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ON DOMINATION OF SIERPINSKI GRAPHS

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Abstract. In this article, we determine the exact values of the double Italian and perfect double Italian domination number of generalized Sierpiński graphs $S(G, t)$, when $t = 2$. In particular, when $G = C_n$ or a graph having exactly one universal vertex or a graph having at least two universal vertices.

1. Introduction

In this paper we study the Roman-$\{3\}$ domination or double Italian domination number and the perfect double Italian domination number of the generalised Sierpiński graph $S(G, t)$. Let us begin with some required terminology. Let $G = (V, E)$ be a graph of order n with vertex set $V = V(G)$ and edge set $E = E(G)$. Let C_n, P_n and K_n be respectively the cycle, path and complete graph on n vertices. A tree is a connected graph with no cycles. The open neighbourhood of a vertex $v \in V(G)$ is the set $N(v) = \{u : uv \in E(G)\}$. The closed neighborhood of a vertex $v \in V(G)$ is $N[v] = N(v) \cup \{v\}$. $|N(v)|$ is called the degree of the vertex $v \in G$ and is denoted by $d(v)$. The maximum degree of a vertex in G is denoted by $\Delta(G)$. A vertex of degree 0 is known as an isolated vertex of G. A vertex of degree 1 is called a leaf vertex. A vertex of degree $n - 1$ is a universal vertex, where $n = |V(G)|$. If S is a non-empty subset of the vertex set V of the graph G then the sub-graph induced by S is defined as the graph having vertex set S and edge set consisting of edges of G having both ends in S. All graphs considered here are simple and undirected. For any graph theoretic terminology, definition or notation not mentioned in this article, the readers may refer to [4, 3, 5, 6, 9].

In [9], Mojdeh and Volkmann introduced the concept of double Italian domination, which is a variant of Roman domination. The origin of Roman domination was motivated by the defense strategies used to defend the Roman Empire during the reign of the Great, Emperor Constantine. Double Italian domination is an optimization of a stronger version of the Roman domination. In [4], Hao et al. initiated the study of perfect double Italian domination. They evaluated the γ^P_{dI} of some standard graphs and examined the corresponding γ_{dI}. In [10], the Roman domination number of generalized Sierpiński graph $S(G, t)$ is studied and general upper bound is found and its tightness observed. In [12], the exact Italian

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domination, perfect Italian domination and double Roman domination number of generalized Sierpiński graph $S(G,2)$ are obtained, where G can be C_n, $n \geq 4$, $K_{1,q}$ or $K_{2,q}, q \geq 2$ and a bistar $B_{m,n}$, $m,n \geq 3$. In [1, 2], a bound for the double Roman domination number of generalized Sierpiński graph is obtained and exact value of $\gamma_{dR}\left(S\left(K_n,2\right)\right)$ is found precisely.

Definition 1.1. For a graph G, a Roman{3}-dominating function or double Italian dominating function (abbreviated DIDF) is a function $f : V \to \{0,1,2,3\}$ having the property that for every vertex $u \in V$, if $f(u) \in \{0,1\}$, then $f(N[u]) \geqslant 3$. Formally, a Roman $\{3\}$-dominating function $f : V \to \{0,1,2,3\}$ has the property that for every vertex $v \in V$, with $f(v) = 0$, there exist at least either three vertices in $V_1 \cap N(v)$ or one vertex in $V_1 \cap N(v)$ and one in $V_2 \cap N(v)$ or two vertices in $V_2 \cap N(v)$ or one vertex in $V_3 \cap N(v)$ and for every vertex $v \in V$, with $f(v) = 1$, there exist at least either two vertices in $V_1 \cap N(v)$ or one vertex in $(V_2 \cup V_3) \cap N(v)$.

The weight of a Roman $\{3\}$ dominating function or double Italian dominating function (See [9]) is the sum $w_f = \Sigma_{v \in V(G)} f(v)$ and the minimum weight of a Roman $\{3\}$-dominating function f is the Roman $\{3\}$-domination number or double Italian domination number, denoted by $\gamma_{R3}(G)$ or $\gamma_{dI}(G)$ and we can write $f = (V_0, V_1, V_2, V_3)$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i = 0,1,2,3$.

