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MAXIMAL SIMPLE PLANAR GRAPH AND FOUR COLOUR THEOREM

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Abstract. The article introduces four concepts: maximal simple planar graphs (mspgs), disc graphs, irreducible graphs, and k-irreducible graphs, along with illustrative examples for each. A complete list of graphs that are both mspgs and regular is provided. The paper presents the following results: A maximal simple planar graph with n vertices has $3(n-2)$ edges and $2(n-2)$ faces; every simple planar graph contains a vertex with degree 5 or less; and the disc graph $D_{m,k}$ has $2m+3(k-1)$ edges and $m+2k-1$ faces. The paper establishes the equivalence between the following statements through logical proofs and rigorous analysis: All planar graphs are 4-colourable; all maximal simple planar graphs are 4-colourable; all irreducible graphs are 4-colourable; and all 5-irreducible graphs are 4-colourable. The core result of the paper is: "Every planar graph is 4-colourable if and only if every 5-irreducible graph is 4-colourable."

1. Introduction

The Four Colour Problem, which asks whether every map on a plane can be coloured with at most four colors so that no two adjacent regions share the same colour, was first conjectured in the mid-19th century. In 1976, Kenneth Appel and Wolfgang Haken proposed a pioneering proof using complex computer-assisted techniques, sparking debate over the complexity of the proof. In 1997, Neil Robertson, Daniel P. Sanders, Paul Seymour, and Robin Thomas proposed an alternative proof that reduces dependence on extensive computer verification. The debate surrounding the use of computer-assisted methods in mathematical proofs continues, despite the confirmation of the Four Colour Theorem by these proofs. However, no one has succeeded in providing a complete rigorous proof of the Four Colour Theorem.

In graph theory, the Four Colour Theorem states that every planar map can be coloured with four colours or, equivalently, every planar graph can be coloured with four colours, as face colouring of a graph corresponds to vertex colouring of its dual graph. This article explores the concept of maximal simple planar graphs and their relationship with the Four Colour Problem. Maximal simple planar graphs are connected, and every simple planar graph is a subgraph of a maximal simple planar graph. Thus, every planar graph is four-colourable if and only if every

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maximal simple planar graph is four-colourable. The article also introduces disc graphs and irreducible graphs as special cases of maximal simple planar graphs. Utilizing these concepts, we examine four equivalent conditions of the Four Colour Theorem, simplifying the vertex colouring cases of graphs within the theorem.

2. Preliminary

Essential definitions, notations, and basic results are provided in this section. Let $V(G)$, $E(G)$, n, e, and f denote the vertex set, edge set, number of vertices, number of edges, and number of faces of the graph G, respectively. A graph is planar if it can be embedded in a plane. A simple graph has no loops and no parallel edges. A graph is simple and planar if it is both simple and planar. A graph is k-colourable if its vertices can be colored with k colors. The chromatic number $\chi(G)$ is the minimum value of k for which G is k-colourable.

Theorem 2.1 (Eulers theorem)**.** *Any regular convex polyhedron with n vertices, e edges, and f faces satisfies the relation $n - e + f = 2$. In other words, a connected planar graph G with n vertices, e edges, and f faces satisfies the above formula.*

3. Maximal Simple Planar Graph (mspg)

Definition 3.1. A simple planar graph G with n vertices is called a *maximal simple planar graph* (mspg) if there does not exist another simple planar graph H with $V(G) = V(H)$ and $E(G)$ is a proper subset of $E(H)$. The set of all maximal simple planar graphs with n vertices is denoted by $\mathcal{I}_n$.

Example 3.2. Complete graphs K_1, K_2, K_3, and K_4 are mspgs.

Proposition 3.3. *Let G be a connected simple planar graph with more than 2 vertices. Then G is an mspg if and only if every face is a 3-cycle (triangle).*

Proof. Let G be a connected maximal simple planar graph (mspg) with $n \geq 3$ vertices. Suppose G has a face φ with a k-cycle as its boundary, where $k > 3$. By introducing new edges, we can partition the k-gon face φ into triangular faces without introducing new vertices, contradicting the maximality of G.

