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LYAPUNOV FUNCTION AND STABILITY OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS WITH FRACTIONAL-LIKE DERIVATIVES

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Abstract. The paper is devoted to the analysis of questions related to the stability of fractional-like stochastic differential equations. Stochastic stability and asymptotically stochastic stability is considered with the use of Lyapunov function. Almost sure exponential stability is established on the Ito formula of the fractional-like derivatives.

1. Introduction

Over the past decades, various variants of fractional derivatives have been widely used in the study of memory properties for complex systems in different areas (see, for example, [1,2]). In [3,4] a new concept, a fractional-like derivative, was introduced and in [5,6] systems of differential equations with these derivatives were considered. In [7] new results are presented for neural networks with fractional discrete time. On the other hand, in recent years, the theory of stability of fractional stochastic differential equations and its applications has been developed. And this is not surprising, since stochasticity is the most important property of the real world, and stability is the highest priority for applied complex systems.

Analysis of the stability of stochastic systems becomes necessary both in theoretical and applied aspects. The mathematical theory of stability of solutions of stochastic differential equations consists of two main directions. These are the direct (second) Lyapunov method [8-10] and Burton's fixed point method [11-13]. The questions of existence, uniqueness and stability of solutions of stochastic partial differential equations were the subject of analysis in [14-16]. New results were also obtained for stochastic integro-differential equations [17-21].

This paper is devoted to the development of Lyapunov-type functions for stochastic differential equations with fractional-like derivatives of the form

$$\mathfrak{D}^{\alpha}_{t_0} X(t) = b(t, X(t)) + \sigma(t, X(t)) \frac{dW(t)}{dt}, t > 0, 0 < \alpha \leq 1 \tag{1.1}$$

$$X(0) = X_0 \tag{1.2}$$

where $\mathfrak{D}^{\alpha}_{t_0}$ is a fractional-like derivative $b, \sigma : [0, +\infty) \times R \to R$ are measurable functions, and $\{W(t), t \in [0, +\infty)\}$ are scalar Brownian motion defined in the complete probability space $(\Omega, \mathfrak{F}, F = \{\mathfrak{F}_t\}_{t \geq 0}, \mathbf{P})$ such that $W(0) = 0, E\{W(t)\} = 0$, $E((W(t) - EW(t))(W(s) - EW(s))) = t - s$.

For each $t \in [0, +\infty)\}$ we denote $\mathfrak{L}_t = \mathbb{L}^2(\Omega, \mathfrak{F}, P)$ as the space of all $\mathfrak{F}_t$ measurable, square-integrable functions $u : \Omega \to \mathbb{R}$ such that $\|u\|^2 = E\{|u|^2\}$.

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The limit $X : [0, +\infty) \to \mathfrak{L}_t$ is said to be $\mathfrak{F}$- adapted, if $X(t) \in \mathfrak{L}_t, t \in [0, +\infty)$.

In [22], in the case of the Caputo fractional derivative, the existence and uniqueness theorem for equation (1.1) was established by the method of contraction mappings. In this case, the functions b and σ are required to satisfy the Lipschitz conditions.

The article is structured as follows. Section 2 presents the definition of a fractional-like derivative and studies the properties of this derivative. Section 3 is devoted to the concept of a fractional-like derivative of Lyapunov-type functions. It is shown that for some simple Lyapunov functions the fractional-like derivative is majorant for the Caputo fractional derivative of these functions. Section 4 contains a fractional-like version of Ito's formula. In Section 5, sufficient conditions for stochastic stability (or stability in probability), asymptotic stochastic stability, and exponential stability are indicated. Finally, section 6 provides concluding remarks.

2. Fractional-like derivatives

Let $\alpha \in (0, 1], R_+ = [0, \infty), t_0 \in R_+$ and $f(t) : [t_0, \infty) \to R$ is a given continuous function.

Definition 2.1. ([3]) For any $\alpha \in (0, 1]$ fractional-like derivative $\mathfrak{D}^{\alpha}_{t_0}(f(t))$ of order $0 < \alpha \leq 1$ of the function $f(t)$ is defined by the equality

$$\mathfrak{D}^{\alpha}_{t_0} f(t) = lim\{\frac{f(t + \theta(t - t_0)^{1-\alpha}) - f(t)}{\theta}, \theta \to 0\}.$$

If $t_0 = 0$, then $\mathfrak{D}^{\alpha}_{t_0}(f(t))$ will take the form

$$\mathfrak{D}^{\alpha}_{t_0}(f(t)) = lim\{\frac{f(t + \theta t^{1-\alpha}) - f(t)}{\theta}, \theta \to 0\}.$$

For the case $t_0 = 0$ we use the notation

$$\mathfrak{D}^{\alpha}_{0}(f(t)) = \mathfrak{D}^{\alpha}(f(t)).$$

If $\mathfrak{D}^{\alpha}$ exists in $(0, b)$ then

$$\mathfrak{D}^{\alpha}(f(0)) = \lim_{t\to 0} \mathfrak{D}^{\alpha}(f(t)).$$

If a fractional-like derivative of a function $f(t)$ of order α exists and is finite on (t_0, ∞), then we say that $f(t)$ is differentiable on (t_0, ∞).