Definition 1.2. A perfect double Italian dominating function (abbreviated, PDIDF or PDID function) is a function $f : V \to \{0,1,2,3\}$ having the property that,

(1) For any vertex $v \in V(G)$, if $f(v) = 0$, then v is either adjacent to at least 3 and at most 4 vertices in V_1 and no vertex in $V_2 \cup V_3$, or is adjacent to at least 1 vertex and at most 2 vertices in V_1 and exactly one vertex in V_2 and no vertex in V_3, or is adjacent to at most one vertex in V_1 and exactly one vertex in V_3 and no vertex in V_2, or is adjacent to two vertices in V_2 and no vertex in $V_1 \cup V_3$.
(2) If $f(v) = 1$,then v either is adjacent to at least 2 vertices and at most 3 vertices in V_1 and no vertex in $V_2 \cup V_3$, or is adjacent to at most one vertex of V_1 and exactly one vertex in V_2 and no vertex in V_3, or is adjacent to exactly one vertex in V_3 and no vertex in $V_1 \cup V_3$. The weight of a perfect double Italian dominating function f is the sum $w_f = f(V) = \Sigma_{v \in V(G)} f(v)$ ie, we have, for $v \in V_0 \cup V_1, 3 \leqslant \Sigma_{u \in N[v]} f(u) \leqslant 4$.And the minimum weight of a perfect double Italian dominating function on G is the perfect double Italian domination number (abbreviated, PDID or PDID number) of G, denoted by $\gamma^P_{dI}(G)$ [4].

Definition 1.3. Let $G = (V,E)$ be a non-empty graph of order $n \geqslant 2$, and t a positive integer.Let V^t denote the set of words of length t on alphabet V. The letters of a word u of length t are denoted by $u_1 u_2 \ldots u_t$. In [9] Klavžar and Milutinović introduced the graph $S\left(K_n,t\right), t \geqslant 1,\ S(t,n)$ in their notation whose vertex set is V^t, where $\{u,\, v\}$ is an edge if and only if there exists $i \in \{1,2,\ldots,t\}$ such that
(i) $u_j = v_j$, if $j < i$
(ii) $u_i \neq v_i$
(iii) $u_j = v_i$ and $v_j = u_i$ if $j > i$.

Later, those graphs were called Sierpiński graphs in [8].

This construction was generalized for any graph $G=(V,E)$, by defining the t-th generalized Sierpiński graph of G, denoted by $S(G,t)$, as the graph with vertex set V^t and edge set

$$\left\{\left\{wu_iu_j^{r-1}, wu_ju_i^{r-1}\right\} : \{u_i,u_j\}\in E, i\neq j; r\in\{1,2,\ldots,t\}; w\in V^{t-r}\right\}.$$

Vertices of the form $xx\ldots x$ are called extreme vertices of $S(G,t)$. Note that for any graph G of order n and any integer $t\geq 2$, $S(G,t)$ has n extreme vertices and, if x has degree $d(x)$ in G, then the extreme vertex $xx\ldots x$ of $S(G,t)$ also has degree $d(x)$ [1, 12, 7, 10]. Figure 1, gives $S(C_6,1)$ and $S(C_6,2)$.

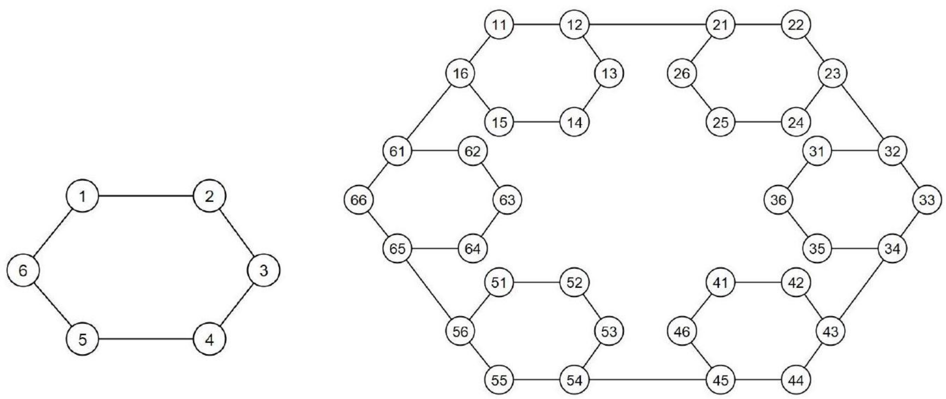

FIGURE 1. $S(C_6,t)$, when $t=1,2$.

