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SPECIFIC CONVOLUTION SUMS AND DIFFERENTIAL COEFFICIENTS UTILIZING SERIES IDENTITIES

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Abstract. In his notebooks, Ramanujan noted various degrees of appealing Eisenstein series relations. In addition, Shaun Cooper mentioned a number of identities involving theta functions and Eisenstein series in his work.This work establishes certain differential identities involving Eisenstein series of various levels, eta-functions, and series identities. Additionally, the Eisenstein series of levels 5 and 7 is being used to evaluate the convolution sum.

1. Introduction

Computational mathematics relies heavily on differential equations and convolution sums. In his work, Ramanujan[3] documented certain differential equations that are derived using theta functions.In his study, Berndt B. C. et al.[4] highlighted the significance in developing differential equations deploying Eisenstein series and η-functions. In Sections 8, 9, and 10, they[4] have developed specific differential equations to demonstrate the identities of orders 14 and 35. Recently, some differential equations involving Eisenstein series and theta function identities were established by Vidya H. C. and Srivatsa Kumar B. R.[7]. Additionally, they used Shaun Cooper's Eisenstein series of various degrees to analyze discrete Convolution sums.

This work offers a suitable approach for creating differential equations incorporating η-functions and series identities. This is accomplished by utilizing some of the Eisenstein series relations documented by Cooper S.[5].We have developed some differential equations in Section 3, and we have assessed convolution sums in Section 4 utilizing the Eisenstein series of levels 5 and 7 as well as Glaisher's identity[6]. Section 2 is devoted to documenting some early findings.

2. Preliminaries

Definition 2.1. [3] For any complex a and q with $|q| < 1$, the q-series is defined by

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n).$$

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For $|ab| < 1$, Ramanujan's general theta-function [3, p.35] is given by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a,-b,ab;ab)_\infty.$$

The special case of a theta function recorded by Ramanujan [3, p.35] is defined by

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = q^{-1/24}\eta(q). \tag{2.1}$$

Definition 2.2. [3] The Ramanujan-type Eisenstein series are defined by

$$P(q) := 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

$$Q(q) := 1 + 240\sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n) q^n.$$

Definition 2.3. [5] The following power series identity

$$Z = \sum_{n=o}^{\infty} h(n) X^n$$

converges in some neighborhood of $X = 0$.

Lemma 2.4. [5] *Let $P(q)$ be the Eisenstein series as defined in Definition 1.2. Then the following identities hold:*

Level	$Z = \sum\limits_{n=o}^{\infty} h(n)X^n$	X
5	$\frac{5P_5-P_1}{4} = \sum\limits_{n=0}^{\infty} \left[\binom{2n}{n} \sum\limits_k \binom{n}{k}^2 \binom{n+k}{n}\right] X^n$	$\frac{\eta_1^4\eta_5^4}{z^2}$
6	$\frac{6P_6-3P_3-2P_2+P_1}{2} = \sum\limits_{n=0}^{\infty} \left[\binom{2n}{n} \sum\limits_k (-8)^{n-k} \binom{n}{k}\binom{k}{l}^3\right] X^n$	$\frac{\eta_1^2\eta_2^2\eta_3^2\eta_6^2}{z^2}$
6	$\frac{6P_6-3P_3+2P_2-P_1}{4} = \sum\limits_{n=0}^{\infty} \left[\binom{2n}{n} \sum\limits_k \binom{n}{k}^2 \binom{2k}{k}\right] X^n$	$\frac{\eta_1^2\eta_2^2\eta_3^2\eta_6^2}{z^2}$
6	$\frac{6P_6+3P_3-2P_2-P_1}{6} = \sum\limits_{n=0}^{\infty} \left[\binom{2n}{n} \sum\limits_k \binom{n}{k}^3\right] X^n$	$\frac{\eta_1^2\eta_2^2\eta_3^2\eta_6^2}{z^2}$
7	$\frac{7P_7-P_1}{6} = \sum\limits_{n=0}^{\infty} \sum\limits_k \left[\binom{n}{k}^2 \binom{2k}{n}\binom{n+k}{n}\right] X^n$	$\left(\frac{\eta_1^2\eta_7^2}{z}\right)^{3/2}$
8	$\frac{8P_8-4P_4-2P_2+P_1}{3} = \sum\limits_{n=0}^{\infty} (-1)^n \left[\binom{2n}{n} \sum\limits_k \binom{n}{k}\binom{2k}{k}\binom{2n-k}{n-k}\right] X^n$	$\frac{\eta_2^4\eta_4^4}{z^2}$
9	$\frac{9P_9-6P_3+P_1}{4} = \sum\limits_{n=0}^{\infty} \left[\binom{2n}{n} \sum\limits_k (-3)^{n-3k} \binom{n}{k}\binom{n-k}{k}\binom{n-2k}{k}\right] X^n$	$\frac{\eta_3^8}{z^2}$
10	$\frac{10P_{10}+5P_5-2P_2-P_1}{12} = \sum\limits_{n=0}^{\infty} \sum\limits_k \left[\binom{n}{k}^4\right] X^n$	$\left(\frac{\eta_1\eta_2\eta_5\eta_{10}}{z}\right)^{4/3}$