Remark 2.2. Definition 2.1 does not satisfy all the conditions that are true for the Riemann-Liouville, Caputo, and other derivatives (see for ex. [5] and the bibliography there).

The following statement holds.

Lemma 2.3. *(see.[5]) Let $\alpha \in (0, 1], f(t), g(t)$ are α - differentiable functions at the point $t > 0$ Then the equalities hold:*
1) $\mathfrak{D}^{\alpha}_{t_0}(af(t) + bg(t)) = a \cdot \mathfrak{D}^{\alpha}_{t_0}(f(t)) + b \cdot \mathfrak{D}^{\alpha}_{t_0}(g(t))$ with all $a, b \in \mathbb{R}$;
2) $\mathfrak{D}^{\alpha}_{t_0}(t^p) = p(t - t_0)^{1-\alpha} t^{p-1}$ with all $p \in \mathbb{R}$;
3) $\mathfrak{D}^{\alpha}_{t_0}(f(t)g(t)) = f(t)\mathfrak{D}^{\alpha}_{t_0}(g(t)) + g(t)\mathfrak{D}^{\alpha}_{t_0}(f(t))$;
4) $\mathfrak{D}^{\alpha}_{t_0}(\frac{f(t)}{g(t)}) = \frac{g(t)\mathfrak{D}^{\alpha}_{t_0}(f(t)) - f(t)\mathfrak{D}^{\alpha}_{t_0}(g(t))}{g^2(t)}$;
5) $\mathfrak{D}^{\alpha}_{t_0}(f(t)) = 0$ for any function $f(t) = c$, where c is an arbitrary constant.

Remark 2.4. Equalities 1) -5) from Lemma 2.3 are similar to the classical results of mathematical analysis for integer orders of derivatives. These statements do not hold for the Riemann-Liouville and other fractional derivatives (See [5]). Part 5) holds for the fractional Caputo derivative.

Lemma 2.5. *(See. [5]). Let $0 < \alpha \leq 1$ and function $h(t) = m(g(t))$ is differentiable with respect to $g(t)$ for all $t \in \mathbb{R}_+$ and function $g(t) - \alpha$ is differentiable for $t \neq t_0$ and $g(t) \neq 0$, then*

$$\mathfrak{D}^{\alpha}_{t_0} g(t) = m'(g(t))\mathfrak{D}^{\alpha}_{t_0}(g(t)).$$

A fractional-like integral of order $0 < \alpha \leq 1$ is introduced using the formula

$$I^{\alpha}_{t_0} f(t) = \int\limits_{t_0}^{t} (s - t_0)^{\alpha - 1} f(s) ds, t > t_0.$$

Lemma 2.6. *(See. [5]). Let $f(t) : (t_0, \infty) \to R - \alpha$ be differentiable for $0 < q \leq 1$.Then for all $t > t_0$ the following ratio is true:*

$$I^{\alpha}_{t_0}(\mathfrak{D}^{\alpha}_{t_0} f(t)) = f(t) - f(t_0).$$

3. Lyapunov function and its fractional-like derivative

An important method for studying the stability of various classes of deterministic and stochastic systems is the second Lyapunov method (see [8]). As a research tool, the second (direct) method uses some special functions called Lapunov functions. The real continuously differentiable function $V : T + B_r \to R$, which satisfies the condition $V(t, 0) = 0$, is said to be a Lyapunov function.Here B_r denotes the ball of radius r centered at the origin in the Euclidean space $\mathbb{R}^n$ with the norm $|x| = (\sum\limits_{i=1}^{n} x_i^2)^{1/2}$, and T denotes the interval of real number line $T = \{a < t < \infty\}$, where a is $-\infty$ or some finite number. We call the derivative $\dot{V}$ of the function $V(t, x)$ by virtue of the equations

$$\dot{x}(t) = b(t, x(t)), b, x \in \mathbb{R}^n \tag{3.1}$$

$$x(t_0) = x_0, b(t, 0) \equiv 0 \tag{3.2}$$

the value

$$\dot{V} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} b_i(t, x) = \frac{\partial V}{\partial t} + (\nabla V, b(t, x)). \tag{3.3}$$

If $x = x(t)$ is the solution of (3.1), then $\dot{V}$ represents the total derivative of complex function $V(t, x(t))$ with respect to time. It should be noted that for calculating $\dot{V}$ there is no need to find the actual solution $x(t)$.

Further, by K we denote the class of functions $\omega_i(u), u \geq 0, i = 0, 1, 2, ...$ - scalar continuous non-decreasing functions such that $\omega_i(0) = 0$ and $\omega_i(u) > 0$ for $u > 0$.

The essence of the classical Lyapunov method lies in the validity of the following three theorems.

Theorem 3.1. *([8]) Let there exist a function $V(t, x)$ such that*

$$\omega_1(|x|) \leq V(t, x), \dot{V}(t, x) \leq 0.$$

Then the trivial solution is Lyapunov stable.

