$\zeta_1^*(q) = \frac{(q^{16}, q^{50}; q^{66})_\infty}{(q^{17}, q^{49}; q^{66})_\infty} = \sum_{n=0}^{\infty} \alpha_n q^n,$$

then $\alpha_{33n+12} = 0$.

Proof. The proof of Theorem 3.2 is identical proof of Theorem 3.1, so we omit the proof.

In the following table , we present the vanishing coefficients of the q-series expressions related to the continued fractions of order 50. Proofs are identical to the proof of Theorem 3.1 ; therefore, we omit them.

TABLE 2. Vanishing Coefficients for Continued Fractions

q-series/Continued Fractions	**Vanishing Coefficients**
$\zeta_2^*(q) = \frac{(q^{14},q^{52};q^{66})_\infty}{(q^{19},q^{47};q^{66})_\infty} = \sum_{n=0}^{\infty} \beta_n q^n$	$\beta_{33n+8} = 0$
$\frac{1}{\zeta_3^*(q)} = \frac{(q^{20},q^{46};q^{66})_\infty}{(q^{13},q^{53};q^{66})_\infty} = \sum_{n=0}^{\infty} \gamma_n q^n$	$\gamma_{33n+8} = 0$
$\zeta_5^*(q) = \frac{(q^{10},q^{56};q^{66})_\infty}{(q^{23},q^{43};q^{66})_\infty} = \sum_{n=0}^{\infty} \epsilon_n q^n$	$\epsilon_{33n+21} = 0$
$\zeta_6^*(q) = \frac{(q^8,q^{58};q^{66})_\infty}{(q^{25},q^{41};q^{66})_\infty} = \sum_{n=0}^{\infty} \varepsilon_n q^n$	$\varepsilon_{33n+5} = 0$
$\frac{1}{\zeta_7^*(q)} = \frac{(q^{26},q^{40};q^{66})_\infty}{(q^7,q^{59};q^{66})_\infty} = \sum_{n=0}^{\infty} \zeta_n q^n$	$\zeta_{33n+5} = 0$
$\frac{1}{\zeta_8^*(q)} = \frac{(q^{28},q^{38};q^{66})_\infty}{(q^5,q^{61};q^{66})_\infty} = \sum_{n=0}^{\infty} \eta_n q^n$	$\eta_{33n+18} = 0$
$\zeta_9^*(q) = \frac{(q^4,q^{62};q^{66})_\infty}{(q^{29},q^{37};q^{66})_\infty} = \sum_{n=0}^{\infty} \lambda_n q^n$	$\lambda_{33n+27} = 0$
$\zeta_{10}^*(q) = \frac{(q^2,q^{64};q^{66})_\infty}{(q^{31},q^{35};q^{66})_\infty} = \sum_{n=0}^{\infty} \xi_n q^n$	$\xi_{33n+32} = 0$
$\frac{1}{\zeta_{11}^*(q)} = \frac{(q^{32},q^{34};q^{66})_\infty}{(q^1,q^{65};q^{66})_\infty} = \sum_{n=0}^{\infty} \omega_n q^n$	$\mu_{33n+32} = 0$

□

4. Some partition-theoretic results

Here, we derive colour partition identities from using Theorem 1.1 and 1.2. First, we give the simple definition of colour partition of a positive integer k and its generating function.
"A partition of a positive integer k, where parts of the same size can be of different colours.
Example: There are 5 two-colour partitions of 2, and its denoted by $\delta_2(2) = 5$. i.e., 2_y, 2_g, $1_y + 1_g$, $1_y + 1_y$ and $1_g + 1_g$, where colour yellow (represented by suffix y)and green (represented by suffix g) the generating function of $\delta_k(n)$ is given by

$$\sum_{n=0}^{\infty} \delta_k(n) q^n = \frac{1}{(q;q)_\infty^k} \tag{4.1}$$

For positive integer a, b and k, the division

$$\frac{1}{(q^a;q^b)_\infty^k} \tag{4.2}$$

is the generating function of the number of partitions of n with parts congruent to a modulo b and each part contains k colours is given by

$$(q^{k\pm};q^c) := (q^k, q^{c-k}; q^c)_\infty \tag{4.3}$$

where k and c are positive integers and $k < c$."