3. Construction of differential equations

Theorem 3.1. *If*

$$S := \frac{1}{q^{1/12}} \frac{f_2 f_3}{f_1 f_6}$$

then the following differential identity holds:

$$q\frac{dS}{dq} + \frac{1}{12}\left(\sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_{k}(-8)^{n-k}\binom{n}{k}\binom{k}{l}^3\right]X^n\right)S = 0$$

where $X = \frac{\eta_1^2\eta_2^2\eta_3^2\eta_6^2}{z^2}$ *and* $z = \frac{6P_6 - 3P_3 - 2P_2 + P_1}{2}$.

Proof. Expressing v in terms of theta function, we obtain

$$S := \frac{1}{q^{1/12}} \frac{(q^2;q^2)_\infty (q^3;q^3)_\infty}{(q;q)_\infty (q^6;q^6)_\infty}$$

Now employing the definition of q-series and then taking logarithm on both sides and differentiating the resulting expression with respect to q, we deduce

$$\frac{q}{S}\frac{dS}{dq} = \sum_{n=1}^{\infty}\frac{6nq^{6n}}{1-q^{6n}} - \sum_{n=1}^{\infty}\frac{3nq^{3n}}{1-q^{3n}} - \sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}} + \sum_{n=1}^{\infty}\frac{nq^{n}}{1-q^{n}} - \frac{1}{12}.$$

Using the definition of Eisenstein series and the first level 6 identity of Lemma 2.4, we deduce the required differential equation. □

Theorem 3.2. *If*

$$S := \frac{1}{q^{1/6}} \frac{f_1 f_3}{f_2 f_6}$$

then the following differential identity holds:

$$q\frac{dS}{dq} + \frac{1}{6}\left(\sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_{k}\binom{n}{k}^2\binom{2k}{k}\right]X^n\right)S = 0,$$

where $X = \frac{\eta_1^2\eta_2^2\eta_3^2\eta_6^2}{z^2}$ *and* $z = \frac{6P_6 - 3P_3 + 2P_2 - P_1}{4}$.

Proof. With the use of the theta function definition, the q-series, applying logarithms on both sides, and differentiation of the resulting relation with regard to q, we may infer

$$\frac{q}{S}\frac{dS}{dq} = \sum_{n=1}^{\infty}\frac{6nq^{6n}}{1-q^{6n}} - \sum_{n=1}^{\infty}\frac{3nq^{3n}}{1-q^{3n}} + \sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}} - \sum_{n=1}^{\infty}\frac{nq^{n}}{1-q^{n}} - \frac{1}{6}.$$

We determine the necessary differential equation by applying the Eisenstein series formulation to the aforementioned relation and then applying the second level 6 identity of Lemma 2.4.

