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STOCHASTIC PROCESSES ASSOCIATED WITH NONLINEAR PDE SYSTEMS WITH CROSS-DIFFUSION

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ABSTRACT. We construct a stochastic model underlying a system of nonlinear parabolic equations with cross diffusion similar to the SKT system. This model is presented as a system of stochastic differential equations with coefficients depending on distributions of the SDE solution functionals. We state conditions of existence and uniqueness of the derived SDE system and the original PDE system and study their solutions. We discuss as well stability and instability of stationary solutions of the Cauchy problem for the SKT type system.

Introduction

Stochastic nature of nonlinear parabolic equations attracted the attention of various authors [1], [2], [3] and others since the seminal paper by McKean [4]. See the recent monograph [5] for a number of references.

On the other hand, in spite of the fact that systems of nonlinear parabolic equations with cross-diffusion arise in various fields such as physics, chemistry, population dynamics and so on, they attracted mainly the attention of people working in PDE theory (see [6], [7] and references there). Though terminology of these works includes such terms as diffusion and cross-diffusion there are not so many results concerning stochastic diffusion processes underlying these systems. Let us mention papers [8], [9], where the authors describe stochastic dynamics of a population composed of M competitive types of individuals, assuming that each type has its own spatial and ecological dynamics depending on the spatial and genetic characteristics of the whole population. Assuming that the motion of each individual (of a given type) is driven by a diffusion process on R^d whose coefficients depend on different individuals around they show that when a size of population goes to infinity, the dynamics of the population is described by a system of non-local parabolic equations with cross diffusion. The limiting system is called a mean-field type model. Thus, the stochastic dynamics of a population is described by an individual-based system.

In contrast to this approach we are interested in stochastic model for the limiting parabolic system itself or in stochastic model underlying a mean field model.

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Stochastic models underlying a number of systems with cross-diffusion and systems of the reaction-diffusion type were constructed in our previous papers [10]-[12].

In this paper we derive a suitable stochastic model associated with the SKT system [13] in one dimensional space, study a solution of this system and investigate effect of the cross diffusion on instability of the stochastic model.

1. Stochastic counterparts of a PDE system with cross-diffusion

Consider a population consisting of individuals belonging to two distinct species (or two classes in the same species) of individuals which interact through diffusion and competition.

A system to describe a dynamics of this kind was proposed by Shigesada, Kawasaki, and Teramoto [13], to study spatial segregation of interacting species.

We are looking for concentrations (number density) $u_m(t,y) \geq 0$ of individuals of two competing species, $m = 1, 2,\ y \in R, t \in [0,T]$.

The dynamics of the population is described by the the Cauchy problem for a system of parabolic equations

$$\frac{\partial u_m}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}[M^2(y,u)u_m] + c_m(y,u)u_m, \quad u_m(0,y) = u_{0m}(y), \tag{1.1}$$

where

$$M_m^2(y,u) = \alpha_m + \alpha_{m1}u_1 + \alpha_{m2}u_2 > 0, \quad c_m(y,u) = c_m - c_{m1}u_1 - c_{m2}u_2$$

and $\alpha_m, \alpha_{mq}, c_m, c_{mq}, m, q = 1, 2$, are positive constants.

Our aim is to construct a weak and a mild solutions of the Cauchy problem (1.1) in terms of a stochastic model associated with this system.

To this end we interpret this system as a system of nonlinear forward Kolmogorov equations for densities of some Markov processes and derive SDEs for these Markov processes.

Let us mention that a specific feature of systems of this type is the following. Although we interpret a solution of this system as a family of densities of some Borel probability measures on R the nonlinearity of the system prevents to write immediately the system of PDEs for required measures themselves. Equations of the type (1.1) are called singular equations. We consider here the Cauchy problem for a system of regular equations of the form

$$\frac{\partial \mu_m}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}[M_m^2[y,\mu]\mu_m] + c_m[y,\mu]\mu_m, \quad \mu_m(0,dy) = \mu_{0m}(dy), \tag{1.2}$$

where $M_m^2[y,\mu] = \alpha_m + \sum_{q=1}^2 \int_R \alpha_{mq}\rho(y-x)\mu_q(dx), \quad c_{mq}[y,\mu] = c_m -$
$- \sum_{q=1}^2 \int_R c_{mq}\rho(y-x)\mu_q(dx)$ and ρ is a mollifier.

To state the problem strictly we need a number of functional spaces. Let $C_b(R)$ be the space of bounded continuous real functions on R and $C_0(R)$, $C_0^\infty(R)$ be the spaces continuous and infinitely differentiable real functions on R^d with compact supports respectively. Let $W^{r,p}(R)$ be the Sobolev space of order r , where r, p are integers. For Euclidian space X we denote by $\mathcal{S}(X)$ Schwartz space consists of C^∞ - functions whose derivatives (including the function itself) decay at infinity faster than any power and by $S'(X)$ its dual called the space of tempered distributions. Recall that $\mathcal{D}(X) = C_0^\infty(X) \subset \mathcal{S}(X)$.