Theorem 3.2. *([8]). If $\omega_1(|x|) \leq V(t,x) \leq \omega_2(|x|), \dot{V}(t,x) \leq -\omega_3(|x|)$, then the trivial solution is asymptotically stable in the sense of Lyapunov.*

Theorem 3.3. *(Chetaev's theorem, [8]). If in the domain $V(t,x) > 0$ the inequality $\dot{V}(t,x) \geq \omega_4(|x|)$ holds, then the trivial solution is unstable.*

The purpose of this paper is to extend the above statements to the case of problem (1.1) - (1.2).

In this regard, note that instead of the usual derivative in Theorems 3.1-3.3, we have to use the following Dini derivative for the Lyapunov function

$$D^+V(t,x) = \lim_{h\to 0+} \frac{1}{h}[V(t+h,x)b(t,x) - V(t,x)]. \tag{3.4}$$

In [6], the concept of a fractional-like derivative introduced the same way for an equation of the form

$$\mathfrak{D}^{\alpha}_{t_0}(x(t)) = b(t, x(t)), \tag{3.5}$$

$$x(t_0) = x_0, \tag{3.6}$$

where $x \in \mathbb{R}^n, b \in C(R_+ \times \mathbb{R}^n, \mathbb{R}^n), t_0 \geq 0$.

Definition 3.4. Let V be continuous and α differentiable function, $V : R_+ \times B_r \to \mathbb{R}^m$ $x(t,t_0,x_0)$ is the solution of problem (3.5)-(3.6).

Then for $(t,x) \in R_+ \times B_r$ the expression

$$^+\mathfrak{D}^{\alpha}_{t_0}V(t,x) = limsup\{\frac{V(t+\theta(t-t_0)^{1-\alpha}, x(t+\theta(t-t_0)^{1-\alpha},t,x)) - V(t,x)}{\theta}, \theta \to 0+\} \tag{3.7}$$

is the upper right fractional-like derivative of the Lyapunov function.

The lower-right, upper-left, and lower-left fractional-like derivatives of the Lyapunov function are determined accordingly.

Lemma 3.5. *Let $V(t,x)$ be continuous, α - differentiable and locally lipschizable function with respect to the second variable x on $R_+ \times B_r$. Then the fractional-like derivative of the function $V(t,x)$ with respect to the solution $x(t,t_0,x_0)$ is defined as*

$$^+\mathfrak{D}^{\alpha}_{t_0}V(t,x) =$$

$$limsup\{\frac{V(t+\theta(t-t_0)^{1-\alpha}, x+\theta(t-t_0)^{1-\alpha}, b(t,x)) - V(t,x)}{\theta}, \theta \to 0+\} \tag{3.8}$$

where $(t,x) \in R_+ \times B_r$.

If $V(t,x(t)) = V(x(t)), 0 < \alpha \leq 1$, function V differentiable with respect to x, and function $x(t)$ α - diferentiable with respect to t for $t > t_0$, then

$$^+\mathfrak{D}^{\alpha}_{t_0}V(t,x) = V'(x(t))V(t,x)|_{(t_0)}.$$

In [6], examples of Lyapunov functions $V_1(x) = x^2(t), x \in R, V_2(x) = x^T x, x \in \mathbb{R}^n$ $V_3(x) = x^T Px, x \in \mathbb{R}^n$ are given and their fractional-like derivatives are calculated, where P is $(n \times n)$ matrix.

The following statement was established.

Lemma 3.6. *Let $x \in R, y \in \mathbb{R}^n$ and P is a constant matrix of order $n \times n$. Then for functions $V_1 = x^2(t), V_2 = y^T(t)y(t) \quad V_3 = y^T(t)Py(t)$ the following estimates hold:*
1) ${}_{t_0}^{c}D_t^\alpha(x^2(t)) \leq^+ \mathfrak{D}_{t_0}^\alpha(x^2(t))$ for $x \in \mathbb{R}$;
2) ${}_{t_0}^{c}D_t^\alpha(y^T(t)y(t)) \leq^+ \mathfrak{D}_{t_0}^\alpha(y^T(t)y(t))$ for $y \in \mathbb{R}^\ltimes$;
3) ${}_{t_0}^{c}D_t^\alpha(y^T(t)Py(t)) \leq^+ \mathfrak{D}_{t_0}^\alpha(y^T(t)Py(t))$ for $y \in \mathbb{R}^\ltimes$;

It follows from Lemma 3.6 that the fractional-like derivative of the Lyapunov function is an upper bound (majorant) for the fractional Caputo derivative of the same Lyapunov functions.

4. Ito's formula for functions with fractional-like derivative

First, we give the definition of the solution to the problem (1,1), (1,2).

