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CERTAIN CLASSES OF BI-UNIVALENT FUNCTIONS RELATED TO PSEUDO STARLIKE AND PSEUDO CONVEX FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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Abstract. The present work deals with bounds of initial coefficients of Taylor-Maclaurin series of a unified class of Pseudo starlike and Pseudo convex functions with respect to symmetric points. Also, we discuss related subclasses of a defined class and point out special cases as corollaries.

1. Introduction

Let A denote the cass of functions of the form

$$\mathfrak{s}(\xi) = \xi + a_2\xi^2 + \cdots \tag{1.1}$$

which are analytic in

$$\Delta = \{\xi : \xi \in \mathbb{C} \text{ and } |\xi| < 1\}.$$

We denote S by the class of univalent functions in Δ. Further, we know that every univalent function has an inverse $\mathfrak{s}^{-1}$, defined by

$$\mathfrak{s}^{-1}(\mathfrak{s}(\xi)) = \xi \qquad (\xi \in \Delta)$$

and

where

$$\mathfrak{s}^{-1}(\zeta) = \zeta - a_2\zeta^2 + (2a_2^2 - a_3)\zeta^3 - (5a_2^3 - 5a_2a_3 + a_4)\zeta^4 + \cdots .$$

A function $\mathfrak{s} \in A$ is said to be bi-univalent in Δ if both $\mathfrak{s}$ and it's inverse $\mathfrak{s}^{-1}$ are univalent in Δ. Let σ denote the class of bi-univalent functions in Δ given by (1.1) (see [20]). For more details of bi-univalent functions one may refer [1, 3, 4, 5, 2, 7, 9, 11, 12, 16, 17, 18, 19, 23].

For analytic functions $\mathfrak{s}$ and $\mathfrak{j}$ in Δ, $\mathfrak{s}$ is said to be subordinate to $\mathfrak{j}$ if there exists an analytic function w such that

$$\zeta(0) = 0, \qquad |\zeta(\xi)| < 1 \quad \text{and} \quad \mathfrak{s}(\xi) = \mathfrak{j}(\zeta(\xi)) \qquad (\xi \in \Delta).$$

It is denoted by

$$\mathfrak{s} \prec \mathfrak{j} \qquad (\xi \in \Delta) \qquad \text{that is} \qquad \mathfrak{s}(\xi) \prec \mathfrak{j}(\xi) \qquad (\xi \in \Delta).$$

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In particular, when $\mathfrak{j}$ is univalent in Δ,

$$\mathfrak{s} \prec \mathfrak{j} \qquad (\xi \in \Delta) \;\Leftrightarrow\; \mathfrak{s}(0) = \mathfrak{j}(0) \quad \text{and} \quad \mathfrak{s}(\Delta) \subset \mathfrak{j}(\Delta).$$

Recently, Babalola [6] defined the class $L_\lambda(\beta)$ of λ-pseudo-starlike functions of order β as follows:

"Suppose $0 \leq \vartheta < 1$ and $\rho \geq 1$ is real. A function $\mathfrak{s} \in A$ given by (1.1) belongs to the class $L_\rho(\vartheta)$ of ρ-pseudo-starlike functions of order ϑ in the unit disk Δ if and only if

$$\Re\left(\frac{\xi\,(\mathfrak{s}'(\xi))^\rho}{\mathfrak{s}(\xi)}\right) > \vartheta \qquad (\xi \in \Delta)\,.$$

In 2013, Babalola See [6] defined the above class $L_\rho(\vartheta)$ of ρ-pseudo-starlike functions of order ϑ" [6].

Remark 1.1. (1) "If $\rho = 1$, we have the class of starlike functions of order ϑ, that is, 1-pseudo-starlike functions of order ϑ.
(2) If $\rho = 2$, we have the class $L_2(\vartheta)$ consists of functions $\mathfrak{s}$ satisfying

$$\Re\left(\mathfrak{s}'(\xi)\frac{\xi\mathfrak{s}'(\xi)}{\mathfrak{s}(\xi)}\right) > \vartheta$$

which is a product combination of geometric expressions for bounded turning and starlike functions.
(3) The class $L_\infty(\vartheta)$ is the singleton subclass of S containing the identity map only.
(4) All pseudo-starlike functions are Bazilevič of type $1 - \frac{1}{\rho}$, order $\vartheta^{\frac{1}{\rho}}$ and univalent in Δ.

We get the aforesaid information from" [6]. "Furthermore, in 2018, Magesh and Bulut [12] considered aforementioned class and discussed certain properties for functions in the class σ" [12].

"A function $\mathfrak{s} \in A$ is said to be starlike with respect to symmetric points, if it satisfies

$$\Re\left(\frac{2\xi\mathfrak{s}'(\xi)}{\mathfrak{s}(\xi) - \mathfrak{s}(-\xi)}\right) > 0, \qquad (\xi \in \Delta).$$

The class of starlike functions with respect to symmetric points is denoted by S_s^* and was introduced by Sakaguchi" [15].