We say that bounded Borel measures $\mu_m, m = 1,2$ are weak solutions of the Cauchy problem (1.2), if for any $h \in \mathcal{S}([0,T) \times R)$, $t \in [0,T]$ integral equalities

$$\int_0^T \int_R \mu_m(s,dy) \left[\frac{\partial h(s,y)}{\partial s} + \frac{1}{2} M_m^2[y,\mu(s)] \frac{1}{2} \frac{\partial^2}{\partial y^2} h(s,y) \right] ds + \tag{1.3}$$

$$+ \int_0^T \int_R \mu_m(s,dy) h(s,y) c_m[y,\mu(s)] ds + \int_R u_m(0,dy) h(y)$$

hold.

Our aim is to construct stochastic models for solutions of the Cauchy problem (1.2). To this end we fix a probability space $(\Omega, \mathcal{F}, P)$ and denote by $w_m(t) \in R, m = 1,2$ independent Wiener processes defined on this probability space. Next we consider a system of stochastic equations

$$d\xi_m(t) = M_m(\xi_m(t), u(t, \xi_m(t))) dw_m(t), \quad \xi_m(0) = \xi_{0m}, \quad m = 1,2, \tag{1.4}$$

where ξ_{0m} are independent random variables with distribution $P(\xi_{0m} \in dy) = \mu_{0m}(dy)$ which do not depend on $w_m(t)$.

Since $u_m(t,y)$ are unknown functions we have to find additional relations to derive a closed system. A natural way is to use the Feynman-Kac formula that is add to (1.4) a relation

$$\int_R h(y) \mu_m(t,dy) = E\left[h(\xi_m(t)) \exp\left\{ \int_0^t c_m(\xi_m(s), u(s, \xi_m(s))) ds \right\} \right], \tag{1.5}$$

where $\mu_m(t,dy) = u_m(t,y)dy$.

In contrast to the case of backward Kolmogorov equations we require that (1.5) would be valid for any test function $h \in C_b(R)$.

To justify this approach assume that (1.5) is valid for any $h \in C_b(R), t \in [0,T]$ and let $c_m(x,u)$ be bounded. Then for each $m = 1,2$ and t the right hand side of (1.5) is a bounded linear functional on the space $C_b(R)$ and hence by the Riesz theorem it defines a unique measure $\mu_m(t,dy)$.

Assume that there exist Markov processes $\xi_m(t), m = 1,2$, satisfying (1.4), denote by $P_m^u(0,x,t,dy) = P\{\xi_m(t) \in dy | \xi_m(0) = x\}$ their transition probabilities and set $\mu_m(t,dy) = P\{\xi_m(t) \in dy\} = u_m(t,y)dy$.

We say that functions $u_m(t,\cdot)$ define a mild solution of (1.1), if for any $h \in C_b(R)$, $t \in [0,T]$ and $m = 1,2$. integral identities

$$\int_R h(y) u_m(t,y) dy = \int_R h(y) \int_R u_{0m}(x) dx P_m^u(0,x,t,dy) +$$

$$+ \int_0^t \int_R \left(\int_R h(y) P_m^u(s,z,t,dy) \right) c_m(z, u(s,z)) u_m(s,z)) dz ds \tag{1.6}$$

hold.

As a result we may consider a closed system (1.4), (1.6).

To make the problem more tractable we consider a mollification of the system (1.1) that is the system (1.2). To this end we choose a mollifier $\rho : R \to R$, that is a compactly supported smooth function on R such that $\int_R \rho(x)dx = 1$ such that

$\lim_{\epsilon\to 0}\rho_\epsilon(x) = \lim_{\epsilon\to 0}\epsilon^{-d}\rho(\frac{x}{\epsilon}) = \delta(x)$ and set $[\rho^*\mu](t,y) = \int_R \rho(y-x)\mu(t,dx) = \int_R \rho(y-x)u(t,x)dx$ provided that the measure $\mu(t,dy)$ has a density $u(t,y)$.