Theorem 4.1. *For any positive integer $n \geq 23$, let $\gamma_1(n)$ be the quantity of methods to divide n into components that are congruent to $\pm 2, \pm 23, \pm 27$ or $\pm 50 \pmod{100}$ such that the parts congruent to ± 2 and $\pm 50 \pmod{100}$ have two colors, $\gamma_2(n)$ be the quantity of methods to divide n into components that are congruent to $\pm 23, \pm 27, \pm 48$ or $\pm 50 \pmod{100}$ such that the parts congruent to ± 48 and $\pm 50 \pmod{100}$ have two colors and $\gamma_3(n)$ be the quantity of methods to divide n into components that are congruent to $\pm 2, \pm 48,$ and $\pm 25 \pmod{100}$ with two colors. Then*

$$\gamma_1(n) - \gamma_2(n-23) - \gamma_3(n) = 0.$$

Proof. Using (1.9), (1.4), (1.5) and replacing q by q^2 in Theorem 2.2, we get

$$\frac{(q^{48\pm};q^{100})_\infty}{(q^{2\pm};q^{100})_\infty} - q^{23}\frac{(q^{2\pm};q^{100})_\infty}{(q^{48\pm};q^{100})_\infty} - \frac{(q^{23\pm,27\pm};q^{100})_\infty (q^{50\pm};q^{100})_\infty^2}{(q^{2\pm,48\pm};q^{100})_\infty (q^{25\pm};q^{100})_\infty^2} = 0. \tag{4.4}$$

Dividing (4.4) by $(q^{2\pm,23\pm,27\pm,48\pm};q^{100})_\infty (q^{50\pm};q^{100})_\infty^2$, we get

$$\frac{1}{(q^{2\pm,50\pm};q^{100})_\infty^2 (q^{23\pm,27\pm};q^{100})_\infty} - \frac{q^{23}}{(q^{48\pm,50\pm};q^{100})_\infty^2 (q^{23\pm,27\pm};q^{100})_\infty} - \frac{1}{(q^{2\pm,48\pm,25\pm};q^{100})_\infty^2} = 0. \tag{4.5}$$

The above quotients of (4.5) represent the generating functions for $\beta_1(n)$, $\beta_2(n)$ and $\beta_3(n)$, respectively. Hence, (4.5) is equivalent to

$$\sum_{n=0}^{\infty} \gamma_1(n) q^n - q^{23} \sum_{n=0}^{\infty} \gamma_2(n) q^n - \sum_{n=0}^{\infty} \gamma_3(n) q^n = 0. \tag{4.6}$$

Where we set $\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = 1$. By matching the coefficients of q^n on each side, we achieve the intended result.
Theorem 4.3 is visualized in the Table 3 below:

Table 3. The case $n = 23$ of Theorem 4.3

$\gamma_1(23) = 1$	$\gamma_2(0) = 1$	$\gamma_3(23) = 0$
$1_r + 1_r + ... + 21 times$		

Theorem 4.2. *For any positive integer $n \geq 31$, let $\xi_1(n)$ be the quantity of methods to divide n into components that are congruent to $\pm 2, \pm 31, \pm 35$ or ± 66 (mod 132) such that the parts congruent to $\pm 2 \pm 66$ (mod 132) have 2 colours. Let $\xi_2(n)$ be the quantity of methods to divide n into components that are congruent to $\pm 31, \pm 35, \pm 64$ or ± 66 (mod 132) such that the parts congruent to ± 64 and ± 66 (mod 132) have 2 colours and $\xi_3(n)$ be the quantity of methods to divide n into components that are congruent to $\pm 2, \pm 33$ and ± 64 (mod 132) with 2 colours. Then,*

$$\xi_1(n) - \xi_2(n-31) - \xi_3(n) = 0.$$

Proof. Using (1.10), (1.4), (1.5) and replacing q by q^2 in Theorem 2.1, we get

$$\frac{(q^{64\pm}; q^{132})_\infty}{(q^{2\pm}; q^{132})_\infty} - q^{31} \frac{(q^{2\pm}; q^{132})_\infty}{(q^{64\pm}; q^{132})_\infty} - \frac{(q^{31\pm,35\pm}; q^{132})_\infty (q^{66\pm}; q^{132})^2_\infty}{(q^{2\pm,64\pm}; q^{132})_\infty (q^{33\pm}; q^{132})^2_\infty} = 0. \quad (4.7)$$

