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# AN APPROACH TO CONNECT THE RAMANUJAN-TYPE EISENSTEIN SERIES N(Q) WITH THETA FUNCTIONS FORMULATED BY BORWEIN

VIDYA H. C., ASHWATH RAO B., AND SMITHA GANESH BHAT*

Abstract. Ramanujan introduced novel categories of remarkable infinite series, referred to as the Ramanujan-type Eisenstein series, in his lost notebook. The present study utilizes the (p, k)-parametrization method developed by Alaca to examine the relationships between cubic theta functions of Borwein and Ramanujan-type Eisenstein series, especially N(q). By using Borwein's theta functions, this method provides an innovative framework for determining Eisenstein series identities. Sum of an infinite series that converges to an infinite product is the primary focus of this study.

## 1. Introduction

Eisenstein series are special types of modular forms, which are holomorphic functions with rich symmetry properties under the action of the modular group. They are expressed in terms of infinite series, often involving lattice points, and serve as foundational objects in number theory, particularly in the study of elliptic functions and modular forms. Borwein's cubic theta functions are specialized theta functions associated with cubic modular transformations. They generalize classical theta functions and have applications in partition theory, q-series, and combinatorics. The connections between Borwein's cubic theta functions and Eisenstein series, explored using parameters introduced by Alaca [1], highlight the deep interplay between modular forms, theta functions, and special functions in mathematics.This work, which emphasizes an analytical approach, is a major step toward comprehending the connection between Borwein's cubic theta functions and Ramanujan-type Eisenstein series. In conclusion, the analytical methodology used in this work goes beyond numerical techniques, enabling the explicit formulation and demonstration of the connections between Borwein's cubic theta functions and Ramanujan-type Eisenstein series. This bridges classical and modern aspects of modular forms, further enriching our understanding of their interconnectedness.

Section 2 provides the foundational background, while Section 3 introduces new identities that reveal novel connections between the Borwein's cubic theta functions and Ramanujan-Eisenstein series, building on but distinct from previous

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*Corresponding Author - Smitha Ganesh Bhat, e-mail- smitha.holla@manipal.edu.

work by Earnest Xia [10]. These findings offer new mathematical perspectives and inspire further exploration in the field.

## 2. Preliminaries

The arithmetic-geometric mean iteration originates from the study of elliptic and theta functions. The Borwein brothers [3, 4] introduced a framework of multi-dimensional theta functions, laying a crucial foundation for further developments.

$$a(q) := \sum_{r,s=-\infty}^{\infty} q^{r^2+rs+s^2},$$

$$b(q) := \sum_{r,s=-\infty}^{\infty} \omega^{r-s} q^{r^2+rs+s^2},$$

$$c(q) := \sum_{r,s=-\infty}^{\infty} q^{\left(r+\frac{1}{3}\right)^2+\left(r+\frac{1}{3}\right)\left(s+\frac{1}{3}\right)+\left(s+\frac{1}{3}\right)^2},$$

where $q$ represents complex number, with $|q| < 1$, and $\omega = e^{2\pi i/3}$, the principal cube root of unity, appears in the context of two-dimensional theta functions. When $q = 0$, the corresponding expressions simplify to $a(q) = 1$, $b(q) = 1$, and $c(q) = 0$.

Euler's binomial theorem, which expands $(1+x)^n$ for any real $n$, underpins much of the Borwein brothers work. They utilized this theorem to derive infinite product representations for the functions $b(q)$ and $c(q)$. Using Euler's binomial theorem, the Borwein brothers likely formulated $b(q)$ and $c(q)$ as products involving terms from binomial expansions, possibly governed by specific coefficient patterns. These representations hold mathematical significance and may reveal deeper properties or behaviors of the functions. Utilizing Euler's binomial theorem as a starting point, the Borwein brothers have formulated representations for both $b(q)$ and $c(q)$ in the form of infinite products, as demonstrated below:

$$b(q) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty},$$

$$c(q) = \frac{3q^{\frac{1}{3}}(q^3;q^3)_\infty^3}{(q;q)_\infty},$$

where

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i).$$

Moreover, the expression for $a(q)$ in terms of $q$-series is documented by the Borwein brothers [3] and Berndt [2]:

$$a(q) = (-q;q^2)_\infty^2(q^2;q^2)_\infty(-q^3;q^6)_\infty^2(q^6;q^6)_\infty + 4q\frac{(q^4;q^4)_\infty(q^{12};q^{12})_\infty}{(q^2;q^4)_\infty(q^6;q^{12})_\infty}.$$

Additionally, they have deduced the basic cubic identity that expresses the relationship between $a(q)$, $b(q)$, and $c(q)$ as follows:

$$a^3(q) = b^3(q) + c^3(q).$$

**Definition 2.1.** In his notebook [6], Srinivasa Ramanujan elucidated the definitions of the Eisenstein Series $L(q)$, $M(q)$ and $N(q)$ as outlined below:

$$L(q) := 1 - 24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}, \tag{2.1}$$

$$M(q) := 1 + 240\sum_{r=1}^{\infty}\frac{r^3q^r}{1-q^r}, \tag{2.2}$$

$$N(q) := 1 - 504\sum_{r=1}^{\infty}\frac{r^5q^r}{1-q^r}. \tag{2.3}$$

**Definition 2.2.** Ramanujan [2] documented a general form of the theta function for any complex numbers $c$ and $d$,

$$f(c,d) := \sum_{m=-\infty}^{\infty} c^{m(m+1)/2}d^{m(m-1)/2}$$

$$:= (-c;cd)_\infty(-d;cd)_\infty(cd;cd)_\infty,$$

where

$$(c;q)_\infty := \prod_{m=0}^{\infty}(1-cq^m), \qquad |q|<1.$$

The specific case of Ramanujan's definition of the theta function [2] is,

$$\varphi(q) := f(q,q) = \sum_{m=-\infty}^{\infty} q^{m^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty.$$

$$f(-q) := f(-q,-q^2) = \sum_{i=-\infty}^{\infty}(-1)^i q^{i(3i-1)/2} = (q;q)_\infty = q^{-1/24}\eta(\tau),$$

where $q = e^{2\pi i\tau}$.

In their noteworthy publication, Alaca et al. [1] introduced the $(p,\ k)$ parametrization of theta functions. This parametrization holds particular significance in the formulation of the duplication and triplication principles, leading to the derivation of specific sum-to-product identities. The parameters $p$ and $k$ are precisely defined as follows:

$$p := p(q) = \frac{\varphi^2(q)-\varphi^2(q^3)}{2\phi^2(q^3)},$$

$$k := k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}.$$

Since $\varphi(0) = 1$, it clear that $p(0) = 0$ and $k(0) = 1$.