Next we consider a system

$$d\xi_m(t) = M_m(\xi_m(t), v(t,\xi_m(t)))dw_m(t), \quad \xi_m(0) = \xi_{0m}, \tag{1.7}$$

with $P\{\xi_m(0)\in dy\} = \mu_{0m}(dy)$, and

$$v_m(t,y) = E\left[\rho(y-\xi_m(t))\exp\left\{\int_0^t c_m(\xi_m(s), v(s,\xi_m(s)))ds\right\}\right]. \tag{1.8}$$

If we assume that there exists a solution to (1.7), (1.8), then we can verify that functions v_m are connected with a weak solution u_m of the Cauchy problem

$$\frac{\partial u_m}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}[M_m^2[y,\rho * u]u_m] + c_m[y,\rho * u]u_m, \quad u_m(0,y) = u_{0m}(y), \tag{1.9}$$

by a relation $v_m(t,y) = \rho * u_m(t,y) = \int_R \rho(y-x)u_m(t,x)dx, \quad m=1,2.$

Note that (1.9) can be treated a a system for densities of bounded Borel measures $\mu_m(t,dy)$ satisfying the Cauchy problem

$$\frac{\partial \mu_m}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[M^2[y,\rho*\mu]\mu_m\right] + c_m[y,\rho*\mu])\mu_m, \quad \mu_m(0,dy) = \mu_{0m}(dy), m=1,2. \tag{1.10}$$

To investigate the system (1.7), (1.8) introduce some additional notations. Denote by $\mathcal{C} = C([0,T];R)$ the space of continuous real-valued functions on $[0,T]$ and consider the process $\xi_m(t,\omega)$ satisfying (1.7) as a value $\omega_m(t)$ of $\omega_m\in\mathcal{C}$ at $t\in[0,T]$. We denote by $\mathcal{F}$ and $\mathcal{F}_t$ $0\le t\le T$ the smallest σ-algebras generated by $\{\xi_m(\tau), 0\le\tau\le T\}$ and $\{\xi_m(\tau), 0\le\tau\le t\}$ respectively. In addition we denote by $\kappa_m(d\omega)$ a probability measure on $\mathcal{C}$ generated by the canonical process $\xi_m(t,\omega)$.

Let $\mathcal{P}_r(\mathcal{C})$ denote the space of Borel probability measures on $\mathcal{C}$ with finite moment of order $r=1,2$ and

$$\mathcal{W}_T^r(\mu,\mu_1) = \inf\left\{\left(\int_{R\times R}\|x-y\|^r\gamma(dx,dy)\right)^{\frac{1}{r}} |\Pi_x^*\gamma = \nu_1, \Pi_y^*\gamma = \nu_2\right\},$$

where $\Pi_x(x,y) = x, \Pi_y(x,y) = y$ for all $x,y\in R$ and $\|\mu\|_r = [\int_R |y|^r\mu(dy)]^{\frac{1}{r}}$.

Consider the space $\mathcal{B}_p = B([0,\infty);\mathcal{P}_r)$ of all $\mathcal{P}_r$ -valued Borel measurable functions $\mu(\cdot)$ satisfying an estimate $\sup_{t\in[0,T]}\|\mu(t)\|_r < \infty$ for all $T<\infty$.

Now we may consider (1.9) and relations

$$u_m^{\kappa_m}(t,y) = E\left[\rho(y-\xi_m(t))\exp\left\{\int_0^t c_m[\xi_m(s), u^{\kappa_m}]ds\right\}\right] \tag{1.11}$$

as a closed system of equations, where κ_m is a distribution of the canonical process $\xi_m(t)$.

Applying the Ito formula we can verify that if $\xi_m(t), u_m^{\kappa_m}(t,y)$ satisfy (1.7), (1.11) then the functions $v_m(t,y) = [\rho * u_m](t,y) = \int_R \rho(y-x)u_m(t,x)dx$ satisfy the Cauchy problem for a system

$$\frac{\partial v_m(t,y)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}(M_m^2(t,y,[\rho * v])v_m) + c_m(t,y,[\rho * v])v_m, \quad v_m(0,y) = v_{0m}.$$

Formally, if we choose $\rho = \delta$ in (1.11) and consider a solution $(\xi_m, u_m^{\kappa_m}(t,y))$ of (1.7), (1.11), we can verify that $u_m^{\kappa_m}$ satisfy (1.1). To this end we apply the Ito formula to the function $\phi(x)$ and the process $\xi_m(t)$ and verify that the functions $u_m^{\kappa_m}$, which are the densities of the measures μ_m, defined in (1.5) satisfy (1.1) in the weak sense.

To investigate (1.7), (1.11). we rewrite (1.11) in the form

$$v_m^{\kappa_m}(t,y) = \int_{\mathcal{C}_m} \rho(y - \omega_m(t)) exp\left\{\int_0^t c_m[\xi(s), v]ds\right\} d\kappa_m(\omega_m), \tag{1.12}$$

where κ_m is the law of the process $\xi_m(\cdot)$ on the canonical space $\mathcal{C}_m = \mathcal{C}$. Note that (1.12) describe the links between the probability measures κ_m on the canonical spaces $\mathcal{C}_m$ and the functions $v_m^{\kappa_m}$ defined on $[0,T]\times R$. Sometimes to make notations more short we omit upper index κ_m.