"A function $\mathfrak{s} \in A$ is said to be convex with respect to symmetric points, if it satisfies

$$\Re\left(\frac{2(\xi\mathfrak{s}'(\xi))'}{(\mathfrak{s}(\xi) - \mathfrak{s}(-\xi)'}\right) > 0, \qquad (\xi \in \Delta).$$

The class of convex functions with respect to symmetric points is denoted by K_s and was introduced by Das and Singh" [8].

In 1982, Goel and Mehrol [10] used subordination technique and generalized the aforesaid classes as follow:

"A function $\mathfrak{s} \in A$ of the form (1.1) is in $S_s^*(A, B)$, $-1 \leq B < A \leq 1$, if it satisfies

$$\frac{2\xi\mathfrak{s}'(\xi)}{\mathfrak{s}(\xi) - \mathfrak{s}(-\xi)} \prec \frac{1 + A\xi}{1 + B\xi} \qquad (\xi \in \Delta).$$

A function $\mathfrak{s} \in A$ of the form (1.1) is in $C_s(A,\ B)$, $-1 \le B < A \le 1$, if it satisfies

$$\frac{2(\xi\mathfrak{s}'(\xi))'}{(\mathfrak{s}(\xi) - \mathfrak{s}(-\xi)'} \prec \frac{1+A\xi}{1+B\xi} \qquad (\xi \in \Delta).$$

One can refer" [10]. In 2004 Ravichandran [13] unifies the above said classes by using Ma-Minda method as follow:

"A function $\mathfrak{s} \in A$ is said to be starlike with respect to symmetric points of Ma - Minda type $S_s^*(\chi_1)$, if it satisfies

$$\frac{2\xi\mathfrak{s}'(\xi)}{\mathfrak{s}(\xi) - \mathfrak{s}(-\xi)} \prec \chi_1(\xi) \qquad (\xi \in \Delta).$$

A function $\mathfrak{s} \in A$ is said to be convex with respect to symmetric points of Ma-Minda type $C_s(\chi_1)$, if it satisfies

$$\frac{2(\xi\mathfrak{s}'(\xi))'}{(\mathfrak{s}(\xi) - \mathfrak{s}(-\xi)'} \prec \chi_1(\xi) \qquad (\xi \in \Delta).$$

Latter, the aforementioned classes are discussed for functions in the class of bi-univalent functions by many renowned researchers also, the works done by the references therein" [1, 4, 5, 7, 9, 16, 17].

Recently, Lashin [11] introduced the following class :

"A function $\mathfrak{s} \in \sigma$ of the form (1.1) is in $S_s^{\sigma}(\eta,\ \chi_1)$, $0 \le \eta \le 1$, if the following analytical criteria holds for $\xi,\ w \in \Delta$

$$(1-\eta)\frac{2\xi\mathfrak{s}'(\xi)}{\mathfrak{s}(\xi) - \mathfrak{s}(-\xi)} + \eta\frac{2\,(\xi\mathfrak{s}'(\xi))'}{(f\mathfrak{s}(\xi) - \mathfrak{s}(-\xi))\ '} \prec \chi_1(\xi)$$

and for $\mathfrak{j} = \mathfrak{s}^{-1}$

$$(1-\eta)\frac{2\zeta\mathfrak{j}'(\zeta)}{\mathfrak{j}(\zeta) - \mathfrak{j}(-\zeta)} + \eta\frac{2\,(\zeta\mathfrak{j}'(\zeta))'}{(\mathfrak{j}(\zeta) - \mathfrak{j}(-\zeta))\ '} \prec \chi_1(\zeta).$$

For $\chi_1(\xi) := \left(\dfrac{1+A\xi}{1+B\xi}\right)^{\gamma}$ and $\chi_1(\zeta) := \left(\dfrac{1+A\zeta}{1+B\zeta}\right)^{\gamma}$ the aforesaid class was studied by Singh [17]. For more details of the subclasses of starlike and convex functions with respect to symmetric points, bi-starlike and bi-convex functions with respect to symmetric points, we may refer and its references" [1, 5, 4, 7, 9, 11, 13, 16, 17, 18, 21, 22].