See that $S(G,1)$ is G itself. If $V=\{1,2,\ldots,n\}$ is the vertex set of G, then $V_i=\{ij : j=1,2,\ldots,n\}$ induces a copy of G in $S(G,2)$ for each $i\in\{1,2,\ldots,n\}$. The sub-graph induced by V_i is denoted by G^i, for $i\in\{1,2,\ldots,n\}$ [12]. The following theorem and observations are useful in this paper.

Theorem 1.4. *For any tree $T, \gamma_{dI}(T)=\gamma_{dR}(T)$..*

Observation 1.1: [9] For any graph $G, \gamma_{dI}(G)\leqslant\gamma_{dR}(G)$ ie, by definition every double Roman dominating function is a double Italian dominating. In fact, double Italian domination is a variant of double Roman domination.

Observation 1.2: [9] If G is a graph of order $n\geq 2$, then $\gamma_{dI}(G)\geq 3$, with equality if and only if G has at least one universal vertex.

Observation 1.3: [9, 4] Let $n\geq 1$. Then

$$\gamma_{dI}(P_n)=\gamma_{dI}^P(P_n)=\begin{cases} n & \text{if } n\equiv 0(\bmod 3)\\ n+1 & \text{otherwise}\end{cases}$$

Observation 1.4: [9, 4] For a cycle C_n, we have $\gamma_{dI}(C_n)=\gamma_{dI}^P(C_n)=n$.

Observation 1.5: [4]. For any graph $G, \gamma_{dI}(G)\leq\gamma_{dI}^P(G)$ by the definition.

Observation 1.6: [9, 12, 11] For any tree T and any positive integer t, $S(T,t)$ is a tree. Then the double Italian domination number of generalised Sierpinski graph $S(B_{m,n},2)$ is $\gamma_{dI}(S(B_{m,n},2))=6(m+n+1), m,n\geqslant 3$.

2. Double Italian and Perfect Double Italian Domination number of $S(C_n, 2)$

In this section, we obtain the exact double Italian and perfect double Italian domination number of $S(C_n, 2)$.

Theorem 2.1. *The double Italian domination number of generalized Sierpiński graph $S(C_n, 2), \gamma_{dI}(S(C_n, 2)) = n(n-1)$ for $n \geqslant 3$.*

Proof. Let $V(C_n) = \{v_1, v_2, \ldots, v_n\}$. Then $S(C_n, 2)$ has the vertex set

$$\{v_i v_j : i, j \in \{1, 2, \ldots, n\}\}$$

and edge set $\{(v_i v_j, v_i v_k) : v_j v_k \in E(C_n)\} \cup \{(v_i v_j, v_j v_i)\} : v_i v_j \in E(C_n)\}$. Now, consider the function,

$$f(v_i v_j) = \begin{cases} 1 & i \in \{1,2,3,\ldots n\}, j = i+1(\bmod n) \\ 2 & i \in \{1,2,3,\ldots n\}, j = i+1+2l(\bmod n), l = 1,2,3,\ldots n-4 \\ 0 & \text{otherwise} \end{cases}$$

Clearly, this is a DIDF on $S(C_n, 2)$. So, $f(S(C_n, 2)) \leqslant n + 2n + 2\left(\frac{n-4}{2}\right)n = n(n-1)$. Conversely, in each $C_n^i, v_i v_i$ is the extreme vertex and $v_i v_{i-1}, v_i v_{i+1}$ are the only vertices adjacent to other copies of C_n. The remaining vertices (other than $v_i v_{i-1}, v_i v_i, v_i v_{i+1}$) form a path on $n-3$ vertices, say P_{n-3}^i. Let f be a DIDF of $S(C_n, 2)$. Note that the extreme vertices $v_i v_i$ and adjacent vertices $v_i v_{i-1}, v_i v_{i+1}$ of all n copies of C_n form a cycle C of length $3n$.