Dividing (4.4) by $(q^{2\pm 31 \pm 35 \pm 64 \pm}; q^{132})_\infty (q^{66\pm}; q^{132})^2_\infty$, we get

$$\frac{1}{(q^{2\pm,66\pm}; q^{132})^2_\infty (q^{31\pm,35\pm}; q^{132})_\infty} - \frac{q^{31}}{(q^{64\pm,66\pm}; q^{132})^2_\infty (q^{31\pm,35\pm}; q^{132})_\infty} - \frac{1}{(q^{2\pm,33\pm,64\pm}; q^{132})^2_\infty} = 0. \quad (4.8)$$

The above quotients of (4.8) represent the generating functions for $\xi_1(n)$, $\xi_2(n)$ and $\xi_3(n)$, respectively. Hence, (4.8) is equivalent to

$$\sum_{n=0}^{\infty} \xi_1(n) q^n - q^{31} \sum_{n=0}^{\infty} \xi_2(n) q^n - \sum_{n=0}^{\infty} \xi_3(n) q^n = 0. \quad (4.9)$$

Where we set $\xi_1(0) = \xi_2(0) = \xi_3(0) = 1$. By matching the coefficients of q^n on each side, we achieve the intended result. Theorem 4.2 is illustrated in the Table 4 below:

Table 4. The case $n = 31$ of Theorem 4.2

$\xi_1(31) = 1$	$\xi_2(0) = 1$	$\xi_3(31) = 0$
$1_b + 1_b + ... + 29 times$		

□

The colour partition identities for the remaining continued fractions of both the orders can be obtianed in similar manner. So, we omit the proof. □

References

1. G. E. Andrews, D. Bressoud, Vanishing coefficients in infinite product expansion, *J. Aust. Math. Soc. Ser.* **27** (1979) 199–202.
2. N. D. Baruah, M. Kaur, Some results on vanishing coefficients in ifinite product series expansion, *Ramanujan J*, **53** (2020) 551–568.
3. N. D. Baruah, N. Saikia, Some mew explicit values of Ramanujan's continued fractions, *Indian J. Math.*, **46(2-3)** (2004) 197-222.
4. B. C. Berndt, Ramanujan's notebooks: Part III. Springer, New York, (1991).
5. B. C. Berndt, H. H. Chan, L. C. Zhang, Some values for the Rogers-Ramanujan continued fraction, *Canad J. Math.* **47(5)** (1995) 897–914.
6. B. C. Berndt, H. H. Chan, L. C. Zhang, Explicit evaluations of the Rogers-Ramanujan continued fraction, *J. Reine Angew Math*, **480** (1996) 141–159.
7. B. C. Berndt, Ramanujan's Notebooks, Part V, Springer, New York, (1998).
8. S. Rajkhowa, N. Saikia, Some results on Ramanujan's continued fractions of order ten and applications, *Indian J. Pure Appl. Math.* https://doi.org/10.1007/s13226-023-00456-5 (2023).
9. S. Rajkhowa, N. Saikia, Theta-function identities of Ramanujan's continued fractions of order fourteen and twenty eight, partition identities and vanishing coefficients, *Funct. Approx. Comment. Math.* **70(2)** (2024) 233-244.
10. S. Ramanujan, Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay, (1957).
11. S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, (1988).
12. L. J. Rogers, Second memoir on the expansion of certain infinite products, *Proc. London Math. Soc.* **25**, (1894) 318–343.
13. M. S. Surekha, On the modular relations and dissections for a continued fraction of order sixteen, *Palestein Journal of Mathematics* **6(1)** (2017) 119-132.
14. G. N. Watson, Theorems stated by Ramanujan (VII), Theorems on continued frations, *J. London Math. Soc.* **4** (1929) 39–48.
15. G. N. Watson, Theorems stated by Ramanujan (IX), Two continued frations, *J. London Math. Soc.* **4** (1929) 231–237.

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