**Lemma 2.3.** [1] *Concerning the previously mentioned Eisenstein series* (2.1)*, the expressions for* $M(q^{i_1})$*, where* $(l_1 = 1,2,3,6,12)$*,* $N(q^{l_2})$*, where* $(l_2 = 1,2,3,6)$*,* $L(q) - l_3L(q^{l_3})$*, where* $(l_3 = 2,3,4,6,12)$*, as well as* $L(-q^{l_4}) - l_5L(q^{l_5})$*, where* $l_4 \in 1,3$ *and* $l_5 \in 1,2,3$*, in terms of the parameters* $p$ *and* $k$*, are articulated as* *follows:*

$$
\begin{aligned}
&M(q) = (1 + 124p(1 + p^6) + 964p^2(1 + p^4) + 2788p^3(1 + p^2) + 3910p^4 + p^8)k^4,\\
&M(q^2) = (1 + 4p(1 + p^6) + 64p^2(1 + p^4) + 178p^3(1 + p^2) + 235p^4 + p^8)k^4,\\
&M(q^3) = (1 + 4p(1 + p^6) + 4p^2(1 + p^4) + 28p^3(1 + p^2) + 70p^4 + p^8)k^4,\\
&M(q^6) = (1 + 4p(1 + p^6) + 4p^2(1 + p^4) - 2p^3(1 + p^2) - 5p^4 + p^8)k^4,\\
&M(q^{12}) = (1 + 4p(1 + p) - 2p^3(1 + p^2) - 5p^4 + p^6(1 + p)/4 + p^8/16)k^4,\\
&L(-q) - L(q) = 3(8p + 12p^2 + 6p^3 + p^4)k^2,\\
&L_{1,2}(q) = (L(-q) - L(q))/48 = (p/2 + 3p^2/4 + 3p^3/8 + p^4/16)k^2,\\
&L_{1,2}(q^3) = (L(-q^3) - L(q^3))/48 = p^3(2 + p)k^2/16,\\
&L(-q) - 2L(q^2) = -(1 - 10p - 12p^2 - 4p^3 - 2p^4)k^2,\\
&L(q) - 2L(q^2) = -(1 + 14p(1 + p^2) + 24p^2 + p^4)k^2,\\
&L(q) - 3L(q^3) = -(1 + 8p(1 + p^2) + 18p^2 + p^4)k^2,\\
&L(q) - 6L(q^6) = -(5 + 22p(1 + p^2) + 36p^2 + 5p^4)k^2,\\
&L(q^2) - 3L(q^6) = -2(1 + 2p(1 + p^2) + 3p^2 + p^4)k^2,\\
&L(q^3) - 2L(q^6) = -(1 + 2p(1 + p^2) + p^4)k^2,\\
&L(q) - 4L(q^4) = -3(1 + 6p + 12p^2 + 8p^3)k^2,\\
&L(q) - 12L(q^{12}) = -(11 + 34p + 36p^2 + 16p^3 + 2p^4)k^2.\\
&N(q) = (1 - 246p(1 + p^{10}) - 5532p^2(1 + p^8) - 38614p^3(1 + p^6)\\
&- 135369p^4(1 + p^4) - 276084p^5(1 + p^2) - 348024p^6 + p^{12})k^6.\\
&N(q^2) = (1 + 6p(1 + p^{10}) - 114p^2(1 + p^8) - 625p^3(1 + p^6) - \frac{4059}{2}p^4(1 + p^4)\\
&- 4302p^5(1 + p^2) - 5556p^6 + p^{12})k^6.\\
&N(q^3) = (1 + 6p(1 + p^{10}) + 12p^2(1 + p^8) - 58p^3(1 + p^6) - 297p^4(1 + p^4)\\
&- 396p^5(1 + p^2) - 264p^6 + p^{12})k^6.\\
&N(q^6) = (1 + 6p(1 + p^{10}) + 12p^2(1 + p^8) + 5p^3(1 + p^6) - \frac{27}{2}p^4(1 + p^4)\\
&- 18p^5(1 + p^2) - 12p^6 + p^{12})k^6.
\end{aligned}
$$

**Lemma 2.4.** *Alaca et al.* [1] *have derived the parametric representations of* $a(q^r), b(q^r), c(q^r)$ *for* $r \in 1, 2, 4, 6$*, as well as* $a(-q)$*,* $b(-q)$*,* $c(-q)$*, expressed in terms of the parameters* $p$ *and* $k$*, and are presented below.*

$$
\begin{aligned}
&a(-q) = (1 - 2p - 2p^2)k,\\
&a(q) = (1 + 4p + p^2)k,\\
&a(q^2) = (1 + p + p^2)k,
\end{aligned}
$$

$$a(q^4) = (1 + p - \frac{1}{2}p^2)k,$$
$$a(q^6) = \frac{1}{3}(1 + p + p^2 + 2^{1/3}((1 - p)(2 + p)(1 + 2p))^{2/3})k,$$
$$b(-q) = 2^{-\frac{1}{3}}((1 - p)(1 + 2p)^4(2 + p))^{\frac{1}{3}}k,$$
$$b(q) = 2^{-\frac{1}{3}}((1 - p)^4(1 + 2p)(2 + p))^{\frac{1}{3}}k,$$
$$b(q^2) = 2^{-2/3}((1 - p)(1 + 2p)(2 + p))^{\frac{2}{3}}k,$$
$$b(q^4) = 2^{-\frac{4}{3}}((1 - p)(1 + 2p)(2 + p)^4)^{\frac{1}{3}}k,$$
$$c(-q) = -2^{\frac{1}{3}}3(p(1 + p))^{\frac{1}{3}}k,$$
$$c(q) = 2^{-\frac{1}{3}}3(p(1 + p)^4)^{\frac{1}{3}}k,$$
$$c(q^2) = 2^{-\frac{2}{3}}3(p(1 + p))^{\frac{2}{3}}k. \quad ,$$
$$c(q^4) = 2^{-\frac{4}{3}}3(p^4(1 + p))^{\frac{1}{3}}k,$$
$$c(q^6) = \frac{1}{3}\left(p^2 + p + 1 - 2^{-2/3}((1 - p)(2 + p)(1 + 2p))^{2/3}\right)k.$$

## 3. Perspectives on Eisenstein Series and Cubic Theta Functions Inspired by Ramanujan

In his notebook [6], Ramanujan documented fascinating series involving the variables $L$, $M$ and $N$, which revealed key identities for infinite series related to theta functions. Building on his work, Xia et al. [10] and Shruti et al. [5] employed computational methods to discover elegant identities involving Eisenstein series and cubic theta functions, particularly between $L(q)$ and $L(q^{n_1})$, where $n_1 \in 2, 3, 4, 6, 12$. Vidya H. C. and Ashwath Rao B. [7], extended their work by considering the Eisenstein series $L(-q^{n_2})$, where $n_2 \in 1, 3$ and established relations among $L(q^{n_1})$ and $L(-q^{n_2})$ and cubic theta functions. Building up on the previous work, more recently Vidya H. C. et al. [8] discovered a relationship between $L(q^{n_1})$, $L(-q^{n_2})$, and $M(q^{n_3})$, where $n_3 \in 1, 2, 3, 6, 12$. Motivated by their work, in this papet our attempt is to produce results relating $L(q^{n_1})$, $L(-q^{n_2})$, $M(q^{n_3})$ and $N(q^{n_4})$ where $n_4 \in 1, 2, 3, 6$. The derived relation allows us to connect the sum of an infinite series with an infinite product, demonstrating the convergence of the series to the corresponding product. In a recent study, a similar result was given by Vidya H. C et al. [9], where they have demonstrated the sum of infinite series converges to infinite product.

Our approach is purely analytical, steering clear of numerical techniques. For computation, we have used MATLAB and the results provide fresh insights into the complex connections between these mathematical structures, deepening the understanding of their relationships within a rigorous context.