When $c_m[y,\mu] = c_m(y) - \sum_{q=1}^d c_{mq}[y,\mu_q] \equiv 0$ we deduce from (1.12) that $v_m(t,y) = \int_R \rho(y-x)\mu_m(t,dx) = [\rho * \mu](t,y)$, where $\mu_m(t)$ is the marginal law of the process $\xi(t) = (\xi_1(t), \xi_2(t)) \in R \times R$. In a general case we consider the system of equations

$$\xi_m(t) = \xi_{0m} + \int_0^t M_m(\xi_m(s), u^{\kappa_m}(s, \xi_m(s)))dw_m(s), \tag{1.13}$$

$$u_m^{\kappa_m}(t,y) = \int_{\mathcal{C}} \rho(y - \xi_m(t,\omega))G(t, \xi_m(\omega)), u^{\kappa})\kappa_m(d\omega), \quad \mathcal{L}(\xi_m) = \kappa_m, \tag{1.14}$$

where $G(t, \xi_m(\omega), \mu) = \exp\left\{\int_0^t c_m[\xi_m(s,\omega), \mu^{\kappa}]ds\right\}$ and ξ_m are the canonical processes $\xi_m : \mathcal{C} \to \mathcal{C}$ defined by $\xi_m(t,\omega_m) = \omega_m(t)$, $t \geq 0, \omega_m \in \mathcal{C}$ and $\kappa_m(d\omega) = \mathcal{L}(\xi_m)$.

Along with (1.13),(1.14) we consider equations

$$\xi_m(t) = \xi_{0m} + \int_0^t M_m[\xi_m(s), \rho * \mu^{\kappa}]dw_m(s), \tag{1.15}$$

where μ^{κ_m} is defined by

$$\int_R \phi(y)\mu_m^{\kappa}(t,dy) = \int_{\mathcal{C}} \phi(\xi_m(t,\omega))G(t, \xi_m(\omega), [\rho * \mu^{\kappa}]]\kappa_m(d\omega), \tag{1.16}$$

for all $\phi \in C_b(R)$ and $\mathcal{L}(\xi_m) = \kappa_m$.

Let us verify that equations (1.15), (1.16) and (1.13), (1.14) are equivalent.

Theorem 1.1. *Assume that functions α_{mk}, c_{mk} are smooth functions of polynomial growth. The existence of a solution to the system (1.15), (1.16) is equivalent to the existence of a solution to (1.13),(1.14). In other words, given a solution (ξ_m, μ_m^{κ}) of (1.15), (1.16) we define a couple (ξ_m, u_m^{κ}) satisfying (1.13), (1.14), where $u_m^{\kappa} = \rho * \mu_m^{\kappa}$. On the other hand given (ξ_m, u_m^{κ}) satisfying (1.13), (1.14) there exist measures μ_m^{κ} such that (ξ_m, μ_m^{κ}) solve (1.15), (1.16).*

Proof. Assume that there exists a couple $(\xi_m(t), u_m^{\kappa}(t,y))$ satisfying (1.15), (1.16) and $u_m^{\kappa}(t,y)$ is bounded for $t \in [0,T]$. As far as $\rho \in L_1(R)$, the Fourier transform $F(u_m^{\kappa})(t,z)$ of the function u_m^{κ} has the form

$$F(u_m^{\kappa})(t,z) = F(\rho)(z)g_m^{\kappa}(t,z), \tag{1.17}$$

where

$$g_m^\kappa(t,z) = \int_{\mathcal{C}} e^{-iz\xi_m(t,\omega)} e^{\int_0^t c_m[\xi_m(s,\omega),[\rho*\mu^\kappa]]ds}\kappa_m(d\omega).$$

By Lebesgue dominated convergence theorem we deduce that the functions $g_m^\kappa(t,z)$ are continuous in z for $t\in[0,T]$. In addition $c_m(u_1,u_2)$ are bounded for bounded $u_m, m=1,2$. Remind that given a sequence of complex numbers β_k and a sequence $x=(x_1,\dots x_k), x_k\in R$ one has for all $y\in R$ and any integer n

$$\sum_{k=1}^n\sum_{q=1}^n \beta_k\bar\beta_q e^{-iz[x_k-x_q]} = \left(\sum_{k=1}^n \beta_k e^{-izx_k}\right)\overline{\left(\sum_{q=1}^n \beta_k\ e^{-iz\cdot x_q}\right)} = |\sum_{k=1}^d e^{-iz\cdot x_k}|^2,$$

and thus g_m^κ is nonnegative definite. Then by the Bochner theorem there exists a unique measure $\mu_m(t)$ on R such that

$$g_m^\kappa(t,z) = \frac{1}{2\pi}\int_R e^{-izy}\mu_m^\kappa(t,dy). \tag{1.18}$$