If $f(v_i v_{i-1}) + f(v_i v_{i+1}) = 0$. Then $f(v_i v_{i-1}) = f(v_i v_{i+1}) = 0$ such that $f(v_i v_i) \geqslant 2$, the remaining vertices form a path of length $n-3 \equiv 0(\bmod 3)$ and $f\left(P_{n-3}^i\right) \geqslant n-3$. So, $f\left(C_n^i\right) \geqslant n-3+2 = n-1$.

If $f(v_i v_{i-1}) + f(v_i v_{i+1}) = 1$. For definiteness let $f(v_i v_{i-1}) = 0$ and $f(v_i v_{i+1}) = 1$ then clearly $f(v_i v_i) \geqslant 2$ and $f\left(P_{n-3}^i - (v_i v_{i+2})\right) \geqslant n-4+1 = n-3$. So, $f\left(C_n^i\right) \geqslant n-3+2+1 = n$.

If $f(v_i v_{i-1}) + f(v_i v_{i+1}) = 2$, let $f(v_i v_{i-1}) = 0$ and $f(v_i v_{i+1}) = 2$ for definiteness then, $f(v_i v_i) \geqslant 1$. Also, $f\left(P_{n-3}^i - (v_i v_{i+2})\right) \geqslant n-3$ hence $f\left(C_n^i\right) \geqslant n$ or consider the case when $f(v_i v_{i-1}) = 1$ and $f(v_i v_{i+1}) = 1$ then $f(v_i v_i) \geqslant 1$ and if $f(v_{i-1} v_i) = 2, f(v_{i+1} v_i) = 2$ then $f\left(P_{n-3}^i - (v_i v_{i+2}), (v_i v_{i-2})\right) \geqslant n-5+1 = n-4$. So, $f\left(C_n^i\right) \geqslant n-4+3 = n-1$.

If $f(v_i v_{i-1}) + f(v_i v_{i+1}) = 3$. Then either $f(v_i v_{i-1}) = 0$ and $f(v_i v_{i+1}) = 3$ or vice versa or $f(v_i v_{i-1}) = 1$ and $f(v_i v_{i+1}) = 2$ or vice versa. For let, $f(v_i v_{i-1}) = 0$ and $f(v_i v_{i+1}) = 3$ then, $f(v_i v_i) = 0$.

Again $f\left(P_{n-3}^i - (v_i v_{i+2})\right) \geqslant n-4+1 = n-3$. So, $f\left(C_n^i\right) \geqslant n-3+3 = n$.If $f(v_i v_{i-1}) = 1$ and $f(v_i v_{i+1}) = 2$ then, $f(v_i v_i) = 0, f\left(P_{n-3}^i - (v_i v_{i+2})\right) \geqslant n-3$ and $f\left(C_n^i\right) \geqslant n$. Here consider the particular case when $f(v_{i-1} v_i) = 2$ and $f\left(P_{n-3}^i - (v_i v_{i+2}), (v_i v_{i-2})\right) \geqslant n-5+1 = n-4$. Hence, $f\left(C_n^i\right) \geqslant n-4+3 =$

$n-1$.

If $f(v_iv_{i-1})+f(v_iv_{i+1}) \geqslant 4$.Then $f\left(P^i_{n-3}-(v_iv_{i+2}),(v_iv_{i-2})\right) \geqslant n-5+1 = n-4$. So, $f\left(C^i_n\right) \geqslant n-4+4=n$.