**Theorem 3.1.** *The connection between Ramanujan-type Eisenstein series and theta functions are as follows:*

$$(i)\ \Big[-\frac{1006}{69}+27v+u\Big]+540\sum_{r=1}^{\infty}\Big[\Big(\frac{u}{7}+\frac{9v}{7}+\frac{68}{483}\Big)\frac{r^5q^r}{1-q^r}-\Big(\frac{8u}{7}$$
$$+\frac{198v}{7}+\frac{100}{483}\Big)\frac{r^5q^{2r}}{1-q^{2r}}-\Big(\frac{198}{161}\Big)\frac{r^5q^{3r}}{1-q^{3r}}+\Big(\frac{2556}{161}\Big)\frac{r^5q^{6r}}{1-q^{6r}}\Big]$$
$$-v\Big(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\frac{r(-q)^r}{1-(-q)^r}\Big)^3+u\Big(-1+24\sum_{r=1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}}$$
$$-\frac{rq^r}{1-q^r}\Big]\Big)^3+\frac{1}{8}\Big(-2+24\sum_{r=1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Big]\Big)^3+v\Big(-3$$
$$+24\sum_{r=1}^{\infty}\Big[\frac{4rq^{4r}}{1-q^{4r}}-\frac{rq^r}{1-q^r}\Big]\Big)^3-\frac{43}{138}\Big(-5+24\sum_{r=1}^{\infty}\Big[\frac{6rq^{6r}}{1-q^{6r}}$$
$$-\frac{rq^r}{1-q^r}\Big]\Big)^3-\frac{1}{138}\Big(-1+24\sum_{r=1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{2rq^{2r}}{1-q^{2r}}\Big]\Big)^3$$
$$=\Big\{\frac{a(q)c^2(q)}{c(q^2)}\Big\}^3. \tag{3.1}$$

$$(ii)\ \Big[-\frac{7}{24}+27v+u\Big]+540\sum_{r=1}^{\infty}\Big(\frac{u}{7}+\frac{9v}{7}-\frac{1}{672}\Big)\frac{r^5q^r}{1-q^r}-\Big(\frac{8u}{7}+\frac{198v}{7}$$
$$-\frac{1}{84}\Big)\frac{r^5q^{2r}}{1-q^{2r}}-\Big(\frac{9}{224}\Big)\frac{r^5q^{3r}}{1-q^{3r}}+\frac{9}{28}\frac{r^5q^{6r}}{1-q^{6r}}-v\Big(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}$$
$$-\frac{r(-q)^r}{1-(-q)^r}\Big)^3+u\Big(-1+24\sum_{r=1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\Big]\Big)^3+v\Big(-3$$
$$+24\sum_{r=1}^{\infty}\Big[\frac{4rq^{4r}}{1-q^{4r}}-\frac{rq^r}{1-q^r}\Big]\Big)^3-\frac{1}{96}\Big(-5+24\sum_{r=1}^{\infty}\Big[\frac{6rq^{6r}}{1-q^{6r}}-\frac{rq^r}{1-q^r}\Big]\Big)^3$$
$$+\frac{1}{96}\Big(-1+24\sum_{r=1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{2rq^{2r}}{1-q^{2r}}\Big]\Big)^3=\Big\{a(q)a(q^2)\Big\}^3. \tag{3.2}$$

$$(iii)\ \Big[-\frac{98}{23}+27u\Big]+540\sum_{r=1}^{\infty}\Big(\frac{9u}{7}-\frac{67}{322}\Big)\frac{r^5q^r}{1-q^r}+\Big(\frac{652}{161}-\frac{198u}{7}\Big)\frac{r^5q^{2r}}{1-q^{2r}}$$

$$+\left(\frac{5535}{322}\right)\frac{r^5q^{3r}}{1-q^{3r}}-\left(\frac{2700}{161}\right)\frac{r^5q^{6r}}{1-q^{6r}}-u\left(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\frac{r(-q)^r}{1-(-q)^r}\right)^3$$
$$-\frac{3}{8}\left(-2+24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\right]\right)^3+u\left(-3+24\sum_{r=1}^{\infty}\left[\frac{4rq^{4r}}{1-q^{4r}}\right.\right.$$
$$\left.\left.-\frac{rq^r}{1-q^r}\right]\right)^3+\frac{1}{46}\left(-5+24\sum_{r=1}^{\infty}\left[\frac{6rq^{6r}}{1-q^{6r}}-\frac{rq^r}{1-q^r}\right]\right)^3$$
$$-\frac{229}{46}\left(-1+24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}-\frac{2rq^{2r}}{1-q^{2r}}\right]\right)^3=\left\{\frac{b^3(q)b(q^2)}{b(-q)b(q^4)}\right\}^3. \tag{3.3}$$

*Proof.* Let us persume that,

$$C_1N(q)+C_2N(q^2)+C_3N(q^3)+C_4N(q^6)+C_5\Big[L(-q)-L(q)\Big]^3$$
$$+C_6\Big[L(q)-2L(q^2)\Big]^3+C_7\Big[L(q)-3L(q^3)\Big]^3+C_8\Big[L(q)-4L(q^4)\Big]^3$$
$$+C_9\Big[L(q)-6L(q^6)\Big]^3+C_{10}\Big[L(q)-12L(q^{12})\Big]^3+C_{11}\Big[2L(q^2)$$
$$-3L(q^3)\Big]^3+C_{12}\Big[3L(q^3)-4L(q^4)\Big]^3=\left\{\frac{a(q)c^2(q)}{c(q^2)}\right\}^3. \tag{3.4}$$

Incorporating Lemma 2.3 and expressing the above relation in terms of (p,k) parametrization and then equalizing the coefficients of $k^3, pk^3, p^2k^3, p^3k^3,\ p^4k^3$, $p^5k^3, p^6k^3, p^7k^3,\ p^8k^3\ p^9k^3, p^{10}k^3, p^{11}k^3$ and $p^{12}k^3$ on either side, we formulate a system

$$\left(\begin{matrix}
1 & 1 & 1 & 1 & 0 & 1 \\
-246 & 6 & 6 & 6 & 0 & 30 \\
-5532 & -114 & 12 & 12 & 0 & -264 \\
-38614 & -625 & -58 & 5 & 13824 & 292 \\
-135369 & -\frac{4059}{2} & -297 & -\frac{27}{2} & 62208 & 2934 \\
-276084 & -4302 & -396 & -18 & 124416 & 5112 \\
-348024 & -5556 & -264 & -12 & 145152 & 5016 \\
-276084 & -4302 & -396 & -18 & 108864 & 3600 \\
-135369 & -\frac{4059}{2} & -297 & -\frac{27}{2} & 54432 & 1908 \\
-38614 & -625 & -58 & 5 & 18144 & 760 \\
-5532 & -114 & 12 & 12 & 3888 & 240 \\
-246 & 6 & 6 & 6 & 486 & 48 \\
1 & 1 & 1 & 1 & 27 & 8
\end{matrix}\right.$$

$$\left.\begin{matrix}
-1 & -8 & -1 & -1 & -8 & -1331 \\
-42 & -192 & -6 & -6 & -48 & -12342 \\
-660 & -1968 & -48 & -12 & -96 & -51216 \\
-4802 & -11200 & -158 & -32 & -40 & -125896 \\
-17019 & -38520 & -603 & -90 & 36 & -204246 \\
-34524 & -81792 & -1044 & -72 & -144 & -230184 \\
-43368 & -105888 & -2112 & -168 & -264 & -184296 \\
-34524 & -81792 & -1044 & -288 & 72 & -105120 \\
-17019 & -38520 & -603 & 180 & 90 & -42084 \\
-4802 & -11200 & -158 & -536 & -292 & -11416 \\
-660 & -1968 & -48 & 384 & -60 & -1968 \\
-42 & -192 & -6 & -96 & 150 & -192 \\
-1 & -8 & -1 & 8 & -125 & -8
\end{matrix}\right)
\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12} \end{pmatrix}
=
\begin{pmatrix} 1 \\ 15 \\ 93 \\ 320 \\ 711 \\ 1125 \\ 1302 \\ 1125 \\ 711 \\ 320 \\ 93 \\ 15 \\ 1 \end{pmatrix}.$$

