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ON SHARP BOUNDS IN THE T. LYONS'S NEO–CLASSICAL INEQUALITY, WHICH IS ESSENTIAL FOR THE THEORY OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In the famous paper of Terry Lyons [1] a new approach to solving a general class of stochastic differential equations with nonsmooth data was proposed. As a byway some known results were refined: considered Brownian paths with "bad" trajectories, some generalization of integrals were proposed not the same as those of Ito, Stratonovich, Skorokhod, more general formulas of change of variables in multidimensional case were proved for new forms of integrals, and a method of successive approximations was studied based on iterated integrals of the new modified type. An interesting fact is that all essential considerations in [1] are based on a seemingly simple inequality that controls the most important estimates in the above paper. This remarkable inequality is now known as the "neo–classsical" one. In this paper we prove that the hypothesis concerning the best constant in this inequality proposed by E.T.R. Love in [2, 3] is not true, and we find the best constant for this inequality. Some special cases and numerical results are also considered.

1. Introduction

A new approach to solving a general class of stochastic differential equations with nonsmooth data was proposed in the well–known fundamental paper by Terry Lyons in [1]. He considered stochastic differential equations of the form

$$dy_t = \sum_k f_k(y_t) dx_t^k, \tag{1.1}$$

and in this equation f_k are given vector fields, x_t are control terms, y_t is the resulting trajectory.

The essential problem and occurring challenge is the following one. If, in accordance with the standard approach, we consider the time t as a parameter and solve this equation as homogeneous, then often enough, the solution will not be continuous and it can exist only as a distribution. In this case the classical theory does not offer any useful methods for determining the solution. Moreover, even for smooth but strongly oscillating problems there are no efficient algorithms for finding solutions numerically. At the same time, this problem arises in many

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branches of mathematics and its applications: control theory, radio engineering problems with noise, Lie algebras, probability theory (multidimensional Brownian trajectories, semi-martingales, random processes), see [4, 5, 6, 7, 8, 9, 10]. For a more detailed description of applications of T.Lyons's approach see [1].

Combining existing methods with new ones in [1] was proposed a very useful and powerful choice of functional spaces for solutions to equation (1.1). These spaces includes norms with p–variations. And using such classes of spaces in [1] solutions to (1.1) and effective numerical methods for it were derived based on deterministic instead of stochastic approach. As a byway some known results were refined: considered Brownian paths with "bad" trajectories, some generalization of integrals were proposed not the same as those of Ito, Stratonovich, Skorokhod, more general formulas of change of variables in multidimensional case were proved for new forms of integrals, and a method of successive approximations was studied based on iterated integrals of the new modified type.

An interesting fact is that all essential considerations in [1] are based on a seemingly simple inequality that controls the most important estimates in the above paper. Exactly, it is the next remarkable inequality

$$\sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p} \leq C(n,p)(1+x)^{n/p}, \tag{1.2}$$

where $p \geq 1$, n is a natural number, $0 \leq x$, and the constant $C(n,p) > 0$. Of course (1.2) is a generalization of the Newton binomial formula (case $p = 1$)

$$\sum_{k=0}^{n} C_n^k x^k = (1+x)^n. \tag{1.3}$$

The binomial coefficient C_x^y is defined using the gamma function or beta function via

$$C_x^y = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)} = \frac{1}{(x+1)\mathrm{B}(x-y+1, y+1)}.$$

The inequality (1.2) became rather famous due to its importance for the stochastic differential equations theory and received a name *neo–classical inequality.* This inequality was studied also in [2, 3, 11, 12, 13].

In this paper we study the sharp constant in the inequality (1.2), namely the best=minimal constant in its rhs $C_{\min}(n,p)$. This problem has some history. In [1] the proof of (1.2) is rather complicated and the constant $C(n,p)$ is not optimal. The next hypothesis was proposed by E.T.R. Love in 1996.

HYPOTHESIS [2, 3]**: the best possible constant in** (1.2) **equals**

$$C_{\min}(n,p) = p. \tag{1.4}$$

In [1] it was declared that this hypothesis is approved by author's numerical calculations. But it occurred that this assertion needs some corrections. In this paper we prove that the hypothesis proposed by E.T.R. Love in [2, 3] is not true, and we find the best constant for the inequality (1.2), also consider some special cases and numerical results.