The finite non-negative Borel measure $\mu_m^\kappa(t,dy)$ can be treated as an element of the tempered Schwartz distribution space $\mathcal{S}'(R)$ such that $F^{-1}(g_m^\kappa)(t)=\mu_m^\kappa(t)$ and for any $\phi\in\mathcal{S}(R)$ the estimate $|\int_R\phi(y)\mu_m^\kappa(t,dy)|\le\|\phi\|_\infty\mu(t,R)<\infty$ holds.

As a result we get from (1.17) and (1.18) that $F(u_m^\kappa)(t,\cdot)=F(\rho)F(\mu_m^\kappa(t))$ and hence

$$u_m^\kappa(t,\cdot)=\rho*\mu_m^\kappa(t). \tag{1.19}$$

To verify that (1.18) yields (1.16) we consider $\langle\phi,\mu_m(t)\rangle=\int_R\phi(y)\mu_m(t,dy), \phi\in\mathcal{S}(R)$ and note that

$$\langle\phi,\mu_m(t)\rangle = \langle\phi, F^{-1}(g_m^\kappa)\rangle = \langle F^{-1}(\phi), g_m^\kappa\rangle =$$

$$= \int_R F^{-1}(\phi)(z)\left(\int_{\mathcal{C}} e^{-iz\xi_m(t,\omega)}e^{\int_0^t c_m(\xi_m^\kappa(s,\omega),u)ds}\kappa_m(d\omega)\right)dz.$$

Next applying the Fubini theorem we have

$$\langle\phi,\mu_m(t)\rangle = \int_{\mathcal{C}}\left(\int_R F^{-1}(\phi)(z)e^{-iz\xi_m(t,\omega)}dz\right)e^{\int_0^t c_m([\xi_m^\kappa(s,\omega)),\rho*\mu]ds}\kappa_m(d\omega)$$

and finally taking into account (1.18) we obtain

$$\langle\phi,\mu_m(t)\rangle = \int_{\mathcal{C}}\phi(\xi_m(t,\omega))e^{\int_0^t c_m([\xi_m^\kappa(s,\omega)),\rho*\mu]ds}\kappa_m(d\omega)$$

which coincides with (1.16).

To prove that (1.16) yields (1.14) assume that $(\xi_m(t),\mu_m)$ satisfy (1.15), (1.16) and set $u_m^\kappa(t,y)=[\rho*\mu_m^\kappa](t,y)$. Since μ_m^κ is finite, we deduce that (1.14) holds setting in (1.16) $\phi=\rho(x-\cdot)$. □

2. Existence and uniqueness theorem

Consider a system

$$\frac{\partial \mu_m}{\partial t} = \frac{1}{2}\Delta[M_m^2[y,\mu]\mu_m] + c_m[y,\mu]\mu_m, \quad \mu_{0m} = \mu_{0m}. \tag{2.1}$$

One may show that one can treat a system (2.1) as a scalar equation with respect to a measure $\mu(t)$ defined on the Borel σ-algebra $\mathcal{B}(R \times R)$ of the space $R \times R$ [14].

Our aim is to construct $\mu(t) \in \mathcal{M}$ such that $\mu(t) = (Law(\xi_1(t)), Law(\xi_2(t)))$, where $\xi_m(t), m = 1,2$, satisfy

$$d\xi_m(t) = M[\xi_m(t), \mu(t)]dw_m(t), \quad \xi_m(0) = \xi_{0m}. \tag{2.2}$$

We say that condition **C 2.1** holds if

$$|M_m[y,\mu]| \leq K_\alpha[1 + \|y\|], \quad |c_m[y,\mu]| \leq C.$$

As above, to make the system closed we add to (2.2) relations

$$u_m(t,y) = E[\rho(y - \xi_m(t))\exp\left\{\int_0^t c_m([\xi_m(s),\mu(s)]ds\right\}.$$

Assume that we are given a probability measure $\kappa \in \mathcal{P}(\mathcal{C} \times \mathcal{C})$ with marginals κ_m and consider equations

$$u_m(t,y) = \int_{\mathcal{C}} \rho(y - \xi_m(t,\omega))e^{\int_0^t c_m[\xi_m(s,\omega),\rho * u]ds}\kappa_m(d\omega). \tag{2.3}$$

We may treat the relations (2.3) as a possibility to associate a probability measure κ on $\mathcal{C}\times\mathcal{C}$ with marginals $\kappa_m, m = 1,2$ to the function $u(x,y) = (u_1(x), u_2(y))$.