Solving for the unknowns, we interestingly note that the system admits infinitely many solutions.

$$C_1 = \left\{ -\frac{68}{483} - \frac{9v}{7} - \frac{u}{7} \right\},\ C_2 = \left\{ \frac{8u}{7} + \frac{198v}{7} + \frac{100}{483} \right\},\ C_3 = \frac{198}{161},$$
$$C_4 = -\frac{2556}{161}, C_5 = -v,\ C_6 = u, C_7 = \frac{1}{8},\ C_8 = v,\ C_9 = -\frac{47}{138},$$
$$C_{10} = 0,\ C_{11} = -\frac{1}{138} \text{ and } C_{12} = 0,$$

where $u, v \in \mathbb{R}$.

It is evident that (i) is derived by substituting these values into (3.4) and subsequently simplifying the expression using the properties and definition of the Eisenstein series.

By maintaining the left-hand side unchanged, altering the right-hand side, and applying similar techniques, the following identities can be established.

$$(ii)\ \Big(-\frac{u}{7}-\frac{9v}{7}+\frac{1}{162}\Big)N(q)+\Big(\frac{8u}{7}+\frac{198v}{7}-\frac{1}{84}\Big)N(q^2)+\frac{9}{224}N(q^3)$$
$$-\frac{9}{28}N(q^6)-v\Big[L(-q)-L(q)\Big]^3+u\Big[L(q)-2L(q^2)\Big]^3+v\Big[L(q)$$
$$-4L(q^4)\Big]^3-\frac{1}{96}\Big[L(q)-6L(q^6)\Big]^3+\frac{1}{96}\Big[2L(q^2)-3L(q^3)\Big]^3$$
$$=\Big\{a(q)a(q^2)\Big\}^3.$$

$$(iii)\ \Big(\frac{67}{322}-\frac{9u}{7}\Big)N(q)+\Big(-\frac{652}{161}+\frac{198u}{7}\Big)N(q^2)-\frac{5535}{322}N(q^3)$$
$$-\frac{2700}{161}N(q^6)-u\Big[L(-q)-L(q)\Big]^3-\frac{3}{8}\Big[L(q)-3L(q^3)\Big]^3$$
$$+u\Big[L(q)-4L(q^4)\Big]^3+\frac{1}{46}\Big[L(q)-6L(q^6)\Big]^3-\frac{229}{46}\Big[2L(q^2)$$
$$-3L(q^3)\Big]^3=\Big\{\frac{b^3(q)b(q^2)}{b(-q)b(q^4)}\Big\}^3.$$

The relationships (3.2) and (3.3), are derived by simplifying the aforementioned relations (ii) and (iii) along with utilizing Definition 2.1. □

**Theorem 3.2.** *The relationship between Ramanujan-type Eisenstein series and theta functions are as follows:*

$$(i)\ \Big[-\frac{7}{24}+\frac{351v}{46}+u\Big]+540\sum_{r=1}^{\infty}\Big(\frac{u}{7}+\frac{3v}{92}-\frac{1}{672}\Big)\frac{r^5q^r}{1-q^r}-\Big(\frac{8u}{7}-\frac{6v}{23}$$
$$+\frac{1}{84}\Big)\frac{r^5q^{2r}}{1-q^{2r}}-\Big(\frac{81v}{92}+\frac{9}{224}\Big)\frac{r^5q^{3r}}{1-q^{3r}}+\Big(-\frac{162v}{23}+\frac{9}{28}\Big)\frac{r^5q^{6r}}{1-q^{6r}}$$
$$+u\Big(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\frac{r(-q)^r}{1-(-q)^r}\Big)^3+\frac{v}{8}\Big(-1+24\sum_{r=1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}}$$
$$-\frac{r(-q)^r}{1-(-q)^r}\Big]\Big)^3-\Big(\frac{v}{92}+\frac{1}{96}\Big)\Big(-2+24\sum_{r=1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Big]\Big)^3$$
$$-\Big(\frac{v}{92}-\frac{1}{96}\Big)\Big(-1+24\sum_{r=1}^{\infty}\Big[\frac{4rq^{4r}}{1-q^{4r}}-\frac{3rq^{3r}}{1-q^{3r}}\Big]\Big)^3+v\Big(-2+24$$

$$\sum_{r=1}^{\infty}\left[\frac{6rq^{6r}}{1-q^{6r}}-\frac{4rq^{4r}}{1-q^{4r}}\right]\Bigg)^3=\Big\{a(q)a(q^2)\Big\}^3. \tag{3.5}$$

$$(ii)\ \left[-\frac{1006}{69}+\frac{351v}{46}+u\right]+540\sum_{r=1}^{\infty}\left(\frac{u}{7}+\frac{3v}{92}+\frac{68}{483}\right)\frac{r^5q^r}{1-q^r}-\Bigg(\frac{8u}{7}$$
$$-\frac{6v}{23}+\frac{100}{483}\Bigg)\frac{r^5q^{2r}}{1-q^{2r}}-\left(\frac{81v}{92}+\frac{198}{161}\right)\frac{r^5q^{3r}}{1-q^{3r}}+\Bigg(-\frac{162v}{23}+\frac{2556}{161}\Bigg)$$
$$\frac{r^5q^{6r}}{1-q^{6r}}+u\Bigg(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\frac{r(-q)^r}{1-(-q)^r}\Bigg)^3+\left(\frac{v}{8}+\frac{1}{8}\right)\Bigg(-1$$
$$+24\sum_{r=1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}-\frac{r(-q)^r}{1-(-q)^r}\right]\Bigg)^3-\left(\frac{v}{92}+\frac{47}{138}\right)\Bigg(-2+24$$
$$\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\right]\Bigg)^3-\left(\frac{v}{92}+\frac{1}{138}\right)\Bigg(-1+24\sum_{r=1}^{\infty}\Bigg[\frac{4rq^{4r}}{1-q^{4r}}$$
$$-\frac{3rq^{3r}}{1-q^{3r}}\Bigg]\Bigg)^3+v\Bigg(-2+24\sum_{r=1}^{\infty}\left[\frac{6rq^{6r}}{1-q^{6r}}-\frac{4rq^{4r}}{1-q^{4r}}\right]\Bigg)^3=\left\{\frac{a(q)c^2(q)}{c(q^2)}\right\}^3. \tag{3.6}$$