2. Estimate of the exact constant in neoclassic binomial inequality

Theorem 2.1. *Let* $0 \le x$, $1 \le p$. *Then for the exact constant in the inequality*

$$\sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p} \le C_{\min}(n,p)(1+x)^{n/p},$$

the equality

$$C_{\min}(n,p) = \frac{1}{2^{n/p}} \sum_{k=0}^{n} \frac{\Gamma\left(\frac{n}{p}+1\right)}{\Gamma\left(\frac{k}{p}+1\right)\Gamma\left(\frac{n-k}{p}+1\right)} = \frac{1}{2^{n/p}} \sum_{k=0}^{n} C_{n/p}^{k/p} \le p, \qquad (2.1)$$

holds. The inequality sign in (2.1) *is strict for all* $p > 1$ *and* $x \ne 1$. *In other words, the optimal constant found in Theorem 2.1 is better than the constant from the hypothesis above, except for the trivial case* $p = 1$ *when we get binomial formula* (1.3).

Proof. Let find the maximum of the function

$$f(x) = \sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p} \cdot (1+x)^{-n/p}.$$

Its derivative has the form

$$f'(x) = \sum_{k=0}^{n} C_{n/p}^{k/p} \left[k/p x^{k/p-1} \cdot (1+x)^{-n/p} - n/p x^{k/p} \cdot (1+x)^{-n/p-1}\right] =$$

$$= \frac{1}{p}(1+x)^{-n/p-1} \sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p-1} \left[k + (k-n)x\right].$$

We have $f'(x) = 0$ when $x = 1$. Indeed, since, obviously $C_{n/p}^{k/p} = C_{n/p}^{(n-k)/p}$

$$f'(1) = \frac{2^{-n/p-1}}{p} \sum_{k=0}^{n} C_{n/p}^{k/p} \left[2k - n\right] =$$

$$= \frac{2^{-n/p-1}}{p} \Big[C_{n/p}^{0/p}(-n) + C_{n/p}^{1/p}(2-n) + C_{n/p}^{2/p}(4-n) + \dots$$

$$\dots + C_{n/p}^{(n-2)/p}(n-4) + C_{n/p}^{(n-1)/p}(n-2) + C_{n/p}^{n/p} n\Big] = 0.$$

When n os odd the corresponding terms cancel each other out, when n is even all the corresponding terms cancel each other out except what's in the middle but it vanishes since $2k - n = 2 \cdot \frac{n}{2} - n = 0$.

Now we show that $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$. We have

$$f'(x) = \frac{1}{p}(1+x)^{-n/p-1} \sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p-1} \left[k + (k-n)x\right] =$$

$$= \frac{1}{p}(1+x)^{-n/p-1} \left(\sum_{k=0}^{n} C_{n/p}^{k/p} k x^{k/p-1} - \sum_{k=0}^{n} C_{n/p}^{k/p} (n-k) x^{k/p}\right).$$

First consider the case when $p > n$. Changing index in the last sum by $n-k \to k$ we get

$$f'(x) = \frac{1}{p}(1+x)^{-n/p-1}\left(\sum_{k=0}^{n} C_{n/p}^{k/p} k x^{k/p-1} - \sum_{k=0}^{n} C_{n/p}^{(n-k)/p} k x^{(n-k)/p}\right) =$$

$$= \frac{1}{p}(1+x)^{-n/p-1}\sum_{k=0}^{n} C_{n/p}^{k/p} k (x^{k/p-1} - x^{(n-k)/p}).$$

It is easy to see now that since $(x^{k/p-1} - x^{(n-k)/p}) > 0$ when $p > n$ we have $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$ since $(x^{k/p-1} - x^{(n-k)/p}) < 0$ for all $k = 0, 1, ..., n$.