Definition 1.2. Let $\mathfrak{s} \in A$ and the functions $\chi_1, \chi_2 : \Delta \to \mathbb{C}$ be convex univalent functions such that

$$\min\{\Re(\chi_1(\xi)), \Re(\chi_2(\xi))\} > 0 \qquad (\xi \in \Delta) \quad \text{and} \quad \chi_1(o) = \chi_2(o) = 1.$$

A function $\mathfrak{s} \in \sigma$ is said to be in the class $M_{\sigma,\,s}^{\chi_1;\,\chi_2}(\eta,\ \rho)$, if for $\xi,\ \zeta \in \Delta$

$$(1-\eta)\frac{2\xi\,[\mathfrak{s}'(\xi)]^{\rho}}{\mathfrak{s}(\xi) - \mathfrak{s}(-\xi)} + \eta\frac{2\left[(\xi\mathfrak{s}'(\xi))'\right]^{\rho}}{(\mathfrak{s}(\xi) - \mathfrak{s}(-\xi))\ '} \in \chi_1\ (\ \Delta\ )$$

and for $\mathfrak{j} = \mathfrak{s}^{-1}$

$$(1-\eta)\frac{2\zeta\,[\mathfrak{j}'(\zeta)]^{\rho}}{\mathfrak{j}(\zeta) - \mathfrak{j}(-\zeta)} + \eta\frac{2\left[(\zeta\mathfrak{j}'(w))'\right]^{\rho}}{(\mathfrak{j}(\zeta) - \mathfrak{j}(-\zeta))\ '} \in \chi_2\ (\ \Delta\ )$$

hold.

Various, results for the special values of the parameters involved given as follows:

(1) When $\eta = 0$, we have $M_{\sigma,\,s}^{\chi_1;\,\chi_2}(0,\ \rho) := S_{\sigma,\,s}^{*;\,\chi_1,\,\chi_2}(\rho)$, if

$$\frac{2\xi[\mathfrak{s}'(\xi)]^{\rho}}{\mathfrak{s}(\xi)-\mathfrak{s}(-\xi)} \in \chi_1\,(\,\Delta\,)\,, \qquad \xi \in \Delta$$

and for $\mathfrak{j} = \mathfrak{s}^{-1}$

$$\frac{2\zeta[\mathfrak{j}'(\zeta)]^{\rho}}{\mathfrak{j}(\zeta)-\mathfrak{j}\,(\,-\zeta\,)} \in \chi_2\,(\,\Delta\,)\,, \qquad \zeta \in \Delta$$

hold.

(2) In particular, when $\eta = 0$ and $\rho = 1$, we have $M_{\sigma,\,s}^{\chi_1;\,\chi_2}(0,\ 1) := S_{\sigma,\,s}^{*;\,\chi_1,\,\chi_2}$, if

$$\frac{2\xi\mathfrak{s}'(\xi)}{\mathfrak{s}(\xi)-\mathfrak{s}(-\xi)} \in \chi_1\,(\,\Delta\,)\,, \qquad \xi \in \Delta$$

and for $\mathfrak{j} = \mathfrak{s}^{-1}$

$$\frac{2\zeta\mathfrak{j}'(\zeta)}{\mathfrak{j}(\zeta)-\mathfrak{j}(-\zeta)} \in \chi_2\,(\,\Delta\,)\,, \qquad \zeta \in \Delta$$

hold.

(3) When $\eta = 1$, we have $M_{\sigma,\,s}^{\chi_1;\,\chi_2}(1,\ \rho) := C_{\sigma,\,s}^{\chi_1;\,\chi_2}(\rho)$, if

$$\frac{2[(\xi\mathfrak{s}'(\xi))']^{\rho}}{[\mathfrak{s}(\xi)-\mathfrak{s}(-\xi)]'} \in \chi_1\,(\,\Delta\,)\,, \qquad \xi \in \Delta$$

and for $\mathfrak{j} = \mathfrak{s}^{-1}$

$$\frac{2[(\zeta\mathfrak{j}'(w))']^{\rho}}{[\mathfrak{j}(\zeta)-\mathfrak{j}\,(\,-\zeta\,)\,]'} \in \chi_2\,(\,\Delta\,)\,, \qquad \zeta \in \Delta$$

hold.

(4) When $\eta = \rho = 1$, we have $M_{\sigma,\,s}^{\chi_1;\,\chi_2}(1,\ 1) := C_{\sigma,\,s}^{\chi_1;\,\chi_2}$, if

$$\frac{2\,(\xi\mathfrak{s}'(\xi))'}{(\mathfrak{s}(\xi)-\mathfrak{s}(-\xi))\,'} \in \chi_1\,(\,\Delta\,)\,, \qquad \xi \in \Delta$$

and for $\mathfrak{j} = \mathfrak{s}^{-1}$

$$\frac{2\,(\zeta\mathfrak{j}'(\zeta))'}{(\mathfrak{j}(\zeta)-\mathfrak{j}(-\zeta))\,'} \in \chi_2\,(\,\Delta\,)\,, \qquad \zeta \in \Delta$$

hold.