*Proof.* Let us persume that,

$$C_1N(q)+C_2N(q^2)+C_3N(q^3)+C_4N(q^6)+C_5\Big[L(-q)-L(q)\Big]^3$$
$$+C_6\Big[L(-q)-2L(q^2)\Big]^3+C_7\Big[L(q)-2L(q^2)\Big]^3+C_8\Big[L(q)-3L(q^3)\Big]^3$$
$$+C_9\Big[2L(q^2)-3L(q^3)\Big]^3+C_{10}\Big[3L(q^3)-4L(q^4)\Big]^3+C_{11}\Big[4L(q^4)-6L(q^6)\Big]^3$$
$$+C_{12}\Big[L(q^4)-3L(q^{12})\Big]^3=\Big\{a(q)a(q^2)\Big\}^3. \tag{3.7}$$

We construct a system by using Lemma 2.3 and rephrasing the relation in terms of the $(p,k)$ parametrization. Then, we equate the coefficients of various terms involving powers of $p$ and $k$, including $k^3$, $pk^3$, $p^2k^3$, $p^3k^3$, $p^4k^3$, $p^5k^3$, $p^6k^3$, $p^7k^3$, $p^8k^3$, $p^9k^3$, $p^{10}k^3$, $p^{11}k^3$, and $p^{12}k^3$ on both sides.

$$
\left(\begin{array}{cccccc}
1 & 1 & 1 & 1 & 0 & 1 \\
-246 & 6 & 6 & 6 & 0 & 30 \\
-5532 & -114 & 12 & 12 & 0 & -264 \\
-38614 & -625 & -58 & 5 & 13824 & 292 \\
-135369 & -\frac{4059}{2} & -297 & -\frac{27}{2} & 62208 & 2934 \\
-276084 & -4302 & -396 & -18 & 124416 & 5112 \\
-348024 & -5556 & -264 & -12 & 145152 & 5016 \\
-276084 & -4302 & -396 & -18 & 108864 & 3600 \\
-135369 & -\frac{4059}{2} & -297 & -\frac{27}{2} & 54432 & 1908 \\
-38614 & -625 & -58 & 5 & 18144 & 760 \\
-5532 & -114 & 12 & 12 & 3888 & 240 \\
-246 & 6 & 6 & 6 & 486 & 48 \\
1 & 1 & 1 & 1 & 27 & 8
\end{array}\right.
$$

$$
\left.\begin{array}{cccccc}
-8 & -27 & -125 & -1331 & -1 & -1 \\
-192 & -486 & -1650 & -12342 & -6 & -6 \\
-1968 & -3888 & -9960 & -51216 & -48 & -12 \\
-11200 & -18144 & -36058 & -125896 & -158 & -32 \\
-38520 & -54432 & -86607 & -204246 & -603 & -90 \\
-81792 & -108864 & -144540 & -230184 & -1044 & -72 \\
-105888 & -145152 & -171120 & -184296 & -2112 & -168 \\
-81792 & -124416 & -144540 & -105120 & -1044 & -288 \\
-38520 & -62208 & -86607 & -42084 & -603 & 180 \\
-11200 & -13824 & -36058 & -11416 & -158 & -536 \\
-1968 & 0 & -9960 & -1968 & -48 & 384 \\
-192 & 0 & -1650 & -192 & -6 & -96 \\
-8 & 0 & -125 & -8 & -1 & 8
\end{array}\right)
\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12} \end{pmatrix}
=
\begin{pmatrix} 27 \\ 486 \\ 3726 \\ 16038 \\ 43173 \\ 76788 \\ 92772 \\ 76788 \\ 43173 \\ 16038 \\ 3726 \\ 486 \\ 27 \end{pmatrix}
$$

Intriguingly, we realize that the system has a limitless number of solutions by solving for the unknowns, which are shown as follows:

$$
C_1 = \left\{\frac{1}{672} - \frac{3v}{92} - \frac{u}{7}\right\},\ C_2 = \left\{\frac{8u}{7} - \frac{6v}{23} - \frac{1}{84}\right\},\ C_3 = \left\{\frac{81v}{92} + \frac{9}{224}\right\},
$$

$$
C_4 = \left\{\frac{163v}{23} - \frac{9}{28}\right\},\ C_5 = u, C_6 = \frac{v}{8}, C_7 = 0,\ C_8 = -\left\{\frac{v}{92} + \frac{1}{96}\right\},
$$

$$
C_9 = 0, C_{10} = -\left\{\frac{v}{92} - \frac{1}{96}\right\},\ C_{11} = v \text{ and } C_{12} = 0,
$$

where $u, v \in \mathbb{R}$.
Substituting the above statistics into (3.7) and simplifying using Definition 2.1 yields equation (3.5).

Modifying the right-hand side of (3.7) leads to the expression designated as equation (ii).

$$(ii)\ \Big(-\frac{u}{7}-\frac{3v}{92}-\frac{68}{483}\Big)N(q)+\Big(\frac{8u}{7}-\frac{6v}{23}+\frac{100}{483}\Big)N(q^2)+\Big(\frac{81v}{92}+\frac{198}{161}\Big)N(q^3)+\Big(\frac{162v}{23}-\frac{2556}{161}\Big)N(q^6)+v\Big[L(-q)-L(q)\Big]^3+\Big(\frac{v}{8}+\frac{1}{8}\Big)\Big[L(-q)-2L(q^2)\Big]^3-\Big(\frac{v}{92}+\frac{47}{138}\Big)\Big[L(q)-3L(q^3)\Big]^3+10\Big[2L(q^2)-3L(q^3)\Big]^3-\Big(\frac{v}{92}+\frac{1}{138}\Big)\Big[3L(q^3)-4L(q^4)\Big]^3+v\Big[4L(q^4)-6L(q^6)\Big]^3=\left\{\frac{a(q)c^2(q)}{c(q^2)}\right\}^3.$$

The key result (3.6) follows directly from (ii) by applying Definition 2.1. □

**Theorem 3.3.** *The relationship between an infinite series and theta functions is described as follows:*

$$(i)\ \Big[-\frac{98}{23}+27v+u\Big]+540\sum_{r=1}^{\infty}\Big(\frac{u}{7}+\frac{9v}{7}-\frac{67}{322}\Big)\frac{r^5q^r}{1-q^r}+\Big(\frac{652}{161}-\frac{8u}{7}-\frac{198v}{7}\Big)\frac{r^5q^{2r}}{1-q^{2r}}+\Big(\frac{5535}{322}\Big)\frac{r^5q^{3r}}{1-q^{3r}}-\Big(\frac{2700}{161}\Big)\frac{r^5q^{6r}}{1-q^{6r}}-u\Big(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\frac{r(-q)^r}{1-(-q)^r}\Big)^3+u\Big(-1+24\sum_{r=1}^{\infty}\Big[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\Big]\Big)^3-\frac{3}{8}\Big(-2+24\sum_{r=1}^{\infty}\Big[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Big]\Big)^3+v\Big(-3+24\sum_{r=1}^{\infty}\Big[\frac{4rq^{4r}}{1-q^{4r}}-\frac{rq^r}{1-q^r}\Big]\Big)^3+\frac{1}{46}\Big(-5+24\sum_{r=1}^{\infty}\Big[\frac{6rq^{6r}}{1-q^{6r}}-\frac{rq^r}{1-q^r}\Big]\Big)^3-\frac{229}{46}\Big(-2+24\sum_{r=1}^{\infty}\Big[\frac{3rq^{6r}}{1-q^{6r}}-\frac{rq^{2r}}{1-q^{2r}}\Big]\Big)^3=\left\{\frac{b^3(q)b(q^2)}{b(-q)b(q^4)}\right\}^3. \tag{3.8}$$