Let now $p \leq n$, then we can combine the first p terms of the sum $\sum\limits_{k=0}^{n} C_{n/p}^{k/p} k x^{k/p-1}$ with the last p terms of the sum $\sum\limits_{k=0}^{n} C_{n/p}^{k/p}(k-n)x^{k/p}$:

$$f'(x) = \frac{1}{p}(1+x)^{-n/p-1}\sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p-1}\left[k + (k-n)x\right] =$$

$$= \frac{1}{p}(1+x)^{-n/p-1}\left(\sum_{k=0}^{n} C_{n/p}^{k/p} k x^{k/p-1} - \sum_{k=0}^{n} C_{n/p}^{k/p}(n-k)x^{k/p}\right) =$$

$$= \frac{1}{p}(1+x)^{-n/p-1}\left(\sum_{k=0}^{p} C_{n/p}^{k/p} k x^{k/p-1} - \sum_{k=n-p}^{n} C_{n/p}^{k/p}(n-k)x^{k/p}\right) +$$

$$+ \frac{1}{p}(1+x)^{-n/p-1}\left(\sum_{k=p}^{n} C_{n/p}^{k/p} k x^{k/p-1} - \sum_{k=0}^{n-p} C_{n/p}^{k/p}(n-k)x^{k/p}\right).$$

Let us show that the last bracket is equal to zero. We can write

$$\left(\sum_{k=p}^{n} C_{n/p}^{k/p} k x^{k/p-1} - \sum_{k=0}^{n-p} C_{n/p}^{k/p}(n-k)x^{k/p}\right) = \left(C_{n/p}^{1} p - C_{n/p}^{0} n\right) +$$

$$+ \left(C_{n/p}^{1/p+1}(p+1) - C_{n/p}^{1/p}(n-1)\right)x^{1/p} + ... + \left(C_{n/p}^{n/p} n - C_{n/p}^{n/p-1} p\right)x^{n/p-1} =$$

$$= \sum_{k=p}^{n}\left(C_{n/p}^{k/p} k - C_{n/p}^{k/p-1}(n+p-k)\right)x^{k/p-1} = 0$$

since

$$C_{n/p}^{k/p} k - C_{n/p}^{k/p-1}(n+p-k) =$$

$$= \frac{\Gamma\left(\frac{n}{p}+1\right)}{\Gamma\left(\frac{k}{p}+1\right)\Gamma\left(\frac{n-k}{p}+1\right)} k - \frac{\Gamma\left(\frac{n}{p}+1\right)}{\Gamma\left(\frac{k}{p}\right)\Gamma\left(\frac{n-k}{p}+2\right)}(n+p-k) =$$

$$= \frac{\Gamma\left(\frac{n}{p}+1\right)}{\Gamma\left(\frac{n-k}{p}+1\right)\Gamma\left(\frac{k}{p}\right)}\left(k \cdot \frac{p}{k} - (n+p-k)\cdot\frac{p}{n-k+p}\right) = 0.$$

Therefore we obtain that

$$f'(x) = \frac{1}{p}(1+x)^{-n/p-1}\left(\sum_{k=0}^{p} C_{n/p}^{k/p} k x^{k/p-1} - \sum_{k=n-p}^{n} C_{n/p}^{k/p}(n-k)x^{k/p}\right) =$$

$$= \frac{1}{p}(1+x)^{-n/p-1}\left(\sum_{k=0}^{p} C_{n/p}^{k/p} k (x^{k/p-1} - x^{(n-k)/p})\right).$$

Since $(x^{k/p-1} - x^{(n-k)/p}) > 0$ for $x < 1$ and $(x^{k/p-1} - x^{(n-k)/p}) < 0$ for $x > 1$ we get $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$ for all $k = 0, 1, ..., p$ when $p \le n$.

So we get for all $x \ge 0$ and for $p \ge 1$ function $f(x)$ has a strict maximum at $x = 1$:

$$f_{\max} = f(1) = \frac{1}{2^{n/p}}\sum_{k=0}^{n} C_{n/p}^{k/p} = \frac{1}{2^{n/p}}\sum_{k=0}^{n} \frac{\Gamma\left(\frac{n}{p}+1\right)}{\Gamma\left(\frac{k}{p}+1\right)\Gamma\left(\frac{n-k}{p}+1\right)}$$

and for $0 \le x \ne 1$

$$f(x) = \sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p} \cdot (1+x)^{-n/p} < f_{\max} = f(1) = \frac{1}{2^{n/p}}\sum_{k=0}^{n} C_{n/p}^{k/p}$$