Conversely, assume each face of the simple planar graph G is a 3-cycle. We aim to prove G is maximal. If G is not maximal, introducing a new edge e without a new vertex would be possible. Since every face of G is a 3-cycle, the graph $G \vee e$ is either nonplanar or not simple, leading to a contradiction. □

Lemma 3.4. *Let G be an mspg with $n \geq 3$ vertices. Then G has $3(n-2)$ edges and $2(n-2)$ faces.*

Proof. Let e and f represent the number of edges and vertices of the graph G, respectively. Since each face of G is a triangle, it is bounded by and contributes three edges to G, while each edge separates and contributes to two faces of G. Hence, we have $3f = 2e$. Substituting $f = \frac{2}{3}e$ into Eulers formula, we obtain: $e = 3(n-2)$ and $f = 2(n-2)$. □

Corollary 3.5. *Every simple planar graph has a vertex with a degree less than or equal to 5.*

Proof. Let δ be the smallest vertex degree of the graph G. The statement is obviously true when $n \leq 6$. If $n \geq 7$, we have

$$e \leq 3(n-2)$$

and

$$n\delta \leq \sum d(v) = 2e \leq 6(n-2) = 6n - 12.$$

Since $0 < \delta$, and $\delta \leq 6 - \frac{12}{n}$, we conclude that $\delta \leq 5$. $\square$

4. Disc Graphs

Definition 4.1. A subgraph of an mspg G, enclosed by an m-cycle, is called a *disc* in G if it contains all the vertices and edges of G that are inside and on an m-cycle. A disc with k interior vertices and $m \geq 3$ boundary vertices is denoted by $D_{m,k}$. A disc without its boundary edges is called an *open disc* and is denoted by $D^0_{m,k}$.

Example 4.2. $D_{4,4}$ and $D_{8,0}$ are discs with perimeters of 4 and 8, respectively. A disc $D_{m,0}$ is called a *hollow disc* with a perimeter of m. All maximal simple planar graphs (mspgs) except K_1 and K_2 are discs with a perimeter of 3. For example, $K_3 = D_{3,0}$ and $K_4 = D_{3,1}$.

Remark 4.3. There exist non-isomorphic discs with the same number of interior vertices and equal perimeter.

Definition 4.4. An mspg with a non-hollow disc $D_{3,k}$ as its proper subgraph is called a *reducible graph.* An *irreducible graph* is an mspg without any non-hollow disc $D_{3,k}$ as its proper subgraph.

Theorem 4.5. *The disc graph $D_{m,k}$ has $2m + 3(k-1)$ edges and $m + 2k - 1$ faces, including outer face, where $m \geq 3$ and $k \geq 0$.*

Proof. The disc $D_{m,k}$ has $m + k$ vertices, and the outer face is an m-gon. By introducing $m - 3$ vertices to the outer face, we form an mspg with $m + k$ vertices and $3[(m + k) - 2]$ edges. Therefore, the number of edges of $D_{m,k}$ is $3[(m + k) - 2] - (m - 3) = 2m + 3(k - 1)$. Suppose f is the number of faces of $D_{m,k+1}$. Eulers theorem implies $(m + k) - [2m + 3(k - 1)] + f = 2$, and so $f = m + 2k - 1$. $\square$

Corollary 4.6.

(1) *The hollow disc $D_{m,0}$ has $2m - 3$ edges and $m - 1$ faces.*
(2) *Every $D_{3,k}$ disc is an mspg.*

Lemma 4.7. *If an mspg graph G is regular, then G is either $K_1, K_2, K_3, K_4, D_{3,3}$, or $D_{3,9}$.*

Proof. Let G be an mspg with n vertices and e edges. Suppose G is a t-regular graph. Clearly, $d(v_i) = t$ for $i = 1, 2, \ldots, n$. Hence, $nt = 2e$. Therefore, $nt = 2[3(n-2)] = 6n-12$. Hence $n = \frac{12}{6-t}$. Putting $t = 2, 3, 4$, and 5, we get $n = 3, 4, 6$, and 12, respectively. There does not exist an mspg with $\delta \geq 6$, where δ is the smallest vertex degree of the graph G. The complete graphs K_1, K_2, K_3, and K_4 are mspgs with $\delta = 0, 1, 2$, and 3, respectively. We conclude that if a graph G is regular as well as an mspg, then G is either $K_1, K_2, K_3, K_4, D_{3,3}$, or $D_{3,9}$. $\square$