$$(ii)\ \Big[-\frac{59}{3}+u+27v\Big]+540\sum_{r=1}^{\infty}\Big[\Big(\frac{5}{42}+\frac{u}{7}+\frac{9v}{7}\Big)\frac{r^5q^r}{1-q^r}-\Big(\frac{8u}{7}+\frac{198v}{7}$$

$$+\frac{8}{21}\Big)\frac{r^5q^{2r}}{1-q^{2r}}-\left(\frac{9}{14}\right)\frac{r^5q^{3r}}{1-q^{3r}}+\left(\frac{144}{7}\right)\frac{r^5q^{6r}}{1-q^{6r}}\Bigg]-v\Bigg(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}$$

$$-\frac{r(-q)^r}{1-(-q)^r}\Bigg)^3+u\Bigg(-1+24\sum_{r=1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\right]\Bigg)^3+\frac{1}{24}\Bigg(-2$$

$$+24\sum_{r=1}^{\infty}\left[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\right]\Bigg)^3+v\Bigg(-3+24\sum_{r=1}^{\infty}\left[\frac{4rq^{4r}}{1-q^{4r}}-\frac{rq^r}{1-q^r}\right]\Bigg)^3$$

$$-\frac{1}{3}\Bigg(-5+24\sum_{r=1}^{\infty}\left[\frac{6rq^{6r}}{1-q^{6r}}-\frac{rq^r}{1-q^r}\right]\Bigg)^3-\frac{2}{3}\Bigg(-2+24\sum_{r=1}^{\infty}\Bigg[\frac{3rq^{6r}}{1-q^{6r}}$$

$$-\frac{rq^{2r}}{1-q^{2r}}\Bigg]\Bigg)^3=\left\{\frac{a(q)c^2(q)}{c(q^2)}\right\}^3. \tag{3.9}$$

(iii)

$$\left[\frac{337}{48}+27v+u\right]+540\sum_{r=1}^{\infty}\left(\frac{u}{7}+\frac{9v}{7}+\frac{5}{168}\right)\frac{r^5q^r}{1-q^r}-\Bigg(\frac{8u}{7}+\frac{198v}{7}-$$

$$\frac{11}{42}\Bigg)\frac{r^5q^{2r}}{1-q^{2r}}-\left(\frac{99}{112}\right)\frac{r^5q^{3r}}{1-q^{3r}}+\frac{45}{7}\frac{r^5q^{6r}}{1-q^{6r}}-v\Bigg(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}$$

$$-\frac{r(-q)^r}{1-(-q)^r}\Bigg)^3+u\Bigg(-1+24\sum_{r=1}^{\infty}\left[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\right]\Bigg)^3+v\Bigg(-3$$

$$+24\sum_{r=1}^{\infty}\left[\frac{4rq^{4r}}{1-q^{4r}}-\frac{rq^r}{1-q^r}\right]\Bigg)^3-\frac{23}{192}\Bigg(-2+24\sum_{r=1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}$$

$$-\frac{rq^r}{1-q^r}\Bigg]\Bigg)^3-\frac{1}{48}\Bigg(-5+24\sum_{r=1}^{\infty}\left[\frac{6rq^{6r}}{1-q^{6r}}-\frac{rq^r}{1-q^r}\right]\Bigg)^3+\frac{23}{24}\Bigg(-2$$

$$+24\sum_{r=1}^{\infty}\left[\frac{3rq^{6r}}{1-q^{6r}}-\frac{rq^{2r}}{1-q^{2r}}\right]\Bigg)^3=\Big\{a(q)a(q^2)\Big\}^3. \tag{3.10}$$

*Proof.* Consider the relation,

$$\begin{aligned} &C_1N(q)+C_2N(q^2)+C_3N(q^3)+C_4N(q^6)+C_5\Big[L(-q)-L(q)\Big]^3 \\ &+C_6\Big[L(-q)-2L(q^2)\Big]^3+C_7\Big[L(q)-2L(q^2)\Big]^3+C_8\Big[L(q)-3L(q^3)\Big]^3 \\ &+C_9\Big[L(q)-4L(q^4)\Big]^3+C_{10}\Big[L(q)-6L(q^6)\Big]^3+C_{11}\Big[L(q)-12L(q^{12})\Big]^3 \\ &+C_{12}\Big[L(q^2)-3L(q^6)\Big]^3=\left\{\frac{b^3(q)b(q^2)}{b(-q)b(q^4)}\right\}^3. \end{aligned} \tag{3.11}$$

Applying Lemma 2.3 and expressing the relation in terms of the $(p,k)$ parametrization. Next, we match the coefficients of terms involving various powers of $p$ and $k$, such as $k^3$, $pk^3$, $p^2k^3$, $p^3k^3$, $p^4k^3$, $p^5k^3$, $p^6k^3$, $p^7k^3$, $p^8k^3$, $p^9k^3$, $p^{10}k^3$, $p^{11}k^3$, and $p^{12}k^3$ on both sides, we develop a system

$$\left(\begin{matrix} 1 & 1 & 1 & 1 & 0 & 1 \\ -246 & 6 & 6 & 6 & 0 & 30 \\ -5532 & -114 & 12 & 12 & 0 & -264 \\ -38614 & -625 & -58 & 5 & 13824 & 292 \\ -135369 & -\frac{4059}{2} & -297 & -\frac{27}{2} & 62208 & 2934 \\ -276084 & -4302 & -396 & -18 & 124416 & 5112 \\ -348024 & -5556 & -264 & -12 & 145152 & 5016 \\ -276084 & -4302 & -396 & -18 & 108864 & 3600 \\ -135369 & -\frac{4059}{2} & -297 & -\frac{27}{2} & 54432 & 1908 \\ -38614 & -625 & -58 & 5 & 18144 & 760 \\ -5532 & -114 & 12 & 12 & 3888 & 240 \\ -246 & 6 & 6 & 6 & 486 & 48 \\ 1 & 1 & 1 & 1 & 27 & 8 \end{matrix}\right.$$

$$\left.\begin{matrix} -1 & -8 & -27 & -125 & -1331 & -1 \\ -42 & -192 & -486 & -1650 & -12342 & -6 \\ -660 & -1968 & -3888 & -9960 & -51216 & -48 \\ -4802 & -11200 & -18144 & -36058 & -125896 & -158 \\ -17019 & -38520 & -54432 & -86607 & -204246 & -603 \\ -34524 & -81792 & -108864 & -144540 & -230184 & -1044 \\ -43368 & -105888 & -145152 & -171120 & -184296 & -2112 \\ -34524 & -81792 & -124416 & -144540 & -105120 & -1044 \\ -17019 & -38520 & -62208 & -86607 & -42084 & -603 \\ -4802 & -11200 & -13824 & -36058 & -11416 & -158 \\ -660 & -1968 & 0 & -9960 & -1968 & -48 \\ -42 & -192 & 0 & -1650 & -192 & -6 \\ -1 & -8 & 0 & -125 & -8 & -1 \end{matrix}\right) \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ -12 \\ 66 \\ -220 \\ 495 \\ -792 \\ 924 \\ -792 \\ 495 \\ -220 \\ 66 \\ -12 \\ 1 \end{pmatrix} \blacksquare$$

The infinitely many solutions for the unknowns are :

$$C_1=\left\{\frac{67}{322}-\frac{u}{7}-\frac{9v}{7}\right\},\ C_2=\left\{\frac{8u}{7}+\frac{198v}{7}-\frac{652}{161}\right\},\ C_3=-\frac{5535}{322},$$
$$C_4=\frac{2700}{161},\ C_5=-u,\ C_6=0,\ C_7=u,\ C_8=-\frac{3}{8},\ C_9=v,$$
$$C_{10}=\frac{1}{46},\ C_{11}=0 \text{ and } C_{12}=-\frac{229}{46},$$

where $u,v\in\mathbb{R}$.
The required result is obtained by substituting these entries into (3.11).