that gives statement of theorem. □

Corollary 2.2. *Let $n = 1$, $p \ge 1$, $x \ge 0$. Then the equality*

$$C_{\min}(1,p) = 2^{1-1/p} \le p, \tag{2.2}$$

holds for the exact constant in the inequality (1.2) *which takes form*

$$1 + x^{1/p} \le 2^{1-1/p}(1+x)^{1/p}.$$

Corollary 2.3. *Let $n = 2$, $p \ge 1$, $x \ge 0$. Then the equality*

$$C_{\min}(2,p) = 2^{1-2/p} + \frac{1}{\sqrt{\pi}}\frac{\Gamma(1/p+1/2)}{\Gamma(1/p+1)}. \tag{2.3}$$

holds for the exact constant in the inequality (1.2) *which takes form*

$$1 + x^{2/p} - 2^{1-2/p}(1+x)^{2/p} \le \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}+1\right)}\left((1+x)^{2/p} - 2^{2/p}\, x^{1/p}\right). \tag{2.4}$$

Proof. Applying the inequality (1.2) when $n = 1$ we can write

$$1 + \frac{\Gamma\left(\frac{2}{p}+1\right)}{\Gamma^2\left(\frac{1}{p}+1\right)}\, x^{1/p} + x^{2/p} \le \frac{(1+x)^{2/p}}{2^{2/p}}\left(2 + \frac{\Gamma\left(\frac{2}{p}+1\right)}{\Gamma^2\left(\frac{1}{p}+1\right)}\right).$$

Using the Legendre formula

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\,\sqrt{\pi}\,\Gamma(2z)$$

we get

$$\frac{\Gamma\left(\frac{2}{p}+1\right)}{\Gamma^2\left(\frac{1}{p}+1\right)} = \frac{2^{2/p}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}+1\right)}$$

and

$$1 + \frac{2^{2/p}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}+1\right)} x^{1/p} + x^{2/p} \le 2^{1-2/p}(1+x)^{2/p} + \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}+1\right)} (1+x)^{2/p}$$

or (2.4). □

Example 2.4. Let consider the graphical representation of the inequality from theorem (2.1). We denote $A(n,p,x) = \sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p}$, $B(n,p,x) = C_{\min}(n,p)(1+x)^{n/p}$, $G(n,p,x) = p(1+x)^{n/p}$. Plots of these three functions for $n = 10$, $x = 1/2$, $p \in [1,15]$ is presented in Figure 1.

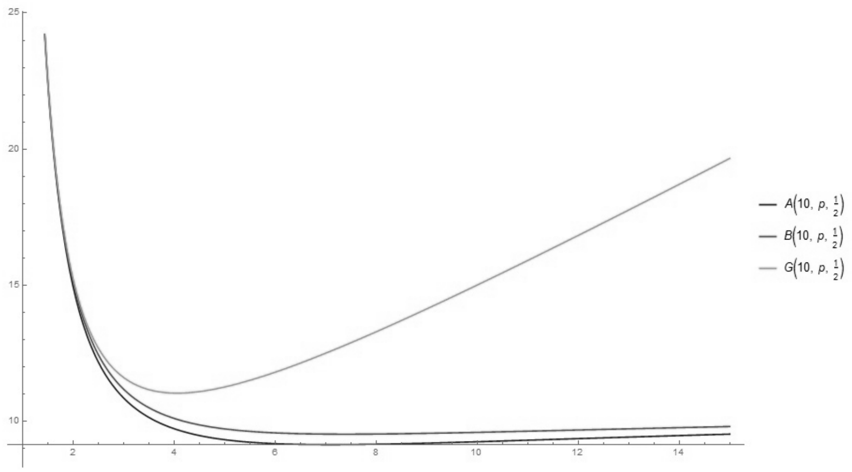

FIGURE 1. Functions $A(n,p,x)$, $B(n,p,x)$, $G(n,p,x)$.

