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OPTIMAL CONTROL OF SOLUTIONS TO SHOWALTER – SIDOROV PROBLEM FOR A BOUSSINESQ – LOVE EQUATION WITH ADDITIVE "NOISE"

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Abstract. In this paper, the problem of optimal control of solutions to the Showalter – Sidorov problem for a high-order Sobolev type equation with additive "noise" is investigated. The existence and uniqueness of a strong solution to the Showalter – Sidorov problem for this equation are proved. Sufficient conditions for the existence and uniqueness of an optimal control of such solutions are obtained. For this, we built the space of "noises". For the differentiation of additive "noise", we use the derivative of a stochastic process in the sense of Nelson – Gliklikh. The article also discusses the stochastic Boussinesq – Love model.

Introduction

Recently, research on Sobolev type equations has expanded considerably. The complete Sobolev type equation

$$Av^{(n)} = B_{n-1}v^{n-1} + \ldots + B_0 v + f \tag{0.1}$$

with the assumption kerA $\neq \{0\}$ has been studied in different aspects for $n \geq 1$ [1 – 6]. Here the operators $A, B_{n-1}, \ldots, B_0$ are linear and continuous, acting from Banach space $\mathfrak{V}$ to Banach space $\mathfrak{G}$, absolute term $f = f(t)$ models the external force.

The lack of equation (0.1) with the deterministic absolute term is that, in natural experiments, the system is subject to random perturbation, for example in the form of white noise. Currently, stochastic ordinary differential equations with various additive random processes are being actively studied [7].

The first results concerning stochastic Sobolev type equations of the first order can be found in [8]. They are based on the extension of the Ito – Stratonovich – Skorokhod method to partial differential equations [9]. In [10] there was studied a stochastic Sobolev type equation of higher order

$$A \overset{o}{\eta}{}^{(n)} = B_{n-1} \overset{o}{\eta}{}^{(n-1)} + \ldots + B_0\eta + w, \tag{0.2}$$

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where w is the stochastic process. It is required to find the random process $\eta(t)$, satisfying (in some sense) equation (0.2) and the initial conditions

$$\overset{o}{\eta}^{(m)}(0) = \xi_m,\ m = 0, 1, \ldots, n-1, \tag{0.3}$$

where ξ_m are given random variables.

At first, w was understood as white noise, which is a generalized derivative of the Wiener process. Later, a new approach to the investigation of equation (0.2) appeared [11] and is being actively developed [12, 13], where "white noise" means the Nelson – Gliklikh derivative of the Wiener process.

Of particular interest is the optimal control problem. Consider the stochastic Sobolev type equation

$$A\overset{o}{\eta}^{(n)} = B_{n-1}\overset{o}{\eta}^{(n-1)} + \ldots + B_0\eta + w + Cu, \tag{0.4}$$

where $\eta = \eta(t)$ is a stochastic process, $\overset{o}{\eta}$ is the Nelson – Gliklikh derivative [14] of process η, $w = w(t)$ is a stochastic process that responds for external influence; u is unknown control function from the Hilbert space $\mathfrak{U}$ of controls.

Supply (0.4) with initial Showalter – Sidorov condition

$$P\left(\overset{o}{\eta}^{(m)}(0) - \xi_m\right) = 0,\ m = 0, ..., n-1. \tag{0.5}$$

We investigate the optimal control problem consisting searching a pair $(\hat{\eta}, \hat{u})$, where $\hat{\eta}$ is a solution to problem (0.4), (0.5), and the control $\hat{u}$ belongs to $\mathfrak{U}_{ad} \subset \mathfrak{U}$, and satisfies the relation

$$J(\hat{\eta}, \hat{u}) = \min_{(\eta,u)} J(\eta, u). \tag{0.6}$$

Here $J(\eta, u)$ is some specially constructed penalty functional and $\mathfrak{U}_{ad}$ is a closed convex set in the Hilbert space $\mathfrak{U}$ of controls.

Let $D \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary ∂D. In the cylinder $D \times \mathbb{R}$ consider the Boussinesq – Love equation

$$(\lambda - \Delta)x_{tt} = \alpha(\Delta - \lambda')x_t + \beta(\Delta - \lambda'')x + u + \omega \tag{0.7}$$

with the boundary condition

$$x(s,t) = 0, \quad (s,t) \in \partial D \times \mathfrak{I}. \tag{0.8}$$

Equation (0.7) describes longitudinal vibrations of an elastic rod subjected to an external load with inertia taken into account; the negative values of the parameter λ do not contradict the physical meaning of the problem.

1. The spaces of "noises". Stochastic K-processes. Phase Space

Let $\Omega \equiv (\Omega, \mathcal{A}, \mathbf{P})$ be a complete probability space, $\mathbb{R}$ be the set of real numbers endowed with the Borel σ-algebra. A measurable mapping $\xi : \Omega \to \mathbb{R}$ is called *a random variable.* The set of random variables having zero expectation ($\mathbf{E}\xi = 0$) and finite variance forms a Hilbert space $\mathbf{L_2}$ with inner product $(\xi_1, \xi_2) = \mathbf{E}\xi_1\xi_2$. Let $\mathcal{A}_0$ be a σ-subalgebra of σ-algebra $\mathcal{A}$. Construct subspace $\mathbf{L_2^0} \subset \mathbf{L_2}$ of random variables measurable with respect to $\mathcal{A}_0$. Denote the orthoprojector by $\Pi : \mathbf{L_2} \to \mathbf{L_2^0}$. Let $\xi \in \mathbf{L_2}$, then $\Pi\xi$ is called *a conditional expectation* of the random variable ξ, and is denoted by $\mathbf{E}(\xi|\mathcal{A}_0)$.