(5) When $\rho = 1$, we have $M_{\sigma,\,s}^{\chi_1;\,\chi_2}(\eta,\ 1) := M_{\sigma,\,s}^{\chi_1;\,\chi_2}(\eta)$, if

$$(1-\eta)\frac{2\xi\mathfrak{s}'(\xi)}{\mathfrak{s}(\xi)-\mathfrak{s}(-\xi)} + \eta\frac{2\,(\xi\mathfrak{s}'(\xi))'}{(\mathfrak{s}(\xi)-\mathfrak{s}(-\xi))\,'} \in \chi_1\,(\,\Delta\,)\,, \qquad \xi \in \Delta$$

and for $\mathfrak{j} = \mathfrak{s}^{-1}$

$$(1-\eta)\frac{2\zeta\mathfrak{j}'(\zeta)}{\mathfrak{j}(\zeta)-\mathfrak{j}(-\zeta)} + \eta\frac{2\,(\zeta\mathfrak{j}'(\zeta))'}{(\mathfrak{j}(w)-\mathfrak{j}(-\zeta))\,'} \in \chi_2\,(\,\Delta\,)\,, \qquad \zeta \in \Delta$$

hold.

In order to derive our main result, we have to recall here the following lemma.

Lemma 1.3. *"Let the function $\chi_1(\xi)$ given by*

$$\chi_1(\xi) = \sum_{n\geq 1} \chi_{1n}\xi^n \qquad (\xi \in \Delta)$$

be convex univalent in Δ. Suppose also that the function $h(\xi)$ given by

$$h(\xi) = \sum_{n\geq 1} h_n\xi^n \qquad (\xi \in \Delta)$$

is holomorphic in Δ. *If*

$$h(\xi) \prec \chi_1(\xi) \qquad (\xi \in \Delta),$$

then

ℕ)”[14].

“In our present section of the estimates for the Taylor-Maclaurin coefficients $|\alpha_2|$ and $|\alpha_3|$ for functions in the above-defined general bi-univalent function class $M_{\sigma,s}^{\chi_1;\chi_2}(\eta, \rho)$, which indeed provides a bridge between the classes of bi-starlike with respect to symmetric points in Δ and bi-convex with respect to symmetric points in Δ. Several related classes are also considered and connections to earlier known results are made motivated by Lashin” [11].

The initial estimates for $\mathfrak{s} \in M_{\sigma,s}^{\chi_1;\chi_2}(\eta, \rho)$ are discussed in the following theorem.

Theorem 1.4. *Let* $\mathfrak{s}(\xi) = \xi + \sum_{n\geq 2} \alpha_n \xi^n$ *be in* $M_{\sigma,s}^{\chi_1;\chi_2}(\eta, \rho)$. *Then*

$$|\alpha_2| \leq \min\left\{\sqrt{\frac{|\chi_1'(o)| + |\chi_2'(o)|}{2(2\rho^2+\rho-1)+4\eta(3\rho^2-1)}},\ \sqrt{\frac{|\chi_1'(o)|^2 + |\chi_2'(o)|^2}{8\rho^2(1+\eta)^2}}\right\}, \tag{1.2}$$

and

$$|\alpha_3| \leq \min\left\{\frac{|\chi_1'(o)|^2 + |\chi_2'(o)|^2}{8\rho^2(1+\eta)^2} + \frac{|\chi_1'(o)| + |\chi_2'(o)|}{2(3\rho-1)(1+2\eta)}; \right.$$
$$\left. \frac{[(1+\eta)(\rho^2-1+2\rho)+\eta(\rho+1)(2\rho-1)]|\chi_1'(o)| + \rho(\rho-1)(1+3\eta)|\chi_2'(o)|}{(3\rho-1)(1+2\eta)[(2\rho^2+\rho-1)+2\eta(3\rho^2-1)]}\right\}. \tag{1.3}$$

Proof. Let $\mathfrak{s} \in M_{\sigma,s}^{\chi_1;\chi_2}(\eta, \rho)$. From Definition 1.2, we thus have

$$(1-\eta)\frac{2\xi\,[\mathfrak{s}'(\xi)]^\rho}{\mathfrak{s}(\xi)-\mathfrak{s}(-\xi)} + \eta\frac{2\left[(\xi\mathfrak{s}'(\xi))'\right]^\rho}{(\mathfrak{s}(\xi)-\mathfrak{s}(-\xi))\,'} \in \chi_1\,(\,\Delta\,) \quad \text{for all} \quad \xi \in \Delta$$

and for $\mathfrak{j} = \mathfrak{s}^{-1}$, we have

$$(1-\eta)\frac{2\zeta\,[\mathfrak{j}'(\zeta)]^\rho}{\mathfrak{j}(\zeta)-\mathfrak{j}(-\zeta)} + \eta\frac{2\left[(\zeta\mathfrak{j}'(\zeta))'\right]^\rho}{(\mathfrak{j}(\zeta)-\mathfrak{j}(-\zeta))\,'} \in \chi_2\,(\,\Delta\,) \quad \text{for all} \quad \zeta \in \Delta.$$