Figure 1 demonstrates the inequalities $A(n,p,x) \le B(n,p,x) \le G(n,p,x)$. We can see that for $p \ge 3$ the upper estimate $G(n,p,x)$ is much rougher than $B(n,p,x)$. If we calculate values $A(n,p,x)$, $B(n,p,x)$, $G(n,p,x)$ for some concrete $p < 3$ we get

$$A(10,1,1/2) = B(10,1,1/2) = G(10,1,1/2) \approx 57.665,$$
$$A(10,1.5,1/2) \approx 22.2691 < B(10,1.5,1/2) \approx 22.3647 < G(10,1.5,1/2) \approx 22.3889,$$
$$A(10,2,1/2) \approx 14.8877 < B(10,2,1/2) \approx 15.0831 < G(10,2,1/2) \approx 15.1875,$$
$$A(10,2.5,1/2) \approx 12.1413 < B(10,2.5,1/2) \approx 12.4126 < G(10,2.5,1/2) \approx 12.6563,$$

So we can see that the estimate $B(n,p,x)$ is better then $G(n,p,x)$ for the $p < 3$ also.

3. Term-by-term estimates

Next, consider term-by-term estimates for the sum in (1.2). In term-based estimates the key inequality is

$$C_{n/p}^{k/p} \le D(n,p) C_n^k, \tag{3.1}$$

where $D(n,p)$ is some constant. Such estimates allow us to sum up the expression on the left in (1.2). This relationship has clear combinatorial interest. Each estimate of the form (3.1) implies a corresponding inequality of the form (1.2) with its own constant.

Theorem 3.1. *If the inequality* (3.1) *holds, then the inequality*

$$\sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p} \le 2^{n(1-1/p)} D(n,p)(1+x)^{n/p}.$$

Proof. Estimating the terms in $\sum\limits_{k=0}^{n} C_{n/p}^{k/p} x^{k/p}$ taking into account (3.1) we get

$$\sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p} \le \sum_{k=0}^{n} D(n,p) C_n^k x^{k/p} =$$

$$= D(n,p)(1+x^{1/p})^n \le D(n,p) 2^{n(1-1/p)} (1+x)^{n/p}.$$

Inequality $(1+x^{1/p})^n \le 2^{n(1-1/p)}$ follows from the monotonicity of power means of the form $M_\alpha(t,y) = \left(\frac{t^\alpha + y^\alpha}{2}\right)^{1/\alpha}$, $\alpha \in \mathbb{R}$ ([14, 15]). Namely for $\alpha_1 < \alpha_2$ we have $M_{\alpha_1}(t,y) \le M_{\alpha_2}(t,y)$ or

$$\left(\frac{t^{\alpha_1} + y^{\alpha_1}}{2}\right)^{1/\alpha_1} \le \left(\frac{t^{\alpha_2} + y^{\alpha_2}}{2}\right)^{1/\alpha_2}.$$

In our case $t = 1$, $y = x$, $\alpha_1 = 1/p < 1 = \alpha_2$.

□

Thus, with the chosen method, the question is reduced to obtaining good estimates for ratios of gamma or beta functions of the form (3.1). There are several ways to obtain such inequalities [15, 16, 17].

Another possibility is based on the use of inequalities of the form

$$C_{n/p}^{k/p} \le B(n,p), \tag{3.2}$$

with some constant $B(n,p)$.

Theorem 3.2. *Let the estimate* (3.2) *be fulfilled under the additional condition* $p > \frac{n+2}{3}$. *Then the inequality*

$$\sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p} \le A(n,p)(1+x)^{n/p}. \tag{3.3}$$

holds with constant

$$A(n,p) = (n+1) 2^{-\frac{n}{p}} B(n,p).$$

Proof. Let (3.2) hold, then, summing up the geometric progression in $\sum\limits_{k=0}^{n} C_{n/p}^{k/p} x^{k/p}$, we obtain the estimate

$$\sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p} \leq B(n,p) \sum_{k=0}^{n} x^{k/p} \leq B(n,p) \frac{1-(x^{1/p})^{n+1}}{1-x^{1/p}}.$$

Now we apply the Tibor Rado inequality [18] with clarifications from [15] of the form

$$R_n(t,y) \leq M_{\frac{n+2}{3}}(t,y), \qquad n \geq 1,$$

where

$$R_n(t,y) = \left(\frac{t^{n+1}-y^{n+1}}{(n+1)(t-y)}\right)^{1/n}, \qquad M_{\frac{n+2}{3}}(t,y) = \left(\frac{t^{\frac{n+2}{3}}+y^{\frac{n+2}{3}}}{2}\right)^{\frac{3}{n+2}}.$$