5. Equivalent Conditions of Four Colour Theorem

Any simple planar graph with n vertices is a subgraph of an mspg with n vertices. Multiple edges and loops do not affect the colouring of vertices. That is, if any maximal simple planar graph with n vertices is k-colourable, then every planar graph with n vertices is k-colourable. Every planar graph is 4-colourable if and only if every maximal simple planar graph is 4-colourable.

Theorem 5.1. *Let $D_{3,k}$ (a subgraph) be a non-hollow disc in an mspg G, and let $v_1, v_2, \ldots, v_k$ be the interior vertices of $D_{3,k}$. Also, let G' be the subgraph of G obtained by removing the interior vertices $v_1, v_2, \ldots, v_k$ from G. Then*

(1) *G' is an mspg.*
(2) *If G' and $D_{3,k}$ are four-colourable, then G is four-colourable.*

Proof.

(1) Let u_1, u_2, and u_3 be the boundary vertices of $D_{3,k}$. Clearly, they are the only common vertices of G' and $D_{3,k}$. In G', the edges u_1u_2, u_2u_3, and u_3u_1 form a triangular face. All other faces of G' are triangles. Hence, G' is an mspg.
(2) By our assumption, G' and $D_{3,k}$ are four-colourable. In G', we can colour u_1, u_2, and u_3 with the colours C_1, C_2, and C_3 respectively. Without loss of generality, we can colour u_1, u_2, and u_3 in $D_{3,k}$ with the same colours in the respective order. G' and $D_{3,k}$ have no other common vertices, and together they form G. Hence, G is four-colourable.

□

Theorem 5.2. *Every maximal simple planar graph is four-colourable if and only if every irreducible graph is four-colourable.*

Proof. Suppose that every maximal simple planar graph is 4-colourable. Every irreducible graph is a maximal simple planar graph, hence it is four-colourable. Note that some maximal simple planar graphs are not irreducible.

Conversely, assume that every irreducible graph is 4-colourable. Then we want to prove that every maximal simple planar graph is 4-colourable. We prove this part by mathematical induction on n. We know that K_1, K_2, K_3, and K_4 are 4-colourable. Hence, the converse statement is true for $m = 1, 2, 3$, and 4. Suppose that the statement is true for $n \leq m$, where $m \geq 4$.

Let G be a maximal simple planar graph with $m + 1$ vertices.

Case I: Suppose G has a proper subgraph isomorphic to a non-hollow disc $D_{3,k}$. Let $v_1v_2v_3v_1$ be the boundary of $D_{3,k}$ with k interior vertices. Remove all k interior vertices from G, obtaining a new graph H with a triangular face f bounded by vertices v_1, v_2, and v_3. All other faces of H are triangles. Hence, H is an mspg with fewer than $m + 1$ vertices. By the induction hypothesis, H and $D_{3,k}$ are four-colourable. Thus, G is four-colourable.

Case II: If G has no proper subgraph isomorphic to a non-hollow disc $D_{3,k}$, by definition, G is an irreducible graph, and according to our assumption, G is four-colourable. □

Theorem 5.3. *If $n \geq 5$, the minimum vertex degree δ of an irreducible graph is 4 or 5.*

Proof. Let G be an irreducible graph with n vertices (where $n \geq 5$). Since G is an mspg, $\delta(G) = 3, 4$, or 5. Suppose there exists a vertex v of degree 3, and let v_1, v_2, and v_3 be the adjacent vertices of v. Hence, there exists a non-hollow disc $D_{3,1}$ with $v_1v_2v_3v_1$ as its boundary. This implies that G has a proper non-hollow disc $D_{3,1}$, which leads to a contradiction. $\square$

6. k-Irreducible Graphs

Definition 6.1. An irreducible graph with minimum vertex degree $\delta = k$ is called a k-irreducible graph.