Definition 4.1. For every $X_{t_0} \in \mathfrak{L}_0$ $\mathfrak{F}$ - adapted random process X is said to be the solution of problem (1.1), (1.2), if the following equality holds for $t_0 \in [0, \infty)$:

$$X(t) = X(t, t_0, X_{t_0}) =$$
$$= X_{t_0} + \int_{t_0}^{t} (s - t_0)^{\alpha-1} b(s, X(s))ds + \int_{t_0}^{t} (s - t_0)^{\alpha-1} \sigma(s, X(s))dW(s). \tag{4.1}$$

We make the following assumptions:
(A1) There is a constant $L > 0$ such that for all

$$X, \tilde{X} \in \mathbb{R}, t \in [0, +\infty)$$

$$|b(t, X) - b(t, \tilde{X})| + |\sigma(t, X) - \sigma(t, \tilde{X})| \leq L|X - \tilde{X}|;$$

(A2) The $\sigma(\cdot, 0)$ function is essentially bounded, i.e.

$$\|\sigma(t, 0)\|_\infty = ess \sup_{t \in [0, +\infty)} |\sigma(t, 0)| < +\infty,$$

and $\sigma(\cdot, 0)$ L_2 integrable, i.e.

$$\int_0^{+\infty} |\sigma(t, 0)|^2 dt < +\infty.$$

Lemma 4.2. *Suppose that (A1) and (A2) are satisfied. Then for $\alpha \in (0, 1)$ problem (1), (2) has the unique solution $X \in \mathfrak{L}_t := \mathbb{L}^2(\Omega, \mathfrak{F}_t, P)$ given in the form (4.1).*

Next, we present a new version of Ito's formula for functions with fractional-like derivatives. This formula defines the rule for differentiating functions of stochastic processes with fractional-like derivatives. Let $W(t), t \geq 0$ be a standard scalar Brownian motion (see introduction) and let $Y \in C^{1,2}(R_t \times R, \mathbb{R})$ denote the family of all real-valued functions $Y(\cdot, Z(\cdot))$ defined and continuously differentiable with respect to $Z \quad R_t \times R$.

Let $Z(t), t \geq t_0$ be Ito process for

$$dz(t) = \tilde{b}(t) + \tilde{\sigma}(t)dW(t),$$

where $\tilde{b} \in \mathbb{L}^1(R_+, R) \quad \tilde{\sigma} \in \mathbb{L}^2(R_+, R)$.

Let us recall the standard one-dimensional Ito formula.

Lemma 4.3. *Let $Y(\cdot) = Y(\cdot, Z(\cdot)) \in C^{1,2}(R_+ \times R, R)$. Then $Y(t), t \geq 0$ is an Ito process given by the equality*

$$dY(t) = [Y_t(t, Z(t)) + Y_Z(t, Z(t))\tilde{b}(t) + \frac{1}{2}Y_{ZZ}(t, Z(t))]dt+$$

$$+Y_Z(t, Z(t))\bar{\sigma}(t)dW(t) \text{ almost surely (a.s.)}$$

Let now $T > 0$. Suppose that $\tilde{X}(t)$ is an Ito process for the equation

$$D^{\alpha}_{t_0}\tilde{X} = b(t) + \sigma(t)\frac{dW(t)}{dt}, t_0 \in [0, T], 0 < \alpha < 1 \tag{4.2}$$

with initial conditions

$$\tilde{X}(t_0) = X_{t_0}. \tag{4.3}$$

Lemma 4.2 and (4.2), (4.3) imply that there exists a unique solution for $t_0 \in [0, T]$ of the form

$$\tilde{X}(t) = X_{t_0} + \int_{t_0}^{t}(s - t_0)^{\alpha-1}b(s)ds + \int_{t_0}^{t}(s - t_0)^{\alpha-1}\sigma(s)dW(s).$$

Note that when $t_0 \in [0, T]$, (4.2) is equivalent to the equation

$$d\tilde{X}(t) = \tilde{X}'(t)dt =$$

$$= (\alpha - 1)[\int_{t_0}^{t}(s - t_0)^{\alpha-1}b(s)ds + \int_{t_0}^{t}(s - t_0)^{\alpha-2}\sigma(s)dW(s)], \tag{4.4}$$

where $(\cdot - t_0)^{\alpha-2}b(\cdot) \in \mathbb{L}^1[0, T]$ and $(\cdot - t_0)^{\alpha-2}\sigma(\cdot) \in \mathbb{L}^2[0, T]$.

We are now ready to present a fractional-like version of Ito's formula.

Theorem 4.4. *Let $Y(\cdot) = Y(\cdot, \tilde{X}(\cdot)) \in C^{1,2}(R_+ \times R, R)$. Then $Y(\cdot)$ is an Ito process given in the form of the following formula*

$$dY(t, \tilde{X}(t)) = Y_t(t, \tilde{X}(t))dt+$$

$$+(\alpha - 1)Y_{\tilde{X}}(t, \tilde{X}(t))\int_{t_0}^{t}(s - t_0)^{\alpha-2}b(s)dsdt+$$

$$+(\alpha - 1)Y_{\tilde{X}}(t, \tilde{X}(t))\int_{t_0}^{t}(s - t_0)^{\alpha-2}\sigma(s)dW(s)dt.$$

Proof.