Then, under the additional condition $\frac{n+2}{3p} < 1$, providing comparison with the arithmetic mean, we obtain

$$\left[\frac{1-\left(x^{1/p}\right)^{n+1}}{(n+1)\left(1-x^{1/p}\right)}\right]^{1/n} = R_n\left(1, x^{1/p}\right) \leq$$

$$\leq \left(\frac{1+\left(x^{1/p}\right)^{\frac{n+2}{3}}}{2}\right)^{\frac{3p}{n+2}\cdot\frac{1}{p}} \leq \left(\frac{1+x}{2}\right)^{1/p},$$

$$\frac{1-x^{\frac{n+1}{p}}}{1-x^{1/p}} \leq (n+1)\cdot 2^{-\frac{n}{p}} \cdot B(n,p).$$

This gives us the inequality (3.3).

Theorem 3.3. *Under the restrictions of theorem 3.2, the inequality*

$$\sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p} \leq H(n,p)(1+x)^{n/p}. \tag{3.4}$$

holds with the constant

$$H(n,p) = \frac{(n+1)\,\Gamma\left(\frac{n}{2p}+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2p}+1\right)} \leq \sqrt{\frac{2}{\pi}}\frac{(n+1)\sqrt{p}}{\sqrt{n}}.$$

Proof. Let consider

$$C_{n/p}^{k/p} = \frac{\Gamma\left(\frac{n}{p}+1\right)}{\Gamma\left(\frac{k}{p}+1\right)\Gamma\left(\frac{n-k}{p}+1\right)}.$$

Here we will use the logarithmic convexity of the gamma function. Namely, if for all $x_1, x_1 \in X$ and for all $p_1 > 0$, $p_2 > 0$, $p_1 + p_2 = 1$ the inequality

$$f(p_1 x_x + p_2 x_2) \leq f^{p_1}(x_1) f^{p_2}(x_2)$$

holds then the function $f(x)$ is called logarithmically convex on X. It is well known [14, 19, 20] that $\Gamma(x)$ has the property of logarithmic convexity. Applying this property to denominator in $C_{n/p}^{k/p}$ for $p_1 = p_2 = \frac{1}{2}$, $x_1 = \frac{k}{p} + 1$, $x_2 = \frac{n-k}{p} + 1$ we get

$$\Gamma\left(\frac{k}{p}+1\right)\Gamma\left(\frac{n-k}{p}+1\right) \geq \left(\Gamma\left(\frac{k}{2p}+\frac{1}{2}+\frac{n-k}{2p}+\frac{1}{2}\right)\right)^2 = \Gamma^2\left(\frac{n}{2p}+1\right)$$

and using the Legendre duplication formula

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\,\sqrt{\pi}\,\Gamma(2z).$$

we obtain

$$C_{n/p}^{k/p} \leq \frac{\Gamma\left(\frac{n}{p}+1\right)}{\Gamma^2\left(\frac{n}{2p}+1\right)} = \frac{2^{\frac{n}{p}}\Gamma\left(\frac{n}{2p}+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n}{2p}+1\right)}.$$

In [16] we can find the inequalities for the ratio of gamma functions with a difference of arguments equal to half of the form

$$\frac{\Gamma\left(\frac{t+1}{2}\right)}{\Gamma\left(\frac{t+2}{2}\right)} < \frac{2}{\sqrt{2t+1}}, \qquad t > 0.$$

In our case putting $t = \frac{n}{p} + 1$ we get

$$C_{n/p}^{k/p} \leq \frac{2^{\frac{n}{p}+1}\sqrt{p}}{\sqrt{\pi}\sqrt{2n+3p}} \leq \frac{2^{\frac{n}{p}+\frac{1}{2}}\sqrt{p}}{\sqrt{\pi}\sqrt{n}} = B(n,p).$$

From theorem 3.2 we obtain

$$\sum_{k=0}^{n} C_{n/p}^{k/p} x^{k/p} \leq (n+1)2^{-\frac{n}{p}} B(n,p)(1+x)^{n/p} =$$

$$= (n+1)2^{-\frac{n}{p}} \cdot \frac{2^{\frac{n}{p}+\frac{1}{2}}\sqrt{p}}{\sqrt{\pi}\sqrt{n}}(1+x)^{n/p} = \sqrt{\frac{2}{\pi}}\frac{(n+1)\sqrt{p}}{\sqrt{n}}(1+x)^{n/p}.$$

That gives the statement of theorem. □

Other estimates of for the ratio $\frac{\Gamma(x+\beta)}{\Gamma(x)}$ can be found in [15, 16, 17]. They are useful for inequalities of considered type.