Thus in all cases, $f\left(C^i_n\right) \geqslant n-1 \forall i$. So, $f(S(C_n,2)) \geqslant n(n-1)$. Hence, $f(S(C_n,2)) = n(n-1)$. □

Corollary 2.2. *The perfect double Italian domination number of generalised Sierpiński graph $S(C_n,2)$ is $\gamma^P_{dI}(S(C_n,2)) = n(n-1)$ for $n \geqslant 3$.*

Proof. Consider the function,

$$f(v_iv_j) = \begin{cases} 1 & i \in \{1,2,3,\dots n\}, j = i+1(\bmod n) \\ 2 & i \in \{1,2,3,\dots n\}, j = i+1+2l(\bmod n), l = 1,2,3,\dots n-4 \\ 0 & \text{otherwise} \end{cases}$$

in Theorem 2.1, it is a PDIDF on $S(C_n,2)$ as well.So, $\gamma^P_{dI}(S(C_n,2)) \leqslant n+2n+2\left(\frac{n-4}{2}\right)n = n(n-1)$ and $\gamma_{dI}(S(C_n,2)) \leqslant \gamma^P_{dI}(S(C_n,2))$.Hence by Theorem 2.1, $n(n-1) \leqslant \gamma^P_{dI}(S(C_n,2))$.So, $\gamma^P_{dI}(S(C_n,2)) = n(n-1)$. □

Remark 2.3. In [12], $\gamma_{dR}(S(C_n,2))$ is found and we get $\gamma_{dI}(S(C_n,2)) < \gamma_{dR}(S(C_n,2))$ for $n \neq 3k+1, k \geqslant 1$. Note that we have used the same technique to prove so.

3. Double Italian domination number and perfect double Italian domination number of $S(G,2)$ with certain conditions

Fir,stly we obtain, the double Italian domination number of $S(G,2)$ where G has a universal vertex followed by determining the perfect double Italian domination number of $S(G,2)$ where G has exactly one universal vertex and then the perfect double Italian domination number of $S(G,2)$ where G has at least two universal vertices.

Theorem 2.3. *If a graph G has a universal vertex,then $\gamma_{dI}(S(G,2)) = 3n-1$.*

Proof. The case when $n = 1,2$ can be proved through inspection.Let $V = \{v_1,v_2,\dots v_n\}$ be the vertex set of G having a universal vertex v_1. Then let $\{v_1v_1,v_2v_1,\dots v_nv_1\}$ be the corresponding vertices on each copy of G, say $G^i, i = 1,2,3,\dots \in S(G,2)$.Then G^1 contains the extreme vertex v_1v_1.Now consider a DIDF function f such that $f(v_iv_1) = 3$ for all $i \neq 1$ and $f(v_1v_1) = 2$.So, $\gamma_{dI}(S(G,2)) \leqslant 3(n-1)+2 = 3n-1$. Conversely, let f be a γ_{dI} - function and to double Italian dominate the extreme vertex v_iv_i in G^i, either $f(v_iv_i) = 2,3$ or if $f(v_iv_i) \in \{0,1\}$ then, $f(N[v_iv_i]) \geqslant 3$ and $f\left(G^i\right) \geq 3$. We know $\gamma_{dI}(S(G,2)) \leqslant 3n-1$ and $f\left(G^i\right) \geq 3 \forall i$ except possibly when $f(v_iv_i) = 2 = f\left(G^i\right)$. So, in the latter case, to double Italian dominate $v_iv_j, j \neq i, v_iv_j$ must be adjacent to all v_jv_i indicating that v_i is a universal vertex in the base graph say, $v_i = v_1$. To double Italian dominate $v_1v_j, j \neq 1, v_jv_1$ must have weight at least 1.If v_jv_1 has weight 1 then G^j contains at least one more vertex with weight at least 2 and if v_jv_1 has weight 2 then G^j contains at least one more vertex with weight at least 1 if not

$v_j v_j$ cannot be double Italian dominated.Therefore, $f\left(G^j\right) \geq 3\ \forall j \neq 1$. Hence, $\gamma_{dI}(S(G,2)) \geqslant 3(n-1)+2 = 3n-1$. So, $\gamma_{dI}(S(G,2)) = 3n-1$. □

Remark 3.2. Clearly $\gamma_{dI}(S(G,2)) = \gamma_{dR}(S(G,2)) = 3n-1$ because if there exists a γ_{dI} - function f such that $f(v) = 1$ for no $v \in V(G)$ then $\gamma_{dI}(G) = \gamma_{dR}(G)$. In particular $\gamma_{dI}\left(S\left(K_n, 2\right)\right) = 3n-1$ and $\gamma_{dI}\left(S\left(K_{1,q}, 2\right)\right) = 3q+2$. Note that for a graph G with no edges and n vertices, $\gamma_{dI}(G) = \gamma_{dI}^P(G) = 2n$.
Remark 3.3. Note that if G contains no universal vertex $\gamma_{dI}(S(G,2)) > 3n-1$. The motivation for the above proof is from [1, 2].