Setting

$$\mathfrak{p}(\xi) =(1-\eta)\frac{2\xi\,[\mathfrak{s}'(\xi)]^\rho}{\mathfrak{s}(\xi)-\mathfrak{s}(-\xi)} + \eta\frac{2\left[(\xi\mathfrak{s}'(\xi))'\right]^\rho}{(\mathfrak{s}(\xi)-\mathfrak{s}(-\xi))\,'} \tag{1.4}$$

and

$$\mathfrak{q}(\zeta) =(1-\eta)\frac{2\zeta\,[\mathfrak{j}'(\zeta)]^\rho}{\mathfrak{j}(\zeta)-\mathfrak{j}(-\zeta)} + \eta\frac{2\left[(\zeta\mathfrak{j}'(\zeta))'\right]^\rho}{(\mathfrak{j}(\zeta)-\mathfrak{j}(-\zeta))\,'}. \tag{1.5}$$

We deduce so that

$$\mathfrak{p}(o) = \chi_1(o) = 1, \qquad \mathfrak{p}(\xi) \in \chi_1\,(\,\Delta\,) \qquad (\xi \in \Delta)$$

and

$$\mathfrak{q}(o) = \chi_2(o) = 1, \qquad \mathfrak{q}(\zeta) \in \chi_2\,(\,\Delta\,) \qquad (\zeta \in \Delta).$$

Therefore, from Definition (1.2), we have

$$\mathfrak{p}(\xi) \prec \chi_1(\xi) \qquad (\xi \in \Delta)$$

and

$$\mathfrak{q}(\zeta) \prec \chi_2(\xi) \qquad (\zeta \in \Delta).$$

"According to Lemma 1.3, we obtain

and

Next, we suppose that

$$\mathfrak{p}(\xi) = 1 + \mathfrak{p}_1\xi \cdots + \mathfrak{p}_n\xi^n + \cdots$$

and

$$\mathfrak{q}(\zeta) = 1 + \mathfrak{q}_1\zeta + \cdots + \mathfrak{q}_n\zeta^n + \cdots .$$

On the other hand, we find from (1.4) and (1.5) that

$$2\rho(1+\eta)\alpha_2 = \mathfrak{p}_1, \tag{1.6}$$

$$2\rho(\rho-1)(1+3\eta)\alpha_2^2 + (3\rho-1)(1+2\eta)\alpha_3 = \mathfrak{p}_2, \tag{1.7}$$

$$-2\rho(1+\eta)\alpha_2 = \mathfrak{q}_1 \tag{1.8}$$

and

$$[2(\rho^2+2\rho-1) + 2\eta(3\rho^2+3\rho-2)]\alpha_2^2 - (3\rho-1)(1+2\eta)\alpha_3 = \mathfrak{q}_2. \tag{1.9}$$

From (1.6) and (1.8), we get

$$\mathfrak{p}_1 = -\mathfrak{q}_1 \tag{1.10}$$

and

$$8\rho^2(1+\eta)^2\alpha_2^2 = \mathfrak{p}_1^2 + \mathfrak{q}_1^2. \tag{1.11}$$

From (1.7) and (1.9), we obtain

$$[2(2\rho^2+\rho-1) + 4\eta(3\rho^2-1)]\alpha_2^2 = \mathfrak{p}_2 + \mathfrak{q}_2. \tag{1.12}$$

Therefore, we find from (1.11) and (1.12) that

$$\alpha_2^2 = \frac{\mathfrak{p}_1^2 + \mathfrak{q}_1^2}{8\rho^2(1+\eta)^2} \tag{1.13}$$

and

$$\alpha_2^2 = \frac{\mathfrak{p}_2 + \mathfrak{q}_2}{2(2\rho^2+\rho-1) + 4\eta(3\rho^2-1)}. \tag{1.14}$$

From (1.13) and (1.14) we have

and

respectively. So, we get the desired estimates on $|\alpha_2|$ as asserted in ((1.2)).

Next, in order to find the bound on $|\alpha_3|$, by subtracting (1.9) from (1.7), we get

$$2(3\rho-1)(1+2\eta)\alpha_3 - 2(3\rho-1)(1+2\eta)\alpha_2^2 = \mathfrak{p}_2 - \mathfrak{q}_2. \tag{1.15}$$

Upon substituting the values of α_2^2 from (1.13) and (1.14) into (1.15), we have

$$\alpha_3 = \frac{\mathfrak{p}_2 - \mathfrak{q}_2}{2(3\rho-1)(1+2\eta)} + \frac{\mathfrak{p}_1^2 + \mathfrak{q}_1^2}{8\rho^2(1+\eta)^2}$$

and

$$\begin{aligned}\alpha_3 &= \frac{\mathfrak{p}_2 - \mathfrak{q}_2}{2(3\rho-1)(1+2\eta)} + \frac{\mathfrak{p}_2 + \mathfrak{q}_2}{2(2\rho^2+\rho-1) + 4\eta(3\rho^2-1)} \\ &= \frac{[(1+\eta)(\rho^2-1+2\rho) + \eta(\rho-1)]\mathfrak{p}_2 + \rho(\rho-1)(1+3\eta)\mathfrak{q}_2}{(3\rho-1)(1+2\eta)[(2\rho^2+\rho-1) + 2\eta(3\rho^2-1)]}\end{aligned}$$

respectively. We thus find that

and

$$\frac{[(1+\eta)(\rho^2-1+2\rho)+\eta(\rho+1)(2\rho-1)]|{\chi_1}'(o)|+\rho(\rho-1)(1+3\eta)|{\chi_2}'(o)|}{(3\rho-1)(1+2\eta)[(2\rho^2+\rho-1)+2\eta(3\rho^2-1)]}.$$

This completes the proof of Theorem 1.4. □

Now we state the following corollaries as a special cases of the aforementioned Theorem 1.4.