Lemma 2.1. *Given a probability measure $\kappa \in \mathcal{P}_2(\mathcal{C}\times\mathcal{C})$ with marginals $\kappa_m(t), m = 1,2$ assume that there exists a unique solution $u = (u_1, u_2)$ of (2.4). Then functions $u_m(t)$ are bounded over a certain interval $[0,T]$.*

Proof. Denote by $K_m^u(t)$ a function for which an estimate $sup_{y\in R}|u_m(t,y)| \leq K_m^u(t), m = 1,2,$ holds and set $K^u(t) = \max(K_1^u(t), K_2^u(t))$. Consider a process $\eta_m(t)$

$$\eta_m(t) = 1 + \int_0^t c_m(u_1(s,\xi_m(s)), u_2(s,\xi_m(s)))\eta_m(s)ds$$

and note that

$$|\eta_m(t)| \leq 1 + \int_0^t [c_m + c_{m1}|u_1(s,\xi_m(s))| + c_{m2}|u_2(s,\xi_m(s))|]|\eta_m(s)|ds$$

and by the Gronwall lemma we get

$$|\eta_m(t)| \leq exp\{\int_0^t [c_m + c_{m1}|u_1(s,\xi_m(s))| + c_{m2}|u_2(s,\xi_m(s))|]ds.$$

This allows to estimate $\sup_y |u_m(t,y)| = K_m^u(t)$

$$K_m^u(t) \leq K_\rho \sum_{m=1}^2 |\eta_m(t)| \leq K_\rho \sum_{m=1}^2 \exp\{\int_0^t [c_m + c_{m1}K_1^u(s) + c_{m2}K_2^u(s)]ds.$$

Set $C = max(c_1, c_2)$, $\beta = \max_{m,q\in\{1,2\}} c_{mq}$ and $\gamma(t) = K_1^u(t) + K_2^u(t)$. Then

$$\gamma(t) \le 2K_\rho \exp\{\int_0^t [C + \beta\gamma(s)]ds.$$

Note that this integral equation is equivalent to the Cauchy problem

$$\frac{d\gamma(t)}{dt} = [C + \gamma(t)]\gamma(t), \quad \gamma(0) = 2K_\rho$$

The solution of this Cauchy problem has the form

$$\gamma(t) = \frac{2CK_\rho e^{Ct}}{C + 2K_\rho - 2K_\rho e^{Ct}}$$

and we can see that if $t \in [0, T_1]$ where $T_1 < \frac{1}{C}\ln\left(1 + \frac{C}{2K_\rho}\right)$ then $\gamma(t) \le K^u < \infty$. □

Theorem 2.2. *Given a probability measure $\kappa \in \mathcal{P}(\mathcal{C}) \times \mathcal{P}(\mathcal{C})$ there exists a unique bounded Lipschitz continuous solution $u = (u_1, u_2)$ of (2.3).*

Proof. Let us start with a priori estimates of solutions to the system (2.3). Namely, let us fix a measure κ on the space $\mathcal{C}^2 = C([0,t]; R) \times C([0,t]; R)$ with marginals κ_m and assume that there exists functions u_m satisfying (2.1).

Denote by $\mathcal{N}_m = \{\zeta : \|\zeta\|_{\infty,1} = \int_{\mathcal{C}} \sup_{t\in[0,T]} \|\zeta(t,\omega)\|\kappa_m(d\omega) < \infty\}$. Denote by $\mathcal{N} = \{\zeta : \|\zeta\|_{\infty,1} = \int_{\mathcal{C}\times\mathcal{C}} \sup_{t\in[0,T]} \|\zeta(t,\omega)\|\kappa(d\omega) < \infty\}$.

The spaces $(\mathcal{N}_m, \|\cdot\|_{\infty,1})$ are Banach spaces. Consider an equivalent norm $\|\cdot\|_{\infty,1}^K$ in the space $\mathcal{N}_m$ of the form $\|Y\|_{\infty,1}^K = E^{\kappa_m}[sup_{s\le T} e^{-Ks}\|Y(s)\|]$.

We define a map $\Gamma_m^{\kappa_m} : \mathcal{C} \to C([0,T] \times R; R)$ by

$$\Gamma_m^{\kappa_m}(\zeta)(t,y) = \int_{\mathcal{C}} \rho(y - \xi_m(t,\omega))G_m(t,\zeta)\kappa_m(d\omega),$$

with $G_m(t,\zeta) = \exp\left\{\int_0^t c_m(u(s,\xi_m(s,\omega)))ds\right\}$ and a map

$$\Psi_m : C([0,T] \times R; R) \to \mathcal{C}_m \quad \text{by} \quad \Psi_m(f)(t,\omega) = f(t,\omega_m(t)).$$

Hence $\Psi_m \circ \Gamma^{\kappa_m} : \mathcal{C} \to \mathcal{C}_m$.