Theorem 2.4. *The perfect double Italian domination number of the generalised Sierpiński graph $S(G,2)$ is given by*

$$\gamma_{dI}^P(S(G,2)) = \begin{cases} 3n-1 & G \in \mathcal{G} \\ 3n & \textit{otherwise} \end{cases}$$

where, $\mathcal{G}$ is the class of graphs with exactly one universal vertex and all other vertices of degree $n-2$ where $n \equiv 1(\mathrm{mod} 4)$, for $n \geqslant 3$.

Proof. Consider $S(G,2)$. Let $V = \{v_1, v_2, \ldots v_n\}$ be the vertex set of G having exactly one universal vertex v_1. Then let $\{v_1v_1, v_2v_1, \ldots v_nv_1\}$ be the corresponding vertices on each copy of G, say $G^i, i = 1,2,3,\ldots n$. in $S(G,2)$. Then G^1 contains the extreme vertex v_1v_1. Consider the function

$$f(u) = \begin{cases} 3 & v_iv_1 \in \{2,3,\ldots n\}, u = v_1v_j \text{ for some } j \neq 1 \\ 0 & \text{otherwise} \end{cases}$$

Clearly, this function is a PDIDF of $S(G,2)$. So, $\gamma_{dI}^P\ S(G,2) \leqslant 3n$. Now conversely, $\gamma_{dI}(S(G,2)) \leqslant \gamma_{dI}^P(S(G,2))$. So, by Theorem 3.1, $3n-1 \leqslant \gamma_{dI}^P(S(G,2))$. If possible let f be a γ_{dI}^P - function such that $f(S(G,2)) = 3n-1$, then G has a universal vertex by Remark 3.3. Since G has exactly one universal vertex $v_1, f\left(v_1v_1\right) = 2 = f\left(G^1\right)$ by Theorem 3.1. If $f\left(v_jv_1\right) = 3$ for some $j \neq 1$, then $f\left(N\left[v_jv_1\right]\right) = 5$. To perfect double italian dominate we must have $3 \leq f\left(N\left[v_jv_1\right]\right) \leq 4$. Hence $1 \leq f\left(v_jv_1\right) \leq 2$ and $f\left(G^j\right) = 3$, for $j \neq 1$. There arise two cases.

Case 1. For some $j, f\left(v_jv_j\right) \neq 0$ If $f\left(v_jv_1\right) = 1$, either $f\left(v_jv_j\right) = 1$ and $f\left(v_jv_k\right) = 1$ such that v_jv_k and v_jv_j are adjacent in G^j or $f\left(v_jv_j\right) = 2$. If $f\left(v_jv_1\right) = 2$, then $f\left(v_jv_j\right) = 1$. Consider G^j, since there exists only one universal vertex in the base graph which corresponds to v_jv_1 in G^j there exists at least one vertex v_jv_s perfect double italian dominated by $v_sv_j,\ s \neq 1, j, k$. ie, v_jv_s is adjacent to v_jv_j.This is true for all such S. Then $v_jv_j, j \neq 1$ is adjacent to all other vertices in G^j, a contradiction.

Case 2. $f\left(v_jv_j\right) = 0$ for all j. If $f\left(v_jv_j\right) = 0$ then either $f\left(v_jv_1\right) = 1$ and $f\left(v_jv_k\right) = 2$ or vice versa such that v_jv_k and v_jv_j are adjacent in G^j. For definiteness let $f\left(v_jv_1\right) = 1$ for any j. First let us prove the following Claim A.

Claim A: $\gamma_{dI}^P(S(G,2)) = 3n-1$ for $G \in \mathcal{G}$ where, $\mathcal{G}$ is the class of graphs with exactly one universal vertex and all other vertices of degree $n-2$ and $n \equiv 1(\mathrm{mod} 4)$, for $n \geqslant 3$.