Consider a set $\mathfrak{I} \subset \mathbb{R}$ and the following two mappings. The first one, $f : \mathfrak{I} \to \mathbf{L_2}$, associates to each $t \in \mathfrak{I}$ the random variable $\xi \in \mathbf{L_2}$. The second one, $g : \mathbf{L_2} \times \Omega \to \mathbb{R}$, associates to each pair (ξ, ω) the point $\xi(\omega) \in \mathbb{R}$. The mapping $\eta : \mathbb{R} \times \Omega \to \mathbb{R}$ given by $\eta = \eta(t, \omega) = g(f(t), \omega)$, where f and g are defined earlier, is called a *stochastic process*. Therefore, the stochastic process $\eta = \eta(t, \cdot)$ is a random variable for each fixed $t \in \mathfrak{I}$, i.e. $\eta(t, \cdot) \in \mathbf{L_2}$, and $\eta = \eta(\cdot, \omega)$ is called *a (sample) path* for each fixed $\omega \in \Omega$. The stochastic process η is called *continuous,* if all its paths are almost sure continuous (i.e. for almost all $\omega \in \Omega$ the paths $\eta(\cdot, \omega)$ are continuous). The set of continuous stochastic processes forms a Banach space, which is denoted by $\mathbf{C}(\mathfrak{I}, \mathbf{L_2})$. Fix $\eta \in \mathbf{C}(\mathfrak{I}, \mathbf{L_2})$ and $t \in \mathfrak{I}$, and denote by $\mathcal{N}_t^\eta$ the σ-algebra generated by the random variable $\eta(t)$. For brevity, $\mathbf{E}_t^\eta = \mathbf{E}(\cdot|\mathcal{N}_t^\eta)$.

Definition 1.1. Let $\eta \in \mathbf{C}(\mathfrak{I}, \mathbf{L_2})$. A random variable

$$\overset{o}{\eta} = \frac{1}{2}\left(\lim_{\Delta t \to 0+} \mathbf{E}_t^\eta \left(\frac{\eta(t + \Delta t, \cdot) - \eta(t, \cdot)}{\Delta t}\right) + \lim_{\Delta t \to 0+} \mathbf{E}_t^\eta \left(\frac{\eta(t, \cdot) - \eta(t - \Delta t, \cdot)}{\Delta t}\right)\right)$$

is called a *Nelson–Gliklikh derivative* $\overset{o}{\eta}$ *of the stochastic process* η *at point* $t \in \mathfrak{I}$, if the limits exist in the sense of the uniform metric on $\mathbb{R}$.

Let $\mathbf{C}^l(\mathfrak{I}, L_2)$, $l \in \mathbb{N}$, be a space of stochastic processes almost sure differentiable in the sense of the Nelson–Gliklikh derivative on $\mathfrak{I}$ up to order l inclusively. The spaces $\mathbf{C}^l(\mathfrak{I}, L_2)$ are called *the spaces of differentiable "noises"*. Let $\mathfrak{I} = \{0\} \cup \mathbb{R}_+$, then a well–known example [15, 16] of a vector in the space $\mathbf{C}^l(\mathfrak{I}, L_2)$ is given by a stochastic process that describes the Brownian motion in Einstein–Smoluchowski model

$$\beta(t) = \sum_{k=1}^{\infty} \xi_k \sin \frac{\pi}{2}(2k+1)t + \xi_0 t,$$

where the independent random variables $\xi_k \in \mathbf{L}_2$ are such that the variances $D\xi_k = [\frac{\pi}{2}(2k+1)]^{-2}$, $k \in \{0\} \cup \mathbb{N}$. As shown in [14], $\overset{o}{\beta}(t) = \frac{\beta(t)}{2t}, t \in \mathbb{R}_+$.

Now let $\mathfrak{V}$ be a real separable Hilbert space with orthonormal basis $\{\varphi_k\}$. Denote by $\mathfrak{V}_\mathbf{K}\mathbf{L}_2$ the Hilbert space, which is a completion of the linear span of *random variables*

$$\eta = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \xi_k \varphi_k$$

by the norm

$$\|\eta\|_{\mathfrak{V}}^2 = \sum_{k=1}^{\infty} \lambda_k \mathbf{D}\xi_k.$$

Here the sequence $\mathbf{K} = \{\lambda_k\} \subset \mathbb{R}_+$ is such that $\sum\limits_{k=1}^{\infty} \lambda_k < +\infty$, $\{\xi_k\} \subset \mathbf{L}_2$ is a sequence of random variables. The elements of $\mathfrak{V}_\mathbf{K}\mathbf{L}_2$ will be called random $\mathbf{K}$-variables. Note that for existence of a random $\mathbf{K}$-variable $\eta \in \mathfrak{V}_\mathbf{K}\mathbf{L}_2$ it is enough to consider a sequence of random variables $\{\xi_k\} \subset \mathbf{L}_2$ having uniformly bounded variances, i.e. $\mathbf{D}\xi_k \le const,\ k \in \mathbb{N}$.