Similarly, applying the same approach, the following identities are derived.

$$(ii)\ \left(-\frac{u}{7}-\frac{9v}{7}-\frac{5}{42}\right)N(q)+\left(\frac{8u}{7}+\frac{198v}{7}+\frac{8}{21}\right)N(q^2)+\frac{9}{14}N(q^3)$$
$$-\frac{144}{7}N(q^6)-v\Big[L(-q)-L(q)\Big]^3-u\Big[L(q)-2L(q^2)\Big]^3+\frac{1}{24}\Big[L(q)$$
$$-3L(q^3)\Big]^3+v\Big[L(q)-4L(q^4)\Big]^3-\frac{1}{3}\Big[L(q)-6L(q^6)\Big]^3-\frac{2}{3}\Big[L(q^2)$$
$$-3L(q^6)\Big]^3=\left\{\frac{a(q)c^2(q)}{c(q^2)}\right\}^3.$$

$$(iii)\ \left(-\frac{5}{168}-\frac{u}{7}-\frac{9v}{7}\right)N(q)+\left(-\frac{11}{42}+\frac{8u}{7}+\frac{198v}{7}\right)N(q^2)-\frac{99}{112}N(q^3)$$
$$+\frac{45}{7}N(q^6)-v\Big[L(-q)-L(q)\Big]^3+u\Big[L(q)-3L(q^3)\Big]^3$$
$$+\frac{23}{192}\Big[L(q)-3L(q^3)\Big]^3+v\Big[L(q)-4L(q^4)\Big]^3-\frac{1}{48}\Big[L(q)-6L(q^6)\Big]^3$$
$$+\frac{23}{24}\Big[L(q^2)-3L(q^6)\Big]^3=\Big\{a(q)a(q^2)\Big\}^3.$$

The fundamental results (3.9) and (3.10) are derived directly employing Definition 2.1 to (ii) and (iii). □

**Theorem 3.4.** *The connection between an infinite series and theta functions is as follows:*

$$(i)\ \left[-\frac{1006}{69}+27v+u\right]+540\sum_{r=1}^{\infty}\Bigg[\left(\frac{u}{7}+\frac{9v}{7}+\frac{68}{483}\right)\frac{r^5q^r}{1-q^r}-\Bigg(\frac{8u}{7}$$

$$+\frac{198v}{7}+\frac{100}{483}\Big)\frac{r^5q^{2r}}{1-q^{2r}}-\Big(\frac{198}{161}\Big)\frac{r^5q^{3r}}{1-q^{3r}}+\Big(\frac{2556}{161}\Big)\frac{r^5q^{6r}}{1-q^{6r}}\Bigg]-v\Bigg(24$$

$$\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\frac{r(-q)^r}{1-(-q)^r}\Bigg)^3+u\Bigg(-1+24\sum_{r=1}^{\infty}\Bigg[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg)^3$$

$$-\frac{1}{138}\Bigg(-1+24\sum_{r=1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{2rq^{2r}}{1-q^{2r}}\Bigg]\Bigg)^3+\Big(\frac{1}{8}\Big)\Bigg(-2+24$$

$$\sum_{r=1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg)^3+v\Bigg(-3+24\sum_{r=1}^{\infty}\Bigg[\frac{4rq^{4r}}{1-q^{4r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg)^3$$

$$-\frac{47}{138}\Bigg(-5+24\sum_{r=1}^{\infty}\Bigg[\frac{6rq^{6r}}{1-q^{6r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg)^3=\Bigg\{\frac{a(q)c^2(q)}{c(q^2)}\Bigg\}^3. \tag{3.12}$$

(ii) $$\Bigg[-\frac{7}{24}+27v+u\Bigg]+540\sum_{r=1}^{\infty}\Bigg(\frac{u}{7}+\frac{9v}{7}-\frac{1}{672}\Bigg)\frac{r^5q^r}{1-q^r}-\Bigg(\frac{8u}{7}$$

$$+\frac{198v}{7}-\frac{1}{84}\Bigg)\frac{r^5q^{2r}}{1-q^{2r}}-\Big(\frac{9}{224}\Big)\frac{r^5q^{3r}}{1-q^{3r}}+\frac{9}{28}\frac{r^5q^{6r}}{1-q^{6r}}-v\Bigg(24$$

$$\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\frac{r(-q)^r}{1-(-q)^r}\Bigg)^3+u\Bigg(-1+24\sum_{r=1}^{\infty}\Bigg[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg)^3$$

$$+v\Bigg(-3+24\sum_{r=1}^{\infty}\Bigg[\frac{4rq^{4r}}{1-q^{4r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg)^3-\frac{1}{96}\Bigg(-5+24\sum_{r=1}^{\infty}\Bigg[\frac{6rq^{6r}}{1-q^{6r}}$$

$$-\frac{rq^r}{1-q^r}\Bigg]\Bigg)^3+\frac{1}{96}\Bigg(-1+24\sum_{r=1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{2rq^{2r}}{1-q^{2r}}\Bigg]\Bigg)^3=\Bigg\{a(q)a(q^2)\Bigg\}^3. \tag{3.13}$$

(iii) $$\Bigg[-\frac{98}{23}+u\Bigg]+540\sum_{r=1}^{\infty}\Bigg(\frac{u}{7}-\frac{67}{322}\Bigg)\frac{r^5q^r}{1-q^r}+\Bigg(\frac{652}{161}-\frac{8u}{7}\Bigg)\frac{r^5q^{2r}}{1-q^{2r}}$$

$$+\Big(\frac{5535}{322}\Big)\frac{r^5q^{3r}}{1-q^{3r}}-\Big(\frac{2700}{161}\Big)\frac{r^5q^{6r}}{1-q^{6r}}+u\Bigg(24\sum_{r=1}^{\infty}\frac{rq^r}{1-q^r}-\frac{r(-q)^r}{1-(-q)^r}\Bigg)^3$$

$$-\frac{3}{8}\Bigg(-1+24\sum_{r=1}^{\infty}\Bigg[\frac{3rq^{3r}}{1-q^{3r}}-\frac{2rq^{2r}}{1-q^{2r}}\Bigg]\Bigg)^3-\Big(\frac{229}{46}\Big)\Bigg(-1+24$$

$$\sum_{r=1}^{\infty}\Bigg[\frac{2rq^{2r}}{1-q^{2r}}-\frac{rq^r}{1-q^r}\Bigg]\Bigg)^3+\frac{1}{46}\Bigg(-2+24\sum_{r=1}^{\infty}\Bigg[\frac{6rq^{6r}}{1-q^{6r}}-\frac{4rq^{4r}}{1-q^{4r}}\Bigg]\Bigg)^3$$