Suppose there exists at least one vertex with degree at most $n-3$, say v_l in G. If there exists no $G^k, k \neq l$, such that $v_l v_k$ is perfect double Italian dominated by $v_k v_l$ ie, in G^l all vertices $v_l v_k$, for all k, is perfect double Italian dominated by G^l alone, a contradiction as base graph G has exactly one universalvertex. So, there exists a $G^k, k \neq l$, such that $v_l v_k$ is perfect double Italian dominated by $v_k v_l$ ie, $f\left(v_k v_l\right) = 2$ because $f\left(v_l v_1\right) = 1$ and $f\left(v_k v_1\right) = 1$ then, there exists at least two vertices $v_k v_j$ and $v_k v_i$ not perfect double Italian dominated by $v_k v_l$ in $G^k, i, j \neq k, l$ and $i \neq j \cdot$ So, $f\left(v_j v_k\right) = 2$ and $f\left(v_i v_k\right) = 2$. Again in G^j and G^i there exists at least one vertex $v_j v_s$ (correspondingly $v_i v_s$) perfect double Italian dominated by $v_s v_j$ (correspondingly $v_s v_i$).We have $f\left(v_s v_j\right) = 2$ and $f\left(v_s v_i\right) = 2$ in $G^s,\ s \neq i, j$. So, $f\left(G^s\right) > 3$, a contradiction. So, all vertices in G are of degree greater than or equal to $n-2$ and exactly one of which is a universal vertex. Now to prove $G \in \mathcal{G}$, the number of edges in G will be $\frac{(n-1)^2}{2}$, which is a positive integer and so n is odd. Due to the symmetry in G, let us assume that v_2 is not adjacent to v_3, v_4 not adjacent to v_5 and so on. We have $f\left(v_2 v_2\right) = 0$ and $f\left(v_2 v_1\right) = 1$, then we let $f\left(v_2 v_j\right) = 2$ for some $j \neq 3$, then there exists some k such that $v_2 v_k$ is perfect double Italian dominated by $v_k v_2$ (as v_k is not adjacent to v_j in base graph G) that is, $f\left(v_2 v_k\right) = 0$ if and only if $f\left(v_k v_2\right) = 2$, then $f\left(v_k v_3\right) = 0$ if and only if $f\left(v_3 v_k\right) = 2$ (as v_2 is not adjacent to v_3 in base graph G) and that forces $f\left(v_3 v_j\right) = 0$ if and only if $f\left(v_j v_3\right) = 2$ then eventually $f\left(v_j v_2\right) = 0$ if and only if $f\left(v_2 v_j\right) = 2$ which is true. This assignment perfect double Italian dominates vertices in four distinct copies G^2, G^3, G^j and G^k together with G^1. Now consider $G^s,\ s \neq 1, 2, 3, j, k$ and repeat the process above, then we will get another four distinct copies of G perfect double Italian dominated and so on. So, n must be such that $n \equiv 1 (\bmod 4)$ (See Figure 2).

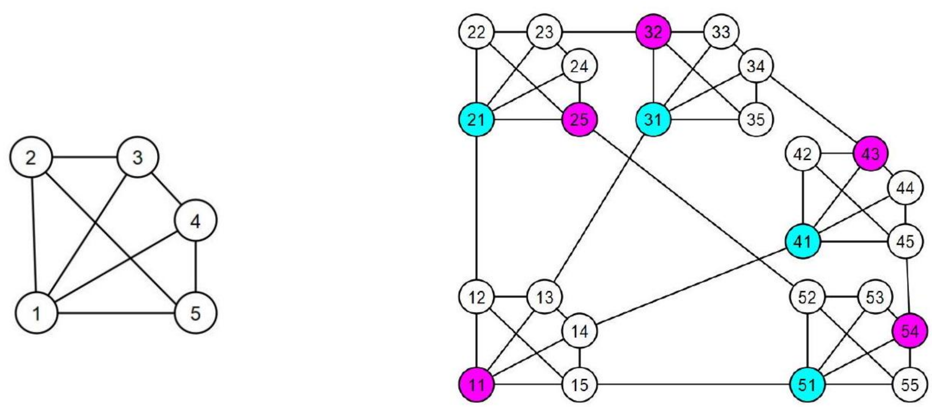

Figure 2. $S(G, t)$, when $t = 1, 2$ and $G \in \mathcal{G}$.