From Lemma 4.3, by virtue of (4.4), we obtain the relation

$$dY(t, \tilde{X}(t)) = \frac{\partial Y(t, \tilde{X}(t))}{\partial t} + \frac{\partial Y(t, \tilde{X}(t))}{\partial \tilde{X}}d\tilde{X}(t)+$$

$$+\frac{1}{2}\frac{\partial Y^2(t,\tilde{X}(t))}{\partial \tilde{X}^2}(d\tilde{X}(t))^2 =$$

$$= Y_t(t,\tilde{X}(t))dt + (\alpha-1)Y_{\tilde{X}}(t,\tilde{X}(t))\int_0^t (s-t_0)^{\alpha-2}b(s)dsdt+$$

$$+(\alpha-1)Y_{\tilde{X}}(t,\tilde{X}(t))\int_{t_0}^t (s-t_0)^{\alpha-2}\sigma(s)dW(s)dt.$$

The theorem is proved.

5. Stochastic stability

It is well known that the question of the stability of some solution of equation (1.1) by means of a change of variables can be reduced to an investigation of the question of the stability of a trivial solution. Therefore, we will assume that

$$b(t,0) \equiv 0, \sigma(t,0) \equiv 0, t \geq 0. \tag{5.1}$$

Under condition (5.1), equation (1.1) has a trivial solution $x(t) \equiv 0$. The stability of a trivial solution of equation (1.1) is understood as its property changes little with a small change in the initial conditions. Depending on the specific understanding of the expression "small change in solution", different definitions of stability are possible.

Here are some of them.

Definition 5.1. A trivial solution to equation (1.1) is called stochastically stable or stable in probability if for each pair $\varepsilon \in (0,1)$ $l > 0$ exist $\delta(\varepsilon, l, 0) > 0$ such that $P\{|X(t)| < l\} \geq 1-\varepsilon, t \geq 0$ whenever $|X_0| < \delta$.

Otherwise, such a solution is called stochastically unstable.

Definition 5.2. A trivial solution (1.1) is called asymptotically stochastically stable if it is stochastically stable, and moreover, for each $\varepsilon \in (0,1)$ exist $\delta_0 = \delta_0(\varepsilon) > 0$ such that $P\{\lim_{t\to+\infty} X(t) = 0\} \geq 1-\varepsilon$, whenever $|X_0| \leq \delta_0$.

Definition 5.3. A trivial solution (1.1) is called exponentially stable almost surely (a.s.) if

$$\lim_{t\to+\infty} sup\frac{1}{t}ln|X(t)| < 0 \text{ a.s.}$$

for all $x_0 \in \mathbb{R}$.

5.1. Lyapunov stability and asymptotic stability. Let $k > 0$ be an arbitrary number. We denote by S_k the sets of functions $S_k = \{X(\cdot) \in \mathbb{R}, |X(\cdot)| < k\}$, by $a \bigwedge b$ minimum of a and b, by $a \bigvee b$ maximum of a and b and by $I_{\{.\}}$ - indicator function.

Let the following condition (V1) be satisfied.

(V1): There is a positive-definite function $V \in C^{1,2}(R_+;[0,+\infty)) \times S_k$ such that for all $(t, X(t) \in [0,+\infty)) \times S_k, \alpha \in (0,1)$

$$L^{\alpha}V(t,X(t)) := V_t(t,X(t)) + (\alpha-1)V_X(t,X(t))\int_{t_0}^{t}(s-t_0)^{\alpha-2}b(s,X(s))ds \tag{5.2}$$

It follows from the definition of the Lyapunov function that $V(t,0)\equiv 0$.

Moreover, there is a continuous non-decreasing function $\mu \in K$, such that

$$V(t,X(t)) \geq \mu(|X(t)|)$$

for all $(t,X(t)) \in [0,+\infty)\times S_k$.

The following statement is true.

Theorem 5.4. *Suppose that conditions (A1), (A2) and (V1) are satisfied and* $0<\alpha<1$*. Then the trivial solution of equation (1.1) is stochastically stable.*

Proof. Let $\varepsilon \in (0,1)$ $l>0$ an arbitrary number such that $l<k$. By the continuity of V and the condition $V(t_0,0)=0$ there exist $\delta=\delta(\varepsilon,l)$ such that

$$\frac{1}{\varepsilon}\sup_{x\in S_\delta} V(t,X(t)) \leq \mu(l) \tag{5.3}$$

Obviously, $\delta<l$. We fix $X_{t_0}\in S_\delta$ and let η the time of the first exit of $X(t)$ S_l, i.e.