□

Theorem 3.3. *If*

$$S := \frac{1}{q^{1/4}} \frac{f_1 f_2}{f_3 f_6}$$

then the following differential identity holds:

$$q\frac{dS}{dq}+\frac{1}{4}\left(\sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_{k}\binom{n}{k}^{3}\right]X^{n}\right)S=0,$$

where $X=\frac{\eta_1^2\eta_2^2\eta_3^2\eta_6^2}{z^2}$ *and* $z=\frac{6P_6+3P_3-2P_2-P_1}{6}$.

Proof. With the use of the theta function definition, the q-series, logarithms on both sides, and differentiation, we obtain

$$\frac{q}{S}\frac{dS}{dq}=\sum_{n=1}^{\infty}\frac{6nq^{6n}}{1-q^{6n}}+\sum_{n=1}^{\infty}\frac{3nq^{3n}}{1-q^{3n}}-\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}-\sum_{n=1}^{\infty}\frac{nq^{n}}{1-q^{n}}-\frac{1}{4}.$$

The requisite differential equation is derived by applying the Eisenstein series formulation and then the third level 6 identity of Lemma 2.4.

□

Theorem 3.4. *If*

$$S:=\frac{1}{q^{1/8}}\frac{f_2f_4}{f_1f_8}$$

then the following differential identity holds:

$$q\frac{dS}{dq}+\frac{1}{8}\left(\sum_{n=0}^{\infty}(-1)^n\left[\binom{2n}{n}\sum_{k}\binom{n}{k}\binom{2k}{k}\binom{2n-k}{n-k}\right]X^{n}\right)S=0,$$

where $X=\frac{\eta_2^4\eta_4^4}{z^2}$ *and* $z=\frac{8P_8-4P_4-2P_2+P_1}{3}$.

Proof. Utilizing the theta function definition, q-series, logarithm on both sides, and differentiation of the resultant expression with regard to q, we arrive at

$$\frac{q}{S}\frac{dS}{dq}=\sum_{n=1}^{\infty}\frac{8nq^{8n}}{1-q^{8n}}-\sum_{n=1}^{\infty}\frac{4nq^{4n}}{1-q^{4n}}-\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}+\sum_{n=1}^{\infty}\frac{nq^{n}}{1-q^{n}}-\frac{1}{8}.$$

The necessary differential equation is also obtained by applying the formulation of the Eisenstein series and subsequently the level 8 identity of Lemma 2.4.

□

Theorem 3.5. *If*

$$S:=\frac{1}{q^{1/6}}\frac{f_3^2}{f_1f_9}$$

then the following differential identity holds:

$$q\frac{dS}{dq}+\frac{1}{6}\left(\sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_{k}(-3)^{n-3k}\binom{n}{k}\binom{n-k}{k}\binom{n-2k}{k}\right]X^{n}\right)S=0,$$

where $X=\frac{\eta_3^8}{z^2}$ *and* $z=\frac{9P_9-6P_3+P_1}{4}$.

Proof. With the incorporation of the theta function definition, the q-series, logarithmic computation on both sides, and differentiation of the resultant equation with respect to q, we can conclude

$$\frac{q}{S}\frac{dS}{dq} = \sum_{n=1}^{\infty}\frac{9nq^{9n}}{1-q^{9n}} - 2\sum_{n=1}^{\infty}\frac{3nq^{3n}}{1-q^{3n}} + \sum_{n=1}^{\infty}\frac{nq^{n}}{1-q^{n}} - \frac{1}{6}.$$

The necessary differential equation is now obtained by stating the aforementioned relation in terms of Eisenstein series and then applying the level 9 identity of Lemma 2.4.

□

Theorem 3.6. *If*

$$S := \frac{1}{q^{1/2}}\frac{f_1 f_2}{f_5 f_{10}}$$

then the following differential identity holds:

$$q\frac{dS}{dq} + \frac{1}{12}\left(\sum_{n=0}^{\infty}\left[\sum_{k}\binom{n}{k}^4\right]X^n\right)S = 0,$$

where $X = \left(\frac{\eta_1\eta_2\eta_5\eta_{10}}{z}\right)^{4/3}$ *and* $z = \frac{10P_{10}+5P_5-2P_2-P_1}{12}$.