Next, consider interval $\mathfrak{I} = (\varepsilon, \tau) \subset \mathbb{R}$. Mapping $\eta : (\varepsilon, \tau) \to \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2$ given by

$$\eta(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\xi_k(t)\varphi_k,$$

where the sequence $\{\xi_k\} \subset \mathbf{C}(\mathfrak{I}, L_2)$, is called a *$\mathfrak{V}$-valued continuous stochastic* **K**-*process*, if the series on the right-hand side converges uniformly on any compact in $\mathfrak{I}$ in the norm $\|\cdot\|_{\mathfrak{V}}$, and paths of process $\eta = \eta(t)$ are almost sure continuous. Continuous stochastic **K**-process $\eta = \eta(t)$ is called *continuously Nelson–Gliklikh differentiable on* $\mathfrak{I}$, if the series

$$\overset{o}{\eta}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \overset{o}{\xi}_k(t)\varphi_k$$

converges uniformly on any compact in $\mathfrak{I}$ in the norm $\|\cdot\|_{\mathfrak{V}}$, and paths of process $\overset{o}{\eta}=\overset{o}{\eta}(t)$ are almost sure continuous. Let $\mathbf{C}(\mathfrak{I}, \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2)$ be a space of continuous stochastic **K**-processes, and $\mathbf{C}^l(\mathfrak{I}, \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2)$ be a space of continuously differentiable up to order $l \in \mathbb{N}$ stochastic **K**-processes. An example of a stochastic **K**-process, which is continuously differentiable up to any order $l \in \mathbb{N}$ inclusively, is a Wiener **K**-process [15, 16]

$$W_{\mathbf{K}}(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)\varphi_k,$$

where $\{\beta_k\} \subset \mathbf{C}^l(\mathfrak{I}, L_2)$ is a sequence of Brownian motions on $\mathbb{R}_+$. Similarly, if $\mathfrak{G}$ is a real separable Hilbert space with orthonormal basis $\{\varphi_k\}$, the spaces $\mathbf{C}(\mathfrak{I}, \mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$ and $\mathbf{C}^l(\mathfrak{I}, \mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$, $l \in \mathbb{N}$, are constructed. Note also that spaces $\mathbf{C}^l(\mathfrak{I}, L_2)$, $\mathbf{C}^l(\mathfrak{I}, \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2)$ and $\mathbf{C}^l(\mathfrak{I}, \mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$, $l \in \mathbb{N}$, are called *the spaces of differentiable* **K**-*"noises"* [15].

2. Stochastic Sobolev type equations of high order with relatively p-bounded operators

By $\vec{B}$ denote the pencil formed by operators $B_{n-1}, \ldots B_0$. The sets $\rho^A(\vec{B}) = \{\mu \in \mathbb{C} : (\mu^n A - \mu^{n-1}B_{n-1} - \ldots - \mu B_1 - B_0)^{-1} \in \mathcal{L}(\mathfrak{G}_{\mathbf{K}}\mathbf{L_2}, \mathfrak{V}_{\mathbf{K}}\mathbf{L_2})\}$ and $\sigma^A(\vec{B}) = \mathbb{C}\backslash\rho^A(B)$ are called an A-resolvent set and an A-spectrum of pencil of operators $\vec{B}$. The operator-function $R^A_\mu(\vec{B}) = (\mu^n A - \mu^{n-1}B_{n-1} - \ldots - \mu B_1 - B_0)^{-1}$ with the domain $\rho^A(\vec{B})$ is called an A-resolvent of pencil $\vec{B}$.

Definition 2.1. The pencil $\vec{B}$ is called polynomially bounded with respect to operator A (or polynomially A-bounded) if there exists a constant $a \in \mathbb{R}_+$ such that for each $\mu \in \mathbb{C}$ the inequality $|\mu| > a$ implies the inclusion $R^A_\mu(\vec{B}) \in \mathcal{L}(\mathfrak{G}_{\mathbf{K}}\mathbf{L_2}, \mathfrak{V}_{\mathbf{K}}\mathbf{L_2})$.

Introduce an additional condition

$$\int_\gamma \mu^k R^A_\mu(\vec{B})d\mu \equiv 0, \quad k = \overline{0, n-2}. \tag{2.1}$$

Construct the set $\sigma^A_n(\vec{B}) = \{\mu \in \mathbb{C} : \mu^n \in \sigma^A(\vec{B})\}$; it is compact in $\mathbb{C}$ due to the compactness of the A-spectrum $\sigma^A(\vec{B})$ of pencil $\vec{B}$. If the pencil $\vec{B}$ is polynomially

A-bounded, and condition (A) is satisfied, then the following operators

$$P = \frac{1}{2\pi i}\int\limits_\gamma \mu^{n-1}R^A_\mu(\vec{B})Ad\mu \in \mathcal{L}(\mathfrak{V}_\mathbf{K}\mathbf{L}_2), Q = \frac{1}{2\pi i}\int\limits_\gamma \mu^{n-1}AR^A_\mu(\vec{B})d\mu \in \mathcal{L}(\mathfrak{G}_\mathbf{K}\mathbf{L}_2)$$

are projectors. Here, $\gamma = \{\mu \in \mathbb{C} : |\mu| = r, r^n > a\}$. Put $\mathfrak{V}^0_\mathbf{K}\mathbf{L}_2(\mathfrak{V}^1_\mathbf{K}\mathbf{L}_2) = \ker P(\operatorname{im} P)$, $\mathfrak{G}^0_\mathbf{K}\mathbf{L}_2(\mathfrak{G}^1_\mathbf{K}\mathbf{L}_2) = \ker Q(\operatorname{im} Q)$. Thus, the spaces $\mathfrak{V}_\mathbf{K}\mathbf{L}_2$ and $\mathfrak{G}_\mathbf{K}\mathbf{L}_2$ can be decomposed into direct sums $\mathfrak{V}_\mathbf{K}\mathbf{L}_2 = \mathfrak{V}^0_\mathbf{K}\mathbf{L}_2 \bigoplus \mathfrak{V}^1_\mathbf{K}\mathbf{L}_2$ and $\mathfrak{G}_\mathbf{K}\mathbf{L}_2 = \mathfrak{G}^0_\mathbf{K}\mathbf{L}_2 \bigoplus \mathfrak{G}^1_\mathbf{K}\mathbf{L}_2$, whereas $\mathfrak{V}^0_\mathbf{K}\mathbf{L}_2 \supset \ker A$. By $A^k(B^k_m)$ define the restriction of operator $A(B_m)$ onto $\mathfrak{V}^k_\mathbf{K}\mathbf{L_2}$, $m = \overline{0, n-1}$, $k = \overline{0,1}$.