Set $\Psi = (\Psi_1, \Psi_2)$ and $\Psi \circ \Gamma : \mathcal{C}^2 \to \mathcal{C}^2$ with $\Psi_m \circ \Gamma_m : \mathcal{C}_m \to \mathcal{C}_m$ and rewrite (2.1) in the form

$$u^\kappa = (\Gamma^\kappa \circ \Psi)(u^\kappa) \quad \text{with} \quad u_m^\kappa = (\Gamma_m^{\kappa_m} \circ \Psi_m)(u^\kappa). \tag{2.4}$$

Assume that there exists a fixed point $Y^\kappa \in \mathcal{C}^2$ for $\Psi \circ \Gamma^\kappa$ such that $Y_m^{\kappa_m} \in \mathcal{C}_m$. Then we have $\Psi \circ \Gamma^\kappa(Y^\kappa) = Y^\kappa$. Finally, choosing $u^\kappa = \Gamma^\kappa(Y^\kappa)$ we obtain

$$u_m^{\kappa_m} = (\Gamma_m^{\kappa_m} \circ \Psi_m)(u^{\kappa_m}). \tag{2.5}$$

Hence $u^\kappa = (u_1^{\kappa_1}, u_2^{\kappa_2})$ satisfies to $\Gamma^\kappa(Y^\kappa) = \Gamma^\kappa \circ \Psi \circ \Gamma^\kappa(Y^\kappa)$ and solves (2.1).

In order to prove the uniqueness of a solution to (2.1) we assume on the contrary that there exist two solutions $u^\kappa = (\Gamma^\kappa)(Y^\kappa)$ and $g^\kappa = (\Gamma^\kappa)(Z^\kappa)$ of this system and for $Y^\kappa = u(t,y), Z^\kappa = g(t,y)$ for any $(t,y) \in [0,T] \times R^d$ we estimate the difference

$$\lambda_m(t) = [\Gamma_m^{\kappa_m}(Z_m^{\kappa_m}) - \Gamma_m^{\kappa_m}(Y_m^{\kappa_m})](t,y) =$$

$$= \int_{\mathcal{C}_m} \rho(y - \xi_m(t,\omega))[G_m(t, Z^{\kappa_m}(\omega)) - G_m(t, Y^{\kappa_m}(\omega))\kappa_m(d\omega)].$$

Note that the process $\eta_m(t) = G_m(t, Z^{\kappa_m}(\omega))$ satisfies an equation

$$\eta_m(t) = 1 + \int_0^t [c_m - c_{m1}(u_1^{\kappa_m}(s, \xi_m(s))) - c_{m2}(u_2^{\kappa_m}(s, \xi_m(s)))]\eta_m(s)ds$$

and estimate a difference $E\|\eta_m(t) - \eta_m^1(t)\|$, where $\eta_m^1(t) = G_m(t, Y^{\kappa_m}(\omega))$,

$$E\|\eta_m(t) - \eta_m^1(t)\| \leq$$

$$\leq L_c \int_0^t e^{Ks} E[\|Z_1^{\kappa_m}(s) - Y_1^{\kappa_m}(s)\| + \|Z_2^{\kappa_m}(s) - Y_2^{\kappa_m}(s)\|]e^{-Ks}ds$$

Keeping in mind that $\sup_{x\in R} |\rho(x)| \leq K_\rho$ we deduce that

$$\begin{aligned}\lambda_m(t) \leq K_\rho L_c e^{K_u T} \int_{\mathcal{C}} \int_0^t e^{Ks}[\|Z_1^{\kappa_m}(s,\omega) - Y_1^{\kappa_m}(s,\omega)\| + \\ + \|Z_2^{\kappa_m}(s,\omega) - Y_2^{\kappa_m}(s,\omega)\|]e^{-Ks}dsd\kappa_m(\omega) \leq \\ \leq K_\rho L_c e^{K_u T} E \int_0^t e^{Ks} \sup_{\theta \leq t} e^{-K\theta}\|Y^{\kappa_m}(\theta) - Z^{\kappa_m}(\theta)\|ds \leq \\ \leq K_\rho e^{K_u T} L_c \frac{e^{Kt} - 1}{K} \left[\|Z_1^{\kappa_m} - Y_1^{\kappa_m}\|_{\infty,1}^K + \right. \\ \left. + \|Z_2^{\kappa_m} - Y_2^{\kappa_m}\|_{\infty,1}^K\right].\end{aligned} \tag{2.6}$$