$$\eta = inf\{t>t_0 : X(t)\in S_l\}.$$

By Theorem 4.4, for any $t>t_0$ we have

$$V(\eta\bigwedge t, X(\eta\bigwedge t)) = V(t_0,X_{t_0})+$$

$$+\int_{t_0}^{\eta\bigwedge t} V_\tau(\tau,X(\tau))d\tau + (\alpha-1)\int_{t_0}^{\eta\bigwedge t} V_X(\tau,X(\tau))\int_{t_0}^{S} V_X(s-t_0)^{\alpha-2}b(s,X(s))dsd\tau+$$

$$(\alpha-1)\int_{t_0}^{\eta\bigwedge t} V_X(\tau,X(\tau))\int_{t_0}^{S} V_X(s-t_0)^{\alpha-2}\sigma(s,X(s))dW(s)d\tau =$$

$$= V(t_0,X_{t_0}) + \int_{t_0}^{\eta\bigwedge t} L^{\alpha}V(\tau,X(\tau))d\tau dt+$$

$$+(\alpha-1)\int_{t_0}^{\eta\bigwedge t} V_X(\tau,X(\tau))\int_{t_0}^{S}(s-t_0)^{\alpha-2}\sigma(s,X(s))dW(s)d\tau. \tag{5.4}$$

Taking the expected value of (5.4) and taking into account that $L^{\alpha}V\leq 0$ we obtain for any $t\geq t_0$ and $0<\alpha<1$

$$(\alpha-1)|E\int_{t_0}^{\eta\bigwedge t} V_X(\tau,X(\tau))\int_{t_0}^{S}(s-t_0)^{\alpha-2}\sigma(s,X(s))dW(s)d\tau| \leq$$

$$\leq (\alpha-1)|E\int\limits_{t_0}^{\eta\bigwedge t} V_X(\tau,X(\tau))\int\limits_{t_0}^{S}(s-t_0)^{\alpha-2}\sigma(s,X(s))dW(s)d\tau|\leq 0$$

Taking into account (5.3), we have

$$P\{\eta\leq t\}\leq\varepsilon.$$

Let $t\to+\infty$, i.e. $P\{\eta\leq+\infty\}\leq\varepsilon$.

Then we have $P\{|X(t)|\leq r\}\geq 1-\varepsilon$ for all $t\geq 0$. By Definition 5.1, the trivial solution (1.1) is stochastically stable.

The theorem is proved.

Now let the following condition (V2) be satisfied.

(V2): There is a positive-definite decreasing function $V\in C^{1,2}([0,+\infty)\times S_k;R_+)$ such that $L^\alpha V<0,\alpha\in(0,1)$, where $L^\alpha V$ defined in (5.2).

From (V2) it follows that $V(t,0)\equiv 0$. In addition, there are continuous non-decreasing functions μ_1,μ_2,μ_3 such that

$$\mu_1(|X(t)|)\leq V(t,X(t))\leq\mu_2(|X(t)|),$$

$$L^\alpha V(t,X(t))\leq-\mu_3(|X(t)|)$$

for all $(t,X(t))\in[0,+\infty)\times S_k$.

Theorem 5.5. *Let conditions (A1), (A2) and (V2) be satisfied. Then the trivial solution (1.1) is asymptotically stochastically stable.*

Proof. By Theorem 5.5, the trivial solution of equation (1.1) is stochastically stable. Further, it can be shown that there exists $\delta_0=\delta_0(\varepsilon)>0$ such that

$$P(\lim_{t\to+\infty}X(t)=0)\geq 1-\varepsilon$$

for $|X_0|<\delta_0,\varepsilon\in(0,1)$.

Based on Definition 5.2, we see that the trivial solution (1.1) is asymptotically stochastically stable.

The theorem is proved.

5.2. Exponential stability almost surely. Now let the following condition (V3) be satisfied.

(V3):

$V\in C^{1,2}([0,+\infty)\times\mathbb{R};R_+)$ and there are constants $c_1>1,c_2\in\mathbb{R},c_3\geq 0$ such that

(1) $c_1|X(t)|\leq V(t,X(t))$,

(2) $L^\alpha V(t,X(t))\leq c_2V(t,X(t))$,

(3) $|V_X(t,X(t))|^2\int\limits_0^t|\sigma(t,X(t))(s-\tau)^{\alpha-2}|^2d\tau\geq c_3V^2(t,X(t))$ for all $X(t)\neq 0,\alpha\in(0,1)$ and $t\geq 0$.

The following statement is true

Theorem 5.6. *Let conditions (A1), (A2) and (V2) be satisfied. Then*

$$\lim_{t\to+\infty} sup\frac{1}{t}ln|X(t)| \leq -\frac{1}{lnc_1}(1-\alpha)(c_2+c_3) \ a.s. \tag{5.5}$$

In particular, if $c_2+c_3>0$, then the trivial solution (1.1) is exponentially stable almost surely