Lemma 2.2. *The operators* $A^k(B^k_m) \in \mathcal{L}(\mathfrak{V}^k_\mathbf{K}\mathbf{L_2}; \mathfrak{G}^k_\mathbf{K}\mathbf{L_2}), k = \overline{0,1}, m = \overline{0, n-1}$*; moreover, there exist the operators* $(B^0_0)^{-1} \in \mathcal{L}(\mathfrak{G}^0_\mathbf{K}\mathbf{L_2}; \mathfrak{V}^0_\mathbf{K}\mathbf{L_2})$ *and* $(A^1)^{-1} \in \mathcal{L}(\mathfrak{G}^1_\mathbf{K}\mathbf{L_2}; \mathfrak{V}^1_\mathbf{K}\mathbf{L_2})$.

Construct the operators $H_0 = (B^0_0)^{-1}A_0, H_m = (B^0_0)^{-1}B^{n-m}_0 \in \mathcal{L}(\mathfrak{V}^0_\mathbf{K}\mathbf{L_2}), m = \overline{1, n-1}$, $S_m = (A^1)^{-1}B^1_m \in \mathcal{L}(\mathfrak{V}^1_\mathbf{K}\mathbf{L_2}), m = \overline{1, n-1}$.

Definition 2.3. Introduce the family of operators $\{K^1_q, K^2_q, \ldots, K^n_q\}$ as follows:

$$K^s_0 = \mathbb{O}, s \neq n, K^n_0 = \mathbb{I},$$
$$K^1_1 = H_0,\ K^2_1 = -H_{n-1}, \ldots, K^s_1 = -H_{n+1-s}, \ldots, K^n_1 = -H_1,$$
$$K^1_q = K^n_{q-1}H_0,\ K^2_q = K^1_{q-1} - K^n_{q-1}H_{n-1}, \ldots,\ K^s_q = K^{s-1}_{q-1} - K^n_{q-1}H_{n+1-s}, \ldots,$$
$$K^n_q = K^{n-1}_{q-1} - K^n_{q-1}H_1, q = 2, 3, \ldots$$

Definition 2.4. The point ∞ is called:

*(*i) *a removable singular point of the* A-resolvent of pencil $\vec{B}$, if $K^1_1 = K^2_1 = \ldots = K^n_1 \equiv \mathbb{O}$;

*(*ii) *a pole* of order $p \in \mathbb{N}$ of the A-resolvent of pencil $\vec{B}$, if $K^s_p \not\equiv \mathbb{O}$, for some s, but $K^s_{p+1} \equiv \mathbb{O}$ for any $s = \overline{1, n}$;

*(*iii) *an essentially singular point* of the A-resolvent of pencil $\vec{B}$, if $K^n_p \not\equiv \mathbb{O}$ for any $p \in \mathbb{N}$.

Consider the linear stochastic Sobolev type equation of higher order (0.2), where the absolute term w will be indicated later. Supplement (0.2) with the initial Showalter – Sidorov condition (0.5) which is the generalization of the condition (0.3) [3] and has advantages over the Cauchy condition in the case of Sobolev type equations. In addition to (0.5), we will consider *the weakened* (in the sense of S.G. Krein) *Showalter – Sidorov condition*

$$\lim_{t\to 0+} P\left(\overset{o}{\eta}{}^{(m)}(t) - \xi_m\right) = 0,\ m = 0, ..., n-1. \tag{2.2}$$

The K-random process $\eta \in \mathbf{C}^n(\mathfrak{I}, \mathfrak{G}_\mathbf{K}\mathbf{L}_2)$ is called *a classical solution of equation* (0.2), if a.s. all its trajectories satisfy equation (0.2) for some K-random process $w \in \mathbf{C}(\mathfrak{I}, \mathfrak{G}_\mathbf{K}\mathbf{L}_2)$. The solution $\eta = \eta(t)$ of (0.2) is called *the classical solution* of problem (0.2), (2.2) if a.s. condition (2.2) is also fulfilled. The classical solutions of problems (0.2), (0.5) and (0.2), (0.3) are defined analogously.

Consider firstly problem (0.3) for the homogeneous equation

$$A\overset{o}{\eta}{}^{(n)} = B_{n-1}\overset{o}{\eta}{}^{(n-1)} + \ldots + B_0\eta. \tag{2.3}$$

In this case (and only in this case) consider $\mathfrak{I} = \mathbb{R}$.

Definition 2.5. The mapping $V^{\bullet} \in C^{\infty}(\mathbb{R}; \mathcal{L}(\mathfrak{V}_{\mathbf{K}}\mathbf{L_2}))$ is called *a propagator* of equation (2.3), if for all $v \in \mathfrak{V}_{\mathbf{K}}\mathbf{L_2}$ the vector-function $\eta(t) = V(t)v$ is a solution of (2.3).

Theorem 2.6. *Let the pencil $\vec{B}$ be polynomially (A,p)-bounded, and condition (A) be satisfied. Then, the family of operators*

$$V_m(t) = \frac{1}{2\pi i}\int\limits_{\gamma} R^A_{\mu}(\vec{B})(\mu^{n-m-1}A - \mu^{n-m-2}B_{n-1} - ... - B_{m+1})e^{\mu t}d\mu, \ m = \overline{0, n-1},$$

define the propagators of equation (2.3).