Corollary 1.5. *Let* $\mathfrak{s}(\xi)=\xi+\sum\limits_{n\geq 2}\alpha_n\xi^n$ *be in* $S^{*,\,\chi_1,\,\chi_2}_{\sigma,\,s}(\rho)$*. Then*

$$|\alpha_2|\leq \min\left\{\sqrt{\frac{|{\chi_1}'(o)|^2+|{\chi_2}'(o)|^2}{8\rho^2}},\ \sqrt{\frac{|{\chi_1}'(o)|+|{\chi_2}'(o)|}{2(2\rho^2+\rho-1)}}\right\},$$

and

$$|\alpha_3|\leq \min\left\{\frac{|{\chi_1}'(o)|+|{\chi_2}'(o)|}{2(3\rho-1)}+\frac{|{\chi_1}'(o)|^2+|{\chi_2}'(o)|^2}{8\rho^2};\right.$$
$$\left.\frac{[(\rho^2-1+2\rho)]|{\chi_1}'(o)|+\rho(\rho-1)|{\chi_2}'(o)|}{(3\rho-1)(2\rho^2+\rho-1)}\right\}.$$

Corollary 1.6. *Let* $\mathfrak{s}(\xi)=\xi+\alpha_2\xi^2+\cdots$ *be in* $S^{*,\,\chi_1,\,\chi_2}_{\sigma,\,s}$*. Then*

$$|\alpha_2|\leq \min\left\{\sqrt{\frac{|{\chi_1}'(o)|^2+|{\chi_2}'(o)|^2}{8}},\ \sqrt{\frac{|{\chi_1}'(o)|+|{\chi_2}'(o)|}{4}}\right\},$$

and

$$|\alpha_3|\leq \min\left\{\frac{|{\chi_1}'(o)|^2+|{\chi_2}'(o)|^2}{8}+\frac{|{\chi_1}'(o)|+|{\chi_2}'(o)|}{4};\ \frac{|{\chi_1}'(o)|}{2}\right\}.$$

Corollary 1.7. *Let* $\mathfrak{s}(\xi)=\xi+\sum\limits_{n\geq 2}\alpha_n\xi^n$ *be in* $C^{\chi_1;\,\chi_2}_{\sigma,\,s}(\rho)$*. Then*

$$|\alpha_2|\leq \min\left\{\sqrt{\frac{|{\chi_1}'(o)|^2+|{\chi_2}'(o)|^2}{32\rho^2}},\ \sqrt{\frac{|{\chi_1}'(o)|+|{\chi_2}'(o)|}{16\rho^2+2\rho-6}}\right\},$$

and

$$|\alpha_3|\leq \min\left\{\frac{|{\chi_1}'(o)|+|{\chi_2}'(o)|}{6(3\rho-1)}+\frac{|{\chi_1}'(o)|^2+|{\chi_2}'(o)|^2}{32\rho^2};\right.$$
$$\left.\frac{[4\rho^2+5\rho-3]|{\chi_1}'(o)|+4\rho(\rho-1)|{\chi_2}'(o)|}{3(3\rho-1)(8\rho^2+\rho-3)}\right\}.$$

Corollary 1.8. *Let* $\mathfrak{s}(\xi)=\xi+\sum_{n\geq 2}\alpha_n\xi^n$ *be in* $C^{\chi_1;\,\chi_2}_{\sigma,\,s}$*. Then*

$$|\alpha_2|\leq \min\left\{\sqrt{\frac{|{\chi_1}'(o)|^2+|{\chi_2}'(o)|^2}{32}},\ \sqrt{\frac{|{\chi_1}'(o)|+|{\chi_2}'(o)|}{12}}\right\},$$

and

$$|\alpha_3|\leq \min\left\{\frac{|{\chi_1}'(o)|^2+|{\chi_2}'(o)|^2}{32}+\frac{|{\chi_1}'(o)|+|{\chi_2}'(o)|}{12};\ \frac{|{\chi_1}'(o)|}{6}\right\}.$$