In Figure 2, the majenta colour vertices represent those vertices with weight 2 and cyan colour vertices represent those with weight 1 and all other vertices are of weight 0 and it represents the γ_{dI}^{P} - function given below. If $n \equiv 3 (\bmod 4)$ the above process of perfect double Italian domination cannot be done for the last remaining two distinct copies of G in $S(G, 2)$. Now consider the function,

$$f(v_iv_j) = \begin{cases} 1 & i \in \{2,3,\dots n\}, j = 1 \\ 2 & i \in \{2,3,4,6,7,8,\dots n-1\}, j = i+1 \text{ or} \\ & i \in \{5,9,13\dots n\}, j = i-3 \text{ or } i = j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Clearly, this is a PDIDF with $f(S(G,2)) = (n-1) + 2n = 3n-1$. So, $\gamma_{dI}^P(S(G,2)) \leqslant 3n-1$ for $G \in \mathcal{G}$ alone, leading to $\gamma_{dI}^P(S(G,2)) = 3n-1$ for $G \in \mathcal{G}$. This proves the Claim A. Then $\gamma_{dI}^P(S(G,2)) > 3n-1$ for $G \notin \mathcal{G}$. Hence $\gamma_{dI}^P(\ S(G,2)) \geqslant 3n$. So, $\gamma_{dI}^P\ S(G,2) = 3n$ otherwise. □

Remark 3.5. In particular, $\gamma_{dI}^P\left(S\left(K_{1,q},2\right)\right) = 3(q+1)$.

Theorem 2.5. *The perfect double Italian domination number of generalised Sierpinski graph $S(G,2)$ such that G has at least two universal vertices is $\gamma_{dI}^P(S(G,2))$ $= 3n-1$.*

Proof. Consider $S(G,2)$. Let $V = \{v_1, v_2, \dots v_n\}$ be the vertex set of G having atleast two universal vertices say, v_1 and v_2. Then let $\{v_1v_1, v_2v_1, \dots v_nv_1\}$ and $\{v_1v_2, v_2v_2, \dots v_nv_2\}$ be the corresponding vertices on each copy of G, say G^i, i $= 1,2,3,\dots$. in $S(G,2)$. Since $\gamma_{dI}(\ S(G,2)) \leqslant \gamma_{dI}^P(\ S(G,2)), 3n-1 \leqslant \gamma_{dI}^P(\ S(G,2))$. Now conversely, consider the function,

$$f(v_iv_j) = \begin{cases} 1 & i \in \{2,3,4,\dots n\}, j = 1 \\ 2 & i \in \{2,3,4,\dots n\}, j = 2, \text{ and } v_1v_1 \\ 0 & \text{otherwise} \end{cases}$$

Clearly f is a PDIDF of $S(G,2)$ such that $f(S(G,2)) = 1(n-1) + 1 + 2(n-1) + 2 = 3n-1$. So, $\gamma_{dI}^P(\ S(G,2)) \leqslant 3n-1$. Hence $\gamma_{dI}^P(\ S(G,2)) = 3n-1$. □

Conclusion

In this paper, we computed the exact value of double Italian and perfect double Italian domination numbers of generalised Sierpiński graph $S(G,2)$ and found that $\gamma_{dI}\left(\ S\left(C_n,2\right)\right) = n(n-1), \gamma_{dI}(S(G,2)) = 3n-1$ where G has a universal vertex,

$$\gamma_{dI}^P(S(G,2)) = \begin{cases} 3n-1, & G \in \mathcal{G} \\ 3n, & \text{otherwise} \end{cases}$$

where $\mathcal{G}$ is the class of graphs with exactly one universal vertex and all other vertices of degree $n-2$ and $n \equiv 1 \pmod 4$, for $n \geqslant 3$ and $\gamma_{dI}^P(\ S(G,2)) = 3n-1$ if G has at least 2 universal vertices.

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