Lemma 2.7. $V_m \in C^{\infty}(\mathbb{R}; \mathcal{L}(\mathfrak{V}_{\mathbf{K}}\mathbf{L_2}; \mathfrak{V}^1_{\mathbf{K}}\mathbf{L_2}))$, $(V_m(t))^{(l)}_t\Big|_{t=0} = \mathbb{O}$ *for* $m \neq l$ *and* $(V_m(t))^{(m)}_t\Big|_{t=0} = P$, *where P is the projector in $\mathfrak{V}_{\mathbf{K}}\mathbf{L_2}$ on $\mathfrak{V}^1_{\mathbf{K}}\mathbf{L_2}$ along $\mathfrak{V}^0_{\mathbf{K}}\mathbf{L_2}$.*

Definition 2.8. The set $\mathfrak{P} \subset \mathfrak{V}_{\mathbf{K}}\mathbf{L_2}$ is called *the phase space* of equation (2.3) if

(i) a.s. every trajectory of the solution $\eta = \eta(t)$ lies in $\mathfrak{P}$ pointwise, i.e. $\eta(t) \in \mathfrak{P}$ a.s. for all $t \in \mathbb{R}$;

(ii) for all random variables $\xi_m \in L_2(\Omega; \mathfrak{P})$, $m = 0, 1, \ldots, n-1$, there exists a unique solution $\eta \in C^n(\mathfrak{I}, \mathfrak{V}_{\mathbf{K}}\mathbf{L_2})$ of (0.3), (2.3).

Theorem 2.9. *If the pencil $\vec{B}$ is polynomially A-bounded, condition (A) is satisfied and ∞ is a pole of order $p \in \{0\} \cup \mathbb{N}$ of A-resolvent, then the phase space of equation (2.3) coincides with the image of projector P.*

Corollary 2.10. *Under the conditions of theorem 2.9 the solution of (0.3), (2.3) is the Gaussian K-random process if the random variables ξ_m, $m = 0, 1, \ldots, n-1$, are Gaussian.*

Lemma 2.11. *Let the pencil $\vec{B}$ be polynomially (A,p)-bounded, and condition (A) be satisfied. Then for all independent random variables $\xi_m \in \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2$, $m = 0, 1, \ldots, n-1$, there exists a.s. a unique solution $\eta \in C^n(\mathfrak{I}, \mathfrak{V}_{\mathbf{K}}\mathbf{L_2})$ of (0.5), (2.3), represented in the form $\eta(t) = \sum\limits_{m=0}^{n-1} V_m(t)\xi_m$, $t \in \mathbb{R}$. If in addition ξ_m, $m = 0, 1, \ldots, n-1$ take values only in $\mathfrak{V}^1_{\mathbf{K}}\mathbf{L_2}$, then this solution is a unique solution of (0.3), (2.3).*

Go back to equation (0.2) and notice that now $\mathfrak{I} = [0, \tau)$. Let the K-random process $w = w(t), t \in [0, \tau)$ be such that

$$(\mathbb{I} - Q)w \in \mathbf{C}^{p+n}(\mathfrak{I}, \mathfrak{G}^0_{\mathbf{K}}\mathbf{L}_2) \text{ and } Qw \in \mathbf{C}(\mathfrak{I}, \mathfrak{G}^1_{\mathbf{K}}\mathbf{L}_2), \tag{2.4}$$

then the K-random process

$$\eta(t) = -\sum_{q=0}^{p} K^n_q (B^0_0)^{-1} \overset{o}{w}{}^{(q)}(t) + \int\limits_0^t V_{n-1}(t-s)(A^1)^{-1}Qw(s)ds \tag{2.5}$$

is a unique classical solution of (0.5), (0.2) with $\xi_m \in \mathfrak{V}^0_{\mathbf{K}}\mathbf{L_2}$, $m = 0, ..., n-1$.

Theorem 2.12. *Let the pencil $\vec{B}$ be polynomially (A,p)-bounded, and condition (A) be satisfied. For any K-random process $w = w(t)$ satisfying (2.4), and for all independent random variables $\xi_m \in \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2,\ m = 0, 1, \ldots, n-1$, independent with w, there exists a.s. a unique solution $\eta \in \mathbf{C}^n(\mathfrak{I}, \mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$ of (0.2), (0.5), represented in the form*

$$\eta(t) = \sum_{m=0}^{n-1} V_m(t)\xi_m - \sum_{q=0}^{p} K_q^n (B_0^0)^{-1} \overset{o}{w}{}^{(q)}(t) + \int_0^t V_{n-1}(t-s)(A^1)^{-1}Qw(s)ds. \tag{2.6}$$

However, "white noise" $w(t) = \overset{o}{W}_K(t) = (2t)^{-1}W_K(t)$ does not satisfy condition (2.4), so it cannot stand on the right-hand side of (0.2). One approach to solving this problem is proposed in [8, 17]. To use this approach, convert the second term on the right-hand side of (2.5) as follows:

$$\begin{aligned}\int_\varepsilon^t V_{n-1}(t-s)(A^1)^{-1}Q\overset{o}{W}_K(s)ds = -V_{n-1}(t-\varepsilon)(A^1)^{-1}QW_K(\varepsilon)+\\ +\int_\varepsilon^t \tfrac{d}{dt}V_{n-1}(t-s)(A^1)^{-1}W_K(s)ds = -V_{n-1}(t-\varepsilon)(A^1)^{-1}QW_K(\varepsilon)+\\ +\int_\varepsilon^t \tilde{V}_{n-2}(t-s)(A^1)^{-1}W_K(s)ds,\end{aligned} \tag{2.7}$$

where $\tilde{V}_{n-2}(t) = \frac{1}{2\pi i}\int_\gamma \mu R_\mu^A(\vec{B})Ae^{\mu t}d\mu$. This integration by parts makes sense for any $\varepsilon \in (0,t), t \in \mathbb{R}_+$ due to definition of the Nelson – Gliklikh derivative. Letting $\varepsilon \to 0$ in (2.7) we get

$$\int_0^t V_{n-1}(t-s)(A^1)^{-1}Q\overset{o}{W}_K(s)ds = \int_0^t \tilde{V}_{n-2}(t-s)(A^1)^{-1}W_K(s)ds. \tag{2.8}$$