Proof. We get the equation in terms of Eisenstein series by applying the definition of theta function, q-series, taking logarithms on both sides, and differentiating the subsequent expression with regard to q. In addition, we derive the essential differential equation through the implementation of Lemma 2.4's level 10 identity. □

4. Convolution Sum

Definition 4.1. For $a, b \in \mathbb{N}$, the convolution sum is defined by

$$U_{a,b}(m) := \sum_{ai+bj=m}\sigma(i)\sigma(j).$$

where $a \le b$ and for any $l, m \in \mathbb{N}$, $\sigma_l(m) = \sum_{u/m} u^l$, and $\sigma_l(m) = 0$ for $m \notin \mathbb{N}$.

For every nonnegative m, the convolution sum $\sum_{r+ks=m}\sigma(r)\sigma(s)$ has been assessed explicitely by A. Alaca et. al.[1, 2] and K. S. Williams et. al. [8]. Also E. X. W. Xia and O. X. M. Yao [9] have determined the illustrations for $\sum_{r+6s=m}\sigma(r)\sigma(s)$ and $\sum_{r+12s=m}\sigma(r)\sigma(s)$. Our proofs are simple and elementory and keys to our proofs are the claims of J. W. L. Glaisher [6],

$$P^2(q) = 1 + \sum_{l=1}^{\infty}(240\sigma_3(l) - 288l\sigma(l))q^l. \tag{4.1}$$

Theorem 4.2. *For any $r, s, l \in \mathbb{N} - \{0\}$, the following identities hold:*

$$i) \sum_{r+5s=l} \sigma(r)\ \sigma(s) = \frac{1}{24}\sigma_1(l) - \frac{1}{20}l\sigma_1(l) + \frac{1}{24}\sigma_3(l) + \frac{25}{24}\sigma_3\left(\frac{l}{5}\right) + \frac{1}{24}\sigma_1\left(\frac{l}{5}\right) - \frac{1}{4}l\sigma_1\left(\frac{l}{5}\right) - \frac{1}{360}A(l)$$

$$ii) \sum_{r+7s=l} \sigma(r)\ \sigma(s) = \frac{5}{168}\sigma_3(l) - \frac{1}{28}l\sigma_1(l) + \frac{35}{24}\sigma_3\left(\frac{l}{5}\right) - \frac{1}{4}l\sigma_1\left(\frac{l}{7}\right) + \frac{1}{24}\sigma_1(l) + \frac{1}{24}\sigma_1\left(\frac{l}{7}\right) - \frac{1}{224}B(l)$$

where
$\sum_{l=1}^{\infty} A(l)q^l = \left[\sum_{l=1}^{\infty} \binom{2l}{l} \left(\sum_k \binom{l}{k}^2 \binom{l+k}{l}\right) X^l\right]^2, X = \frac{\eta_1^4\eta_5^4}{z^2}, z = \frac{5P_5 - P_1}{4}$
and $\sum_{l=1}^{\infty} B(l)q^l = \left[\sum_{l=1}^{\infty} \left(\sum_k \binom{l}{k}^2 \binom{2k}{l}\binom{l+k}{l}\right) X^l\right]^2, X = \left(\frac{\eta_1^2\eta_7^2}{z}\right)^{3/2}, z = \frac{7P_7 - P_1}{6}$.

Proof. i) On squaring the level 5 identity of Lemma 2.4, we get

$$P^2(q) + 25P^2(q^5) - 10P(q)P(q^5) = 16\left[\sum_{l=1}^{\infty}\left[\binom{2l}{l}\sum_k \binom{l}{k}^2\binom{l+k}{l}\right]X^l\right]^2.$$

By using (4.1) and the Eisenstein series concept, we can now conclude (i) by equating the coefficients of q^l on both sides.
Likewise, by using (4.1) and the notion of Eisenstein series to square the level 7 identity of Lemma 2.4, and then equating the coefficients of q^l on both sides, we arrive at (ii).

□

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