Proof. We fix any $X_0 \neq 0$. From Theorem 5.5 and (V3)(2), (3) for $\alpha \in (0,1)$ we have

$$lnV(t,X(t)) = lnV(t_0,X_{t_0}) + \int_0^t \frac{(V_s(s,X(s)))}{V(s,X(s))}ds+$$

$$+(\alpha-1)\int_0^t \frac{V_X(s,X(s))\int_0^s b(s,X(s))(s-\tau)^{\alpha-2}d\tau}{V(s,X(s))}ds+$$

$$+(\alpha-1)\int_0^t \frac{V_X(s,X(s))\int_0^s \sigma(s,X(s))(s-\tau)^{\alpha-2}dW(\tau)}{V(s,X(s))}ds \leq$$

$$\leq lnV(0,X(0)) + \int_0^t \frac{L^\alpha V_s(s,X(s))}{L(s,X(s))}ds+$$

$$+(\alpha-1)\int_0^t \frac{V_X(s,X(s))\int_0^s \sigma(\tau,X(\tau))(s-\tau)^{\alpha-2}dW(\tau)}{V(s,X(s))}ds.$$

We introduce the notation

$$M(t) = \int_0^t V_X(s,X(s))(\int_0^s \sigma(\tau,X(\tau))(s-\tau)^{\alpha-2}dW(\tau)/V(s,X(s)))ds.$$

Then let $n = 1, 2, ...$ For an arbitrary $\varepsilon \in (0,1)$ using (V3) (3) we can get

$$P\{\sup_{0\leq t\leq n}|M(t)+\varepsilon\int_0^t \frac{V_x^2(s,X(s))\int_0^t |\sigma(\tau,X(\tau))(s-\tau)^{\alpha-2}|^2d\tau}{V^2(s,X(s))}ds| \geq c_3t\} \leq \varepsilon$$

Using the Borel-Cantelli theorem (see eg [8]), we obtain almost surely

$$M(t) \leq c_3t - \varepsilon\int_0^t \frac{V_x^2(s,X(s))\int_0^s g(\tau,X(\tau))(s-\tau)^{\alpha-2}|^2d\tau}{V^2(s,X(s))}ds = (1-\varepsilon)c_3t. \tag{5.6}$$

Thus, using (V3) (3) and (5.6), we have

$$lnV(t,X(t)) \leq lnV(0,X(0)) - (1-\alpha)[c_2+(1-\varepsilon)c_3]t.$$

Then we get

$$\frac{1}{t}lnV(t,X(t)) \leq -(1-\alpha)[c_2+(1-\varepsilon)c_3]+\frac{lnV(0,X(0))}{t}.$$

Thus

$$\lim_{t\to+\infty} sup\frac{1}{t}lnV(t,X(t)) \leq -(1-\alpha)[c_2+(1-\varepsilon)c_3].$$

Using now (V3) (1) we obtain

$$\lim_{t\to+\infty} sup\frac{1}{t}lnc_1|X(t)| \leq \lim_{t\to+\infty} sup\frac{1}{t}lnV(t,X(t)) \leq -(1-\alpha)[c_2+(1-\varepsilon)c_3] \leq$$

$$\leq lnV(t_0X_{t_0}) + \int_{t_0}^{t} \frac{L^\alpha V(\tau,X(\tau)}{L(\tau,X(\tau))}d\tau +$$

$$+(1-\alpha)\int_{t_0}^{t} V_X(\tau,X(\tau))\int_{t_0}^{S}(s-t_0)^{\alpha-2}\sigma(s,X(s))dW(s)/V(s,X(s))d\tau.$$

Let us introduce the notation

$$M_1(t) = \int_{0}^{t} V_X(\tau,X(\tau))\int_{t_0}^{S}(s-t_0)^{\alpha-2}\sigma(s,X(s))dW(s)/V(s,X(s))d\tau.$$

Let $n=1,2,...$ For arbitrary $\varepsilon \in (0,1)$ using (V3) (3) we obtain

$$P\{\sup_{0\leq t\leq n}|M_1(t)+\varepsilon\int_{t_0}^{t}\frac{V_X^2(\tau,X(\tau))\int_{t_0}^{S}|(s-t_0)^{\alpha-2}\sigma(s,X(s))|^2ds}{V^2(\tau,X(\tau))}d\tau| \geq c_3t\} \leq \varepsilon$$

Using the properties of the σ -algebra $\mathfrak{F}$, we obtain the inequality

$$M_1(t) \leq c_3t - \varepsilon\int_{t_0}^{t}\frac{V_X^2(\tau,X(\tau))\int_{t_0}^{S}|(s-t_0)^{\alpha-2}\sigma(s,X(s))|^2ds}{V^2(\tau,X(\tau))}d\tau = (1-\varepsilon)c_3t \text{ a.s.} \tag{5.7}$$

Thus, using (V3) (3) and (5.7), we have

$$lnV(t,X(t)) \leq lnV(t_0,X_{t_0}) - (1-\alpha)[c_2+(1-\varepsilon)c_3t].$$

Then we get

$$\frac{1}{t}lnV(t,X(t)) \leq -(1-\alpha)[c_2+(1-\varepsilon)c_3t]+\frac{lnV(t_0,X_{t_0})}{t}.$$

Thus

$$\lim_{t\to+\infty}\frac{1}{t}lnV(t,X(t)) \leq -(1-\alpha)[c_2+(1-\varepsilon)c_3]$$