Corollary 2.13. *Let the pencil $\vec{B}$ be polynomially (A,p)-bounded, and condition (A) be satisfied, $W_k \in \mathbf{C}(\mathfrak{I}, \mathfrak{G}^1_{\mathbf{K}}\mathbf{L}_2)$. Let $\mathfrak{I} \subset \overline{\mathbb{R}}_+$. For all independent random variables $\xi_m \in \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2,\ m = 0, 1, \ldots, n-1$, independent from W_K, there exists a.s. a unique solution $\eta \in \mathbf{C}^n(\mathfrak{I}, \mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$ of the problem (0.5) for the equation*

$$A\overset{o}{\eta}{}^{(n)} = B_{n-1}\eta^{n-1} + \ldots + B_0\eta + \overset{o}{W}_K, \tag{2.9}$$

given by

$$\eta(t) = \sum_{m=0}^{n-1} V_m(t)\xi_m + \int_0^t \tilde{V}_{n-2}(t-s)(A^1)^{-1}QW_K(s)ds.$$

Theorem 2.14. *Let the pencil $\vec{B}$ be polynomially (A,p)-bounded, and condition (A) be satisfied. For all random variables $\xi_m \in \mathfrak{V}_{\mathbf{K}}\mathbf{L_2}$, independent from W_K,* *there exists a.s. unique solution $\eta = \eta(t)$ of (2.2), (2.8) given by*

$$\eta(t) = \sum_{m=0}^{n-1} V_m(t)\xi_m + \int_0^t \tilde{V}_{n-2}(t-s)(A^1)^{-1}QW_K(s)ds -$$

$$-\sum_{q=0}^{p} K_q^n (B_0^0)^{-1}(\mathbb{I}-Q) \overset{o}{W}{}_K^{(q+1)}(t).$$

3. Strong solutions

Let $L_2(\mathfrak{I};\mathfrak{V}_\mathbf{K}\mathbf{L_2})$ be a space of stochastic processes whose paths are square-integrable on $\mathfrak{I}$.

Definition 3.1. A vector function

$$\eta \in H^n(\mathfrak{V}_\mathbf{K}\mathbf{L_2}) = \{\eta \in L_2(\mathfrak{I};\mathfrak{V}_\mathbf{K}\mathbf{L_2}) : \overset{o}{\eta}{}^{(n)} \in L_2(\mathfrak{I};\mathfrak{V}_\mathbf{K}\mathbf{L_2})\}$$

is called a *strong solution* of equation (0.2), if it a.s. turns the equation to identity almost everywhere on interval $(0,\tau)$. A strong solution $\eta = \eta(t)$ of equation (0.2) is called a *strong solution to problem* (0.2), (0.5) if condition (0.5) a.s. holds.

This is well defined by virtue of the continuity of the embedding $H^n(\mathfrak{V}_\mathbf{K}\mathbf{L_2}) \hookrightarrow C^{n-1}(\mathfrak{I};\mathfrak{V}_\mathbf{K}\mathbf{L_2})$. The term "strong solution" has been introduced to distinguish a solution of equation (0.2) in this sense from the solution (2.6), which is usually said to be "classical". Note that the classical solution (2.6) is also a strong solution to problem (0.2), (0.5).

Let us construct the spaces

$$H^{p+n}(\mathfrak{G}_\mathbf{K}\mathbf{L_2}) = \{v \in L_2(\mathfrak{I};\mathfrak{G}_\mathbf{K}\mathbf{L_2}) : \overset{o}{v}{}^{(p+n)} \in L_2(\mathfrak{I};\mathfrak{G}_\mathbf{K}\mathbf{L_2}), p \in \{0\}\cup\mathbb{N}\}.$$

The space $H^{p+n}(\mathfrak{G}_\mathbf{K}\mathbf{L_2})$ is a Hilbert space with inner product

$$[v,w] = \sum_{q=0}^{p+n} \int_0^\tau \left\langle v^{(q)}, w^{(q)} \right\rangle_{\mathfrak{G}_\mathbf{K}\mathbf{L_2}} dt.$$

Let $w \in H^{p+n}(\mathfrak{G}_\mathbf{K}\mathbf{L_2})$. Introduce the operators

$$A_1 w(t) = -\sum_{q=0}^{p} K_q^n (B_0^0)^{-1}(\mathbb{I}-Q) \overset{o}{w}{}^{(q)}(t),$$

$$A_2 w(t) = \int_0^t V_{n-1}(t-s)(A^1)^{-1}Qw(s)ds, t \in (0,\tau)$$

and the function

$$k(t) = \sum_{m=0}^{n-1} V_m(t)\xi_m.$$

Lemma 3.2. *Let the operator B be (A,p)-bounded, $p \in \{0\}\cup\mathbb{N}$. Then*

(i) $A_1 \in \mathcal{L}(H^{p+n}(\mathfrak{G}_\mathbf{K}\mathbf{L_2}); H^n(\mathfrak{V}_\mathbf{K}\mathbf{L_2}))$;

(ii) for arbitrary $\xi_m \in \mathfrak{V}_\mathbf{K}\mathbf{L}_2$, $m = \overline{0,n-1}$ the vector function $k \in C^n([0,\tau);\mathfrak{V}_\mathbf{K}\mathbf{L_2})$;

(iii) $A_2 \in \mathcal{L}(H^{p+n}(\mathfrak{G}_\mathbf{K}\mathbf{L_2}); H^n(\mathfrak{V}_\mathbf{K}\mathbf{L_2}))$.