Thus, using the results of this paper, one can refine some estimates and conclusions from [1]. Note the remaining unsolved problem of finding the lower bound if the sign in the inequality (1.2) is replaced by the opposite sign. Also an interesting question is to classify a sum in the LHS of the neo–classical inequality (1.2), it looks like some truncated Fox–Wright function [7, 21, 22] but in fact doesn't coincide with any of them.

References

1. Lyons T.: *Differential equations driven by rough signals.* Revista Matematica Iberoamericana, 14, no. 2 (1998) 215–310.
2. Love E. R.: *On an inequality conjectured by T.J. Lyons.* J. Inequal. and Appl., 2 (1998) 229–233.
3. Love E. R.: *An inequality conjectured by T. J. Lyons.* Integral Transforms and Special Functions, 10, Issue 3–4 (2000) 283–288.

4. Love E.R.; Prasad G.; Sahai A.: *An improved estimate of the rate of the integrated Meyer-König and Zeller operators for functions of bounded variation.* J. Math. Anal. Appl., 187, no. 1 (1994) 1–16.
5. Bhattacharya R.N.; Rao R.R.: *Normal approximation and asymptotic expansions.* Society for Industrial and Applied Mathematics, Philadelphia, 2010.
6. Chow Y.S.; Teicher H.: *Probability Theory.* Springer-Verlag, New York, 1988.
7. Shishkina E.L.; Sitnik S.M.: *Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics.* (Mathematics in Science and Engineering), Elsevier, Academic Press, 2020.
8. Gliklikh Yu. E.: *Ordinary and stochastic differential geometry as a tool for mathematical physics.*Kluwer Academic Publishers, Dordrecht, 1996.
9. Gliklikh Yu. E.: *Global Analysis in Mathematical Physics. Geometric and Stochastic Methods.* Springer-Verlag, New York, 1997.
10. Gliklikh Yu. E.: *Global and Stochastic Analysis with Applications to Mathematical Physics.* Springer-Verlag, London, 2011.
11. Sitnik S.M.: *On generalization of binomial theorem which arising in differential equations theory.* Vestnik Voronezhskogo Instituta MVD Rossii, 1 (2004) 143–147.
12. Hara K.; Hino M.: *Fractional order Taylor's series and the neo–classical inequality.* arXiv:1001.1775v1 (2010).
13. Li J.–H.: *On Lyons' inequality and estimates of convergence rates of approximation via Meyer-König and Zeller operators.* Master thesis in National Central University (2004).
14. Hardy G.H; Littlewood J.E.; Polya G.: *Inequalities.* Cambridge Univ. Press, New York, 1934.
15. Sitnik S.M.: *Generalized Young and Cauchy–Bunyakowsky inequalities with applications: a survey.* arXiv:1012.3864, (2010). — 51 pp.
16. Qi F.: *Bounds for the ratio of two Gamma functions.* Journal of Inequalities and Applications, Hindawi Publishing Corporation, Article ID 493058 (2010). — 84 pp.
17. Merkle M.: *Logarithmic convexity and inequalities for the gamma function.* J. Math. Analysis Appl., 203 (1996) 369–380.
18. Rado T.: *On convex functions.* Trans. Amer. Math. Soc., 37 (1935) 266–285.
19. Mollerup J.; Bohr H.: *Lærebog i Kompleks Analyse, vol. III.*, Analysis Textbook, Copenhagen (1922).
20. Sitnik S.M.: *Inequalities for Bessel functions.* Dokl. Math., 51, no.1 (1995) 25–28.
21. Srivastava H.M.; Manocha H.L.: *A treatise on generating functions.* Halsted Press, New York, 1984.
22. Gorenflo R.; Kilbas A.A.; Rogosin S.V.: *Mittag-Leffler Functions, Related Topics and Applications (2nd edition).* (Springer Monographs in Mathematics), Springer–Verlag Berlin Heidelberg, Berlin, 2020.

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