Using now (V3) (1) we obtain

$$\lim_{t\to+\infty} sup\frac{1}{t}lnc_1|X(t)| \leq \lim_{t\to+\infty}\frac{1}{t}lnV(t,X(t)) \leq$$

$$(1-\alpha)[c_2+(1-\varepsilon)c_3].$$

Finally, we get

$$\lim_{t\to+\infty} sup\frac{1}{t}ln|X(t)| \leq -\frac{1}{lnc_1}(1-\alpha)[c_2+(1-\varepsilon)c_3].$$

Since ε is arbitrary, we obtain the estimate 5.5
Note that $c_1 > 1$. Then if $c_2 + c_3 > 0$, we will obtain

$$-\frac{1}{lnc_1}(1-\alpha)(c_2+c_3) < 0.$$

By Definition 5.3, the trivial solution (1.1) is exponentially stable almost surely.

6. Conclusion

For deterministic and stochastic systems of differential equations with fractional derivatives of Riemann-Liouville, Caputo, Grunwald-Letnikov, definitions of Lyapunov functions with fractional orders of derivatives are given in [5,6].

However, the practical calculation of these derivatives is associated with great difficulties due to the absence of a chain rule for them. In this paper, the direct Lyapunov method is carried over to scalar stochastic differential equations with fractional-like derivatives for which the chain rule for differentiating complex functions is satisfied. For this purpose, a new fractional-like version of the Ito formula is introduced. It is important that the fractional-like derivatives are majorants for the Caputo derivatives.

References

[1] B.Bayor, D.F.M.Torres: Existence of solution to a local fractional nonlinear differenial equations, *J.Appl.Mech.Tech.Phys*, 55(2) (2014), 191-198
[2] A.Chadha, S.N.Boca: Existence and exponential stability for neutral stochastic fractional differential equations with impulses driven by Poisson, Stochastic, 90(5) (2018), 663-681
[3] T.Abdeljawad: On comfortable fractional calculus, J.Comput.Apll.Math., 279 (2015), 57-66
[4] R.Khalil, M.Al-Horani, A. Yousef, M.Sababheh: A new definition of fractional derivative, J.Comput.Apll.Math., 264 (2014), 65-70
[5] A.A.Martynyuk: On the stability of a system of equations with fractional derivatives with respect to two measures, J.Math.Sci., 217 (2016), 468-475
[6] A.A.Martynyuk, I.M.Stamova: Fractional-like derivative of Lyapunov-type functions and applications to stability analysis of motion, Electronic Journal of Differential Equations, 208(62), (2018), 1-12
[7] V.Lakshmikantham, S.Leela, A.A.Martytyuk: Stability analysis of nonlinear systems, Mariel Dekker, (1998)
[8] V. N. Afanasyev, V. B. Kolmanovsky, V. R. Nosov: Mathematical theory of control systems design, Moscow, Vishaya shcola, (1989)
[9] N.Aguila-Camacho, M.A.Duarte-Mermoud, J.A. Gallegos: Lyapunov functions for fractional order systems, Commun.Nonlinear Sci.Numer.Simul., 19 (2014), 2951-2957
[10] R.Z. Has'minskii: Stability of systems of differential equations under random perturbations, M. Nauka, (1969)
[11] T.A.Burton: Fractional differential equations and Lyapunov functionals, Nonlinear Anal., Theory Methods Appl., 74 (2011) 5648-5662
[12] T.A.Burton, T.Furumochi: Krasnoselski's fixed point theorem and stability, Nonlinear Anal.Theory Methods, 49 (2002), 445-454
[13] T.Burton, B.Zhang: Fixed points and fractional differential equations, Fixed Point Theor, 13 (2013), 313-325

[14] M.Ilolov, Kh.S.Kuchakshoev, J.Sh.Rahmatov: Fractional stochastic evolution equations: White noise model, Communication on Stochastic Analysis, 14(3-4) (2020), 55-69
[15] M.Ilolov, V.N.Kozobrod: Periodic random perturbations of differential equations with parameter, Dokladi AN Taj.SSR, 31(3) (1988), 147-150
[16] R.Sakthivel, P.Revathi, Y.Pen: Existence of solutions for nonlinear fractional stochastic differential equations, Nonlinear Anal.Theory Methods Appl., 81 (2013), 70-86
[17] Yu.E.Gliklich: Global and Stochastic Anakysis with Applications to Mathematical Physics, Sringer, London, (2011)
[18] J.C.Trigeasson, N.Maamzi, J.Sabatier, A.Dusfaloup: A Lyapunov approach to the stabilityof fractional differential equations,, Signal Process., 91 (2011), 437-445
[19] B.Oksendal: Stochastic Differential Equations: an Introduction with Application, 5th ed.Springer, Berlin, (2000)
[20] A.Yu.Veretennikov: On weak solutions of highly degenerate SDEs, Automation and Remote Control, 83(3) (2020), 398-410
[21] V.Lakshmikantham, S.Leela, M.Sambandham: Lyapunov Theory theory for fractional differential equations, Commun.Appl.Anal., 12 (2008), 365-376

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