Corollary 1.9. *Let* $\mathfrak{s}(\xi)=\xi+\sum\limits_{n\geq 2}\alpha_n\xi^n$ *be in* $M^{\chi_1;\,\chi_2}_{\sigma,\,s}(\eta)$*. Then*

$$|\alpha_2|\leq \min\left\{\sqrt{\frac{|\chi_1{}'(o)|^2+|\chi_2{}'(o)|^2}{8(1+\eta)^2}},\ \sqrt{\frac{|\chi_1{}'(o)|+|\chi_2{}'(o)|}{4(1+2\eta)}}\right\},$$

and

$$|\alpha_3|\leq \min\left\{\frac{|\chi_1{}'(o)|+|\chi_2{}'(o)|}{4(1+2\eta)}+\frac{|\chi_1{}'(o)|^2+|\chi_2{}'(o)|^2}{8(1+\eta)^2};\ \frac{|\chi_1{}'(o)|}{2(1+2\eta)}\right\}.$$

We note that, for the different choices of the functions χ_1 and χ_2, we get interesting known or new subclasses of the analytic function class σ. For example, if we set

$$\chi_1(\xi)=\left(\frac{1+\xi}{1-\xi}\right)^{\vartheta}\qquad\text{and}\qquad\chi_2(\zeta)=\left(\frac{1-\zeta}{1+\zeta}\right)^{\vartheta}\qquad(0<$$

Corollary 1.10. *Let* $\mathfrak{s}(\xi)=\xi+\sum\limits_{n\geq 2}\alpha_n\xi^n$ *be in* $SS^{*}_{\sigma,\,s}(\rho,\,\vartheta)$*. Then*

$$|\alpha_2|\leq \min\left\{\frac{\vartheta}{\rho},\ \sqrt{\frac{\vartheta}{2\rho^2+\rho-1}}\right\}\quad \textit{and}\quad |\alpha_3|\leq \min\left\{\frac{2\vartheta}{3\rho-1}+\frac{\vartheta^2}{\rho^2};\ \frac{2\vartheta}{(3\rho-1)}\right\}.$$

Corollary 1.11. *Let* $\mathfrak{s}(\xi)=\xi+\sum\limits_{n\geq 2}\alpha_n\xi^n$ *be in* $SS^{*}_{\sigma,\,s}(\vartheta)$*. Then*

$$|\alpha_2|\leq \min\left\{\vartheta,\ \sqrt{\frac{\vartheta}{2}}\right\},\qquad \textit{and}\qquad |\alpha_3|\leq \min\left\{\vartheta+\vartheta^2;\ \vartheta\right\}.$$

Corollary 1.12. *Let* $\mathfrak{s}(\xi)=\xi+\sum\limits_{n\geq 2}\alpha_n\xi^n$ *be in* $SC^{\chi_1;\,\chi_2}_{\sigma,\,s}(\rho,\,\vartheta)$*. Then*

$$|\alpha_2|\leq \min\left\{\frac{\vartheta}{2\rho},\ \sqrt{\frac{2\vartheta}{8\rho^2+\rho-3}}\right\}$$

and

$$|\alpha_3|\leq \min\left\{\frac{2\vartheta}{3(3\rho-1)}+\frac{\vartheta^2}{4\rho^2};\ \frac{2\vartheta}{3(3\rho-1)}\right\}.$$

Corollary 1.13. *Let* $\mathfrak{s}(\xi)=\xi+\sum\limits_{n\geq 2}\alpha_n\xi^n$ *be in* $SC^{\chi_1;\,\chi_2}_{\sigma,\,s}(\vartheta)$*. Then*

$$|\alpha_2|\leq \min\left\{\frac{\vartheta}{2},\ \sqrt{\frac{\vartheta}{3}}\right\}\qquad \textit{and}\qquad |\alpha_3|\leq \min\left\{\frac{\vartheta}{3}+\frac{\vartheta^2}{4};\ \frac{\vartheta}{3}\right\}.$$

Corollary 1.14. *Let* $\mathfrak{s}(\xi)=\xi+\sum\limits_{n\geq 2}\alpha_n\xi^n$ *be in* $SM^{\chi_1;\,\chi_2}_{\sigma,\,s}(\eta,\,\rho,\,\vartheta)$*. Then*

$$|\alpha_2|\leq \min\left\{\frac{\vartheta}{\rho(1+\eta)},\ \sqrt{\frac{2\vartheta}{2\eta(3\rho^2-1)+2\rho^2+\rho-1}}\right\}$$

and

$$|\alpha_3|\leq \min\left\{\frac{2\vartheta}{(3\rho-1)(1+2\eta)}+\frac{\vartheta^2}{\rho^2(1+\eta)^2};\ \frac{2\vartheta}{(3\rho-1)(1+2\eta)}\right\}.$$