Theorem 3.3. *Let the operator B be (A,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. For any K-random process $w = w(t)$ satisfying (2.4), and for all independent random variables $\xi_m \in \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2$, $m = 0, 1, \ldots, n-1$, independent from w, there exists a.s. a unique strong solution to problem (0.2), (0.5).*

4. Optimal control

Let $\mathfrak{U}$ be a real separable Hilbert space with orthonormal basis $\{\varphi_k\}$. Consider the Showalter – Sidorov problem (0.5) for linear inhomogeneous Sobolev type equation with additive "noise" (0.4).

Introduce the control space

$$\overset{o}{H}{}^{p+n}(\mathfrak{U}) = \{u \in L_2(0,\tau;\mathfrak{U}) : u^{(p+n)} \in L_2(0,\tau;\mathfrak{U}), u^{(q)}(0) = 0, q = \overline{0,p}\},$$

$p \in \{0\} \cup \mathbb{N}$. It is a Hilbert space with inner product

$$[v,w] = \sum_{q=0}^{p+n} \int_0^\tau \left\langle v^{(q)}, w^{(q)} \right\rangle_{\mathfrak{U}} dt.$$

In the space $\overset{o}{H}{}^{p+n}(\mathfrak{U})$ we single out a closed convex subset $\overset{o}{H}{}^{p+n}_{\partial}(\mathfrak{U})$, which will be called *the set of admissible controls.*

Definition 4.1. A vector function $\hat{u} \in \overset{o}{H}{}^{p+n}_{\partial}(\mathfrak{U})$ is called an *optimal control of solutions to problem* (0.4), (0.5), if relation (0.6) holds.

We need to prove the existence of a unique control $\hat{u} \in \overset{o}{H}{}^{p+n}_{\partial}(\mathfrak{U})$, minimizing the penalty functional

$$J(\eta,u) = \mu \sum_{q=0}^{n} \int_0^\tau || \overset{o}{\eta}{}^{(q)} - \overset{o}{\tilde{\eta}}{}^{(q)} ||^2_{\mathfrak{V}_{\mathbf{K}}\mathbf{L}_2} dt + \nu \sum_{q=0}^{p+n} \int_0^\tau \left\langle N_q \overset{o}{u}{}^{(q)}, \overset{o}{u}{}^{(q)} \right\rangle_{\mathfrak{U}} dt. \quad (4.1)$$

Here $\mu, \nu > 0$, $\mu + \nu = 1$, $N_q \in \mathcal{L}(\mathfrak{U})$, $q = 0, 1, \ldots, p+n$, are self-adjoint positively defined operators, and $\tilde{\eta}(t)$ is the target state of the system.

Theorem 4.2. *Let the operator B be (A,p)-bounded, $p \in \{0\} \cup \mathbb{N}$. Then for arbitrary $w \in H^{p+n}(\mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$ there exists a unique optimal control of solutions to problem (0.4), (0.5).*

Proof. By Theorem 5, for arbitrary $w \in H^{p+n}(\mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$, $\xi_m \in \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2$, and $u \in H^{p+n}(\mathfrak{U})$ there exists a unique strong solution $\eta \in H^n(\mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$ to problem (0.4), (0.5), given by

$$\eta(t) = (A_1 + A_2)(w + Cu)(t) + k(t), \quad (4.2)$$

where the operators A_1, A_2 and the vector function k are defined in Lemma 3.2.

Fix $w \in H^{p+n}(\mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$ and $\xi_m \in \mathfrak{V}_{\mathbf{K}}\mathbf{L}_2$, and consider function (4.2) as a mapping $D : u \mapsto \eta(u)$. The mapping $D : H^{p+n}(\mathfrak{U}) \to H^n(\mathfrak{G}_{\mathbf{K}}\mathbf{L}_2)$ is continuous. Therefore, the penalty functional depends only on $u : J(\eta,u) = J(u)$.

We write out the functional (4.1) in the form

$$J(u) = \mu \, \|\eta(t,u) - \tilde{\eta}\|^2_{H^n(\mathfrak{V}_{\mathbf{K}}\mathbf{L}_2)} + \nu[v,u],$$

where $v^{(q)}(t) = N_q u^{(q)}(t),\ q = 0, \dots, p+n$. Hence it follows that

$$J(u) = \pi(u,u) - 2\lambda(u) + \mu \left\| \tilde{\eta} - \eta(t,0) \right\|^2_{H^n(\mathfrak{Y}_\mathbf{K}\mathbf{L_2})},$$

where

$$\pi(u,u) = \mu \left\| \eta(t,u) - \eta(t,0) \right\|^2_{H^n(\mathfrak{Y}_\mathbf{K}\mathbf{L_2})} + \nu[v,u]$$

is a bilinear continuous coercive form on $H^{p+n}(\mathfrak{U})$ and

$$\lambda(u) = \mu \left\langle \tilde{\eta} - \eta(t,0), \eta(t,u) - \eta(t,0) \right\rangle_{H^n(\mathfrak{Y}_\mathbf{K}\mathbf{L_2})}$$

is a linear continuous form on $H^{p+n}(\mathfrak{U})$. Therefore, the assumptions of theorem in [18, p. 13, Theorem 1.1] are satisfied. The proof of the theorem is complete.