Example 6.2. K_1, K_2, K_3, and K_4 are the only 0-irreducible graph, 1-irreducible graph, 2-irreducible graph, and 3-irreducible graph respectively. $D_{3,3}$ is the smallest 4-irreducible graph.

Theorem 6.3. *Every irreducible graph is four-colourable if and only if every 5-irreducible graph is four-colourable.*

Proof. Assume that every irreducible graph is four-colourable. Every 5-irreducible graph is an irreducible graph and therefore it is four-colourable.

Conversely, assume that every 5-irreducible graph is four-colourable. We want to prove that every irreducible graph is four-colourable. We prove this part by induction on n.

If $n = 6$, then $D_{3,3}$ is the irreducible graph, clearly a 4-irreducible graph. Let v_1, v_3, v_5 be the boundary vertices and v_2, v_4, v_6 be the interior vertices of $D_{3,3}$. Also, let (v_1, v_4), (v_2, v_5), (v_3, v_6) be the non-adjacent pairs of vertices. We can colour v_1 and v_4 by C_1, v_2 and v_5 by C_2, v_3 and v_6 by C_3. Hence, it is 3-colourable, implying $D_{3,3}$ is four-colourable.

We assume that the statement is true for n up to m. Let G be an irreducible graph with $m + 1$ vertices and $\delta(G) = k$, where $k = 4$ or 5.

Case I: If $\delta = 4$, there exists a vertex v in G such that $d(v) = 4$. So v is adjacent to exactly 4 vertices, say v_1, v_2, v_3, and v_4. Remove the vertex v from G, and we get a simple planar graph (need not be maximal), denoted as H, with m vertices. By the induction hypothesis, it is four-colourable. The maximal property of G implies the existence of the 4-cycle $v_1v_2v_3v_4v_1$ as its boundary. All other faces of H are triangles.

In G, either of the pairs (v_1, v_3) or (v_2, v_4) are non-adjacent. Otherwise, G would have a subgraph isomorphic to K_5, contradicting the planarity of G. Without loss of generality, we assume that v_2 and v_4 are non-adjacent vertices of G. By fusing the vertices v_2 and v_4 together, we obtain a new vertex u and a new planar graph (not simple), denoted as H_1. The graph H_1 has $m - 1$ vertices. Multiple edges do not affect the colouring of H_1.

By the induction hypothesis, H_1 is four-colourable. In a colouring of H_1, we can colour the vertices v_1, u, and v_3 by at most 3 colours, say C_1, C_2, and C_3, respectively. So we can colour the vertices of H using four colours as follows: v_2 and v_4 in H by C_2 since they are non-adjacent, all other vertices of H by the colours used in H_1, respectively. Thus, we can colour v with the fourth colour. Hence, G is four-colourable.

Case II: If $\delta = 5$, then G is a 5-irreducible graph. By our initial assumption (that is, every 5-irreducible graph is four-colourable), G is four-colourable. □

Theorem 6.4. *Every planar graph is four colourable if and only if every 5-irreducible graph is 4-colourable.*

7. Conclusion

This paper introduces and explores fundamental concepts in maximal simple planar graphs, including disc graphs, irreducible graphs, and k-irreducible graphs, through definitions and illustrative examples. The paper establishes several results, such as the equations for the edge and face counts of maximal simple planar graphs and disc graphs. Furthermore, it identifies all maximal simple planar graphs that are regular graphs. The paper proposes three theorems equivalent to the Four Colour Theorem: (1) Every maximal simple planar graph is 4-colourable. (2) Every irreducible graph is 4-colourable. (3) Every 5-irreducible graph is 4-colourable. The findings presented in this paper lay the foundation for further research in graph theory and have implications for various applications in computer science, mathematics, and related fields. Future work on maximal simple planar graphs and disc graphs includes studying paths and cycles, investigating adjacency matrices and their properties, exploring connectivity, spanning trees, different types of labeling, Hamiltonian and Euler graphs, matchings, directed graphs, maximal planar networks, and topology properties.

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