Corollary 1.15. *Let* $\mathfrak{s}(\xi) = \xi + \sum_{n\geq 2} \alpha_n \xi^n$ *be in* $SM^{\chi_1;\,\chi_2}_{\sigma,\,s}(\eta,\,\vartheta)$*. Then*

$$|\alpha_2| \leq \min\left\{\frac{\vartheta}{1+\eta},\, \sqrt{\frac{\vartheta}{1+2\eta}}\right\} \qquad \text{and} \qquad |\alpha_3| \leq \min\left\{\frac{\vartheta}{1+2\eta} + \frac{\vartheta^2}{(1+\eta)^2};\; \frac{\vartheta}{1+2\eta}\right\}.$$

Similarly, if we let

$$\chi_1(\xi) = \frac{1+(1-2\vartheta)\xi}{1-\xi} \qquad \text{and} \qquad \xi,\,\zeta \in \Delta),$$

Corollary 1.16. *Let* $\mathfrak{s}(\xi) = \xi + \sum_{n\geq 2} \alpha_n \xi^n$ *be in* $S^*_{\sigma,\,s}(\rho,\,\vartheta)$*. Then*

$$|\alpha_2| \leq \min\left\{\frac{1-\vartheta}{\rho},\, \sqrt{\frac{1-\vartheta}{2\rho^2+\rho-1}}\right\} \quad \text{and} \quad |\alpha_3| \leq \min\left\{\frac{2(1-\vartheta)}{3\rho-1} + \frac{(1-\vartheta)^2}{\rho^2};\; \frac{2(1-\vartheta)}{(3\rho-1)}\right\}.$$

Corollary 1.17. *Let* $\mathfrak{s}(\xi) = \xi + \sum_{n\geq 2} \alpha_n \xi^n$ *be in* $S^*_{\sigma,\,s}(\vartheta)$*. Then*

$$|\alpha_2| \leq \min\left\{1-\vartheta,\, \sqrt{\frac{1-\vartheta}{2}}\right\} \qquad \text{and} \qquad |\alpha_3| \leq \min\left\{1-\vartheta+(1-\vartheta)^2;\; 1-\vartheta\right\}.$$

Corollary 1.18. *Let* $\mathfrak{s}(\xi) = \xi + \sum_{n\geq 2} \alpha_n \xi^n$ *be in* $C^{\chi_1;\,\chi_2}_{\sigma,\,s}(\rho,\,\vartheta)$*. Then*

$$|\alpha_2| \leq \min\left\{\frac{1-\vartheta}{2\rho},\, \sqrt{\frac{2(1-\vartheta)}{8\rho^2+\rho-3}}\right\} \qquad \text{and} \qquad |\alpha_3| \leq \min\left\{\frac{2(1-\vartheta)}{3(3\rho-1)} + \frac{(1-\vartheta)^2}{4\rho^2};\; \frac{2(1-\vartheta)}{3(3\rho-1)}\right\}.$$

Corollary 1.19. *Let* $\mathfrak{s}(\xi) = \xi + \sum_{n\geq 2} \alpha_n \xi^n$ *be in* $C^{\chi_1;\,\chi_2}_{\sigma,\,s}(\vartheta)$*. Then*

$$|\alpha_2| \leq \min\left\{\frac{1-\vartheta}{2},\, \sqrt{\frac{1-\vartheta}{3}}\right\} \qquad \text{and} \qquad |\alpha_3| \leq \min\left\{\frac{1-\vartheta}{3} + \frac{(1-\vartheta)^2}{4};\; \frac{1-\vartheta}{3}\right\}.$$

Corollary 1.20. *Let* $\mathfrak{s}(\xi) = \xi + \sum_{n\geq 2} \alpha_n \xi^n$ *be in* $M^{\chi_1;\,\chi_2}_{\sigma,\,s}(\eta,\,\rho,\,\vartheta)$*. Then*

$$|\alpha_2| \leq \min\left\{\frac{1-\vartheta}{\rho(1+\eta)},\, \sqrt{\frac{2(1-\vartheta)}{2\eta(3\rho^2-1)+2\rho^2+\rho-1}}\right\}$$

and

$$|\alpha_3| \leq \min\left\{\frac{2(1-\vartheta)}{(3\rho-1)(1+2\eta)} + \frac{(1-\vartheta)^2}{\rho^2(1+\eta)^2};\; \frac{2(1-\vartheta)}{(3\rho-1)(1+2\eta)}\right\}.$$

Corollary 1.21. *Let* $\mathfrak{s}(\xi) = \xi + \sum_{n\geq 2} \alpha_n \xi^n$ *be in* $M^{\chi_1;\,\chi_2}_{\sigma,\,s}(\eta,\,\vartheta)$*. Then*

$$|\alpha_2| \leq \min\left\{\frac{1-\vartheta}{1+\eta},\, \sqrt{\frac{1-\vartheta}{1+2\eta}}\right\} \qquad \text{and} \qquad |\alpha_3| \leq \min\left\{\frac{1-\vartheta}{1+2\eta} + \frac{(1-\vartheta)^2}{(1+\eta)^2};\; \frac{1-\vartheta}{1+2\eta}\right\}.$$

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