5. The Boussinesq – Love Equation

Consider the Boussinesq – Love equation (0.7) with the boundary condition (0.8). To reduce problem (0.7), (0.8) to equation

$$A\overset{o}{\eta}^{(2)} = B_1 \overset{o}{\eta} + B_0\eta + \omega + Cu \tag{5.1}$$

put

$$\mathfrak{V} = \{x \in W_2^{l+2}(D) : x(s) = 0, s \in \partial D\}, \quad \mathfrak{G} = W_2^l(D),$$

where $W_2^l(D)$ is the Sobolev space. Define the operators A, B_1 and B_0 by the formulas $A = \lambda - \Delta$, $B_1 = \alpha(\Delta - \lambda')$, $B_0 = \beta(\Delta - \lambda'')$, and $C = \mathbb{I}$. For each $l \in \{0\} \cup \mathbb{N}$, $A, B_1, B_0 \in \mathcal{L}(\mathfrak{V}_\mathbf{K}\mathbf{L_2}, \mathfrak{G}_\mathbf{K}\mathbf{L_2})$.

By $\{\lambda_k\}(= \sigma(\Delta))$ denote the eigenvalues of the Dirichlet problem for the Laplace operator Δ numbered in nonascending order with multiplicities, and by $\{\varphi_k\}$ denote the corresponding eigenfunctions orthonormal in $L^2(D)$. Since $\{\varphi_k\} \subset C^\infty(D)$, we have

$$\mu^2 A - \mu B_1 - B_0 = \sum_{k=1}^{\infty} [(\lambda - \lambda_k)\mu^2 + \alpha(\lambda' - \lambda_k)\mu + \beta(\lambda'' - \lambda_k)] < \varphi_k, \cdot > \varphi_k,$$

where $< \cdot, \cdot >$ is the inner product on $L^2(D)$.

The following assertion was proved in [19].

Lemma 5.1. *Let one of the following conditions be satisfied:*
(i) $\lambda \notin \sigma(\Delta)$;
(ii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda \neq \lambda')$;
(iii) $(\lambda \in \sigma(\Delta)) \wedge (\lambda = \lambda') \wedge (\lambda \neq \lambda'')$.
Then the pencil $\vec{B}$ *is polynomially A-bounded.*

In cases (i), (iii) the pencil $\vec{B}$ is $(A,0)$-bounded. In case (ii) ∞ is essentially singular point. Therefore we exclude thus case from further considerations.

Consider the Showalter – Sidorov theorem

$$\sum_{\lambda \neq \lambda_k} < \varphi_k, \xi(s,0) - \xi_0(s) > \varphi_k = 0, \quad \sum_{\lambda \neq \lambda_k} < \varphi_k, \xi_t(s,0) - \xi_1(s) > \varphi_k = 0. \tag{5.2}$$

By Theorem 3.3, we have the following assertion.

Theorem 5.2. *Let one of assumptions (i) and (iii) of Lemma 5.1 be satisfied. Then for arbitrary* $\xi_0, \xi_1 \in \mathfrak{V}_{\mathbf{K}}\mathbf{L_2}$ *and* $\omega \in H^2(\mathfrak{V}_{\mathbf{K}}\mathbf{L_2})$*, there exists a unique strong solution of problem (0.8), (5.2) for the equation*

$$(\lambda - \Delta) \overset{o}{\xi}_{tt} = \alpha(\Delta - \lambda') \overset{o}{\xi}_t + \beta(\Delta - \lambda'')\xi + \omega,$$

which can be represented in the form

$$\xi(t) = -\sum_{\lambda=\lambda_k} \frac{< \varphi_k, \omega(t) >}{\beta(\lambda'' - \lambda_k)} \varphi_k +$$

$$+ \sum{}' \left[\frac{\mu_k^1(\lambda - \lambda_k) + \alpha(\lambda' - \lambda_k)}{(\lambda - \lambda_k)(\mu_k^1 - \mu_k^2)} e^{\mu_k^1 t} + \frac{\mu_k^2(\lambda - \lambda_k) + \alpha(\lambda' - \lambda_k)}{(\lambda - \lambda_k)(\mu_k^2 - \mu_k^1)} e^{\mu_k^2 t} \right] < \varphi_k, \xi_0 > \varphi_k +$$

$$+ \sum{}' \frac{e^{\mu_k^1 t} - e^{\mu_k^2 t}}{(\mu_k^1 - \mu_k^2)} < \varphi_k, \xi_1 > \varphi_k + \sum{}' \int_0^t \frac{e^{\mu_k^1(t-\tau)} - e^{\mu_k^2(t-\tau)}}{(\lambda - \lambda_k)(\mu_k^1 - \mu_k^2)} < \varphi_k, \omega(\tau) > \varphi_k d\tau,$$

$$t \in (-T, T),$$

where the prime on the sums indicates the absence of terms with indices k *such that* $\lambda = \lambda_k$.

Let us proceed to the optimal control problem. We fix $0 < \tau < T$ and introduce the control space

$$H^2(\mathfrak{U}) = \{u \in L_2(0, \tau; \mathfrak{U}) : \ddot{u} \in L_2(0, \tau; \mathfrak{U})\}.$$

In the space $H^2(\mathfrak{U})$, we single out a closed convex subset $H^2_\partial(\mathfrak{U})$, which will be the set of admissible controls.

Theorem 5.3. *Let the assumptions of Theorem 5.2 be satisfied. Then for arbitrary* $\xi_0, \xi_1 \in \mathfrak{V}_{\mathbf{K}}\mathbf{L_2}$ *there exists a unique optimal control* $\hat{u} \in H^2_\partial(\mathfrak{U})$ *of solutions of problem (0.8), (5.2) for Eq. (0.7) minimizing the functional (4.1).*

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