$$= \left\{ \frac{b^3(q)b(q^2)}{b(-q)b(q^4)} \right\}^3. \tag{3.14}$$

*Proof.* Assume that

$$\begin{aligned}
&C_1N(q) + C_2N(q^2) + C_3N(q^3) + C_4N(q^6) + C_5\Big[L(-q) - L(q)\Big]^3 \\
&+ C_6\Big[L(q) - 2L(q^2)\Big]^3 + C_7\Big[2L(q^2) - 3L(q^3)\Big]^3 + C_8\Big[L(q) - 3L(q^3)\Big]^3 \\
&+ C_9\Big[3L(q^3) - 4L(q^4)\Big]^3 + C_{10}\Big[L(q) - 4L(q^4)\Big]^3 + C_{11}\Big[4L(q^4) - 6L(q^6)\Big]^3 \\
&+ C_{12}\Big[L(q) - 6L(q^6)\Big]^3 = \left\{ \frac{a(q)c^2(q)}{c(q^2)} \right\}^3.
\end{aligned} \tag{3.15}$$

Reformulating the relation in terms of the $(p,k)$ parametrization and using Lemma 2.3, we match the coefficients of various terms involving powers of $p$ and $k$, including those of the form $k^3$, $pk^3$, $p^2k^3$, $p^3k^3$, $p^4k^3$, $p^5k^3$, $p^6k^3$, $p^7k^3$, $p^8k^3$, $p^9k^3$, $p^{10}k^3$, $p^{11}k^3$, and $p^{12}k^3$, and build a system

$$\left(\begin{array}{cccccc}
1 & 1 & 1 & 1 & -1 & -1 \\
-246 & 6 & 6 & 6 & -42 & -6 \\
-5532 & -114 & 12 & 12 & -660 & -48 \\
-38614 & -625 & -58 & 5 & -4802 & -158 \\
-135369 & -\frac{4059}{2} & -297 & -\frac{27}{2} & -17019 & -603 \\
-276084 & -4302 & -396 & -18 & -34524 & -1044 \\
-348024 & -5556 & -264 & -12 & -43368 & -2112 \\
-276084 & -4302 & -396 & -18 & -34524 & -1044 \\
-135369 & -\frac{4059}{2} & -297 & -\frac{27}{2} & -17019 & -603 \\
-38614 & -625 & -58 & 5 & -4802 & -158 \\
-5532 & -114 & 12 & 12 & -660 & -48 \\
-246 & 6 & 6 & 6 & -42 & -6 \\
1 & 1 & 1 & 1 & -1 & -1
\end{array}\right.$$

$$\left.\begin{array}{cccccc}
-8 & -1 & -27 & -8 & -125 & -1331 \\
-192 & -6 & -486 & -48 & -1650 & -12342 \\
-1968 & -12 & -3888 & -96 & -9960 & -51216 \\
-11200 & -32 & -18144 & -40 & -36058 & -125896 \\
-38520 & -90 & -54432 & 36 & -86607 & -204246 \\
-81792 & -72 & -108864 & -144 & -144540 & -230184 \\
-105888 & -168 & -145152 & -264 & -171120 & -184296 \\
-81792 & -288 & -124416 & 72 & -144540 & -105120 \\
-38520 & 180 & -62208 & 90 & -86607 & -42084 \\
-11200 & -536 & -13824 & -292 & -36058 & -11416 \\
-1968 & 384 & 0 & -60 & -9960 & -1968 \\
-192 & -96 & 0 & 150 & -1650 & -192 \\
-8 & 8 & 0 & -125 & -125 & -8
\end{array}\right)
\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \\ C_9 \\ C_{10} \\ C_{11} \\ C_{12} \end{pmatrix}
=
\begin{pmatrix} 27 \\ 486 \\ 3726 \\ 16038 \\ 43173 \\ 76788 \\ 92772 \\ 76788 \\ 43173 \\ 16038 \\ 3726 \\ 486 \\ 27 \end{pmatrix}.$$

On solving the above system for $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}$, we get:

$$C_1 = \left\{ -\frac{68}{483} - \frac{u}{7} - \frac{9v}{7} \right\},\ C_2 = \left\{ \frac{8u}{7} + \frac{198v}{7} + \frac{100}{483} \right\},\ C_3 = -\frac{198}{161},$$

$$C_4 = -\frac{2556}{161},\ C_5 = -v,\ C_6 = u,\ C_7 = -\frac{1}{138},\ C_8 = \frac{1}{8},\ C_9 = 0,$$

$$C_{10} = v,\ C_{11} = 0 \text{ and } C_{12} = -\frac{47}{138},$$

where $u, v \in \mathbb{R}$.
Substituting these values into (3.15) gives (3.12).
Similarly, modifying the right-hand side of (3.15) and simplifying, leads to equations (ii) and (iii).

$$(ii)\ \left( -\frac{u}{7} - \frac{9v}{7} + \frac{1}{672} \right) N(q) + \left( \frac{8u}{7} + \frac{198v}{7} - \frac{1}{84} \right) N(q^2) + \frac{9}{224} N(q^3) - \frac{9}{28} N(q^6) - v\Big[L(-q) - L(q)\Big]^3 + u\Big[L(q) - 2L(q^2)\Big]^3 + \frac{1}{96}\Big[2L(q^2) - 3L(q^3)\Big]^3 + v\Big[L(q) - 4L(q^4)\Big]^3 - \frac{1}{96}\Big[L(q) - 6L(q^6)\Big]^3 = \Big\{a(q)a(q^2)\Big\}$$

$$(iii)\ \left( \frac{67}{322} - \frac{u}{7} \right) N(q) + \left( -\frac{652}{161} + \frac{8u}{7} \right) N(q^2) - \frac{5535}{322} N(q^3) + \frac{2700}{161} N(q^6) + u\Big[L(-q) - L(q)\Big]^3 - \left\{ \frac{229}{46} \right\} \Big[L(q) - 2L(q^2)\Big]^3 - \frac{3}{8}\Big[2L(q^2) - 3L(q^3)\Big]^3 + \frac{1}{46}\Big[4L(q^4) - 6L(q^6)\Big]^3 = \left\{ \frac{b^3(q)b(q^2)}{b(-q)b(q^4)} \right\}^3.$$

By using the definition of Eisenstein series to the above relations (ii) and (iii) and further simplifying, we obtain equations (3.13) to (3.14). □

## 4. Conclusion

This study focuses on the sum of an infinite series that converges to an infinite product, highlighting a fascinating mathematical phenomenon commonly encountered in number theory, analysis, and modular forms. These relationships are not only visually striking but also carry deep implications for understanding the interplay between series and product representations of special functions. This process not only confirms the validity of derived relations but also highlights the utility of the Eisenstein series in transforming and simplifying complex summations or series into more interpretable forms. The convergence of the Ramanujan-type Eisenstein series to an infinite product represents a significant result in number theory and complex analysis, revealing profound connections between theta functions, modular forms, and infinite series.

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Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education,, Manipal- 576104, India.

*Email address*: vidya.rao@manipal.edu

Department of Computer Science and Engineering, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India.

*Email address*: ashwath.rao.b@gmail.com

Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education,, Manipal- 576104, India.

*Email address*: smitha.holla@manipal.edu