Finally, we obtain that

$$\sum_{m=1}^2 \lambda_m(t) \leq K_\rho e^{K_u T} L_c \frac{e^{Kt} - 1}{K} \sum_{m=1}^2 \left[\|Z_1^{\kappa_m} - Y_1^{\kappa_m}\|_{\infty,1}^K + \|Z_2^{\kappa_m} - Y_2^{\kappa_m}\|_{\infty,1}^K\right].$$

Next since

$$(\Psi \circ \Gamma^\kappa)(Z^\kappa)(t) = \Gamma^\kappa(Z^\kappa)(t, \xi(t)), (\Psi \circ \Gamma^\kappa)(Y^\kappa)(t) = \Gamma^\kappa(Y^\kappa)(t, \xi(t))$$

we get

$$\begin{aligned}E\left[\sum_{m=1}^2 \sup_{0\leq t\leq T} e^{-Kt}|\Psi_m \circ \Gamma_m^{\kappa_m}(Z^\kappa)(t) - \Psi_m \circ \Gamma_m^\kappa(Y^{\kappa_m})(t)|\right] = \\ = E\left[\sum_{m=1}^2 \sup_{0\leq t\leq T} e^{-Kt}|\Gamma_m^{\kappa_m}(Z^\kappa)(t, \xi_m(t)) - \Gamma_m^{\kappa_m}(Y^{\kappa_m})(t, \xi_m(t))|\right] = \\ = \sum_{k=1}^2 \|\Gamma_m^{\kappa_m}(Z^{\kappa_m}) - \Gamma_m^{\kappa_m}(Y^{\kappa_m})\|_{\infty,1} \leq \\ \leq 2K_\rho e^{K_u T} L_c \frac{1}{K} \left[\|Z_1^{\kappa_m} - Y_1^{\kappa_m}\|_{\infty,1}^K + \|Z_2^{\kappa_m} - Y_2^{\kappa_m}\|_{\infty,1}^K\right].\end{aligned}$$

As a result we get that

$$\sum_{m=1}^2 \|\lambda_m\|_{\infty,1}^K \leq K_\rho e^{K_u T} L_c \frac{1}{K} \sum_{m=1}^2 \|\lambda_m\|_{\infty,1}^K$$

and choosing $K > K_\rho L_c$ we deduce that $\Psi \circ \Gamma^\kappa$ is a contraction in the product space $(\mathcal{C}, \|\cdot\|_{\infty,1}^K) \times (\mathcal{C}, \|\cdot\|_{\infty,1}^K)$. Hence by Banach fixed point theorem we prove the existence a nd uniqueness of a solution to (2.1) under the condition that measures $\kappa_m, m = 1, 2$, are fixed. Moreover the functions $u_m(t, y)$ are Lipschitz continuous. This yields that there exists a unique bounded solution $\xi_m^\kappa(t)$ of the SDE (2.1). In addition by Burkholder-Davies-Gundy inequality we may verify that $E[sup_{t\le T_1}|\xi_m(t)|^2] \le C[1 + E|\xi_{0m}|^2]$. Thus the law $N_m(\kappa_m) = \mathcal{L}(\xi_m)$ belongs to $\mathcal{P}_2(\mathcal{C})$. This yields that the process $\xi(t) = (\xi_1(t), \xi_2(t))$ has the law $N(\kappa) = \mathcal{L}(\xi)$ which belongs to $\mathcal{P}_2(\mathcal{C} \times \mathcal{C}.)$ □

Next we consider N as a map acting in $\mathcal{P}_2(\mathcal{C} \times \mathcal{C})$. To prove that N is a contraction in the Wasserstein metrics consider solutions u^κ and u^{κ_1} of (1.16) corresponding to measures κ and κ^1.

Below we need an auxiliary estimate.

Lemma 2.3. *Assume* **C 1** *holds and the measures* κ_m *are given. Then the estimate*

$$\|u^\kappa(t, y) - u^{\kappa_1}(t, y_1)\|^2 \le M_{K_u}(t)[\|y - y_1\|^2 + \mathcal{W}_{T_1}(N(\kappa), N(\kappa^1)) \tag{2.7}$$

holds.

Proof. Let $\kappa \in \mathcal{P}_2(\mathcal{C} \times \mathcal{C})$. Then

$$\|u^\kappa(t, y) - u^{\kappa_1}(t, y_1)\|^2 \le \beta_1 + \beta_2,$$

where $\beta_1^2 = 2\|u^\kappa(t, y) - u^\kappa(t, y_1)\|^2$, $\beta_2^2 = 2\|u^\kappa(t, y_1) - u^{\kappa_1}(t, y_1)\|^2$. Note that

$$\beta_1^2 \le \int_{\mathcal{C}} |\rho(y - \xi_m(t, \omega)) - \rho(y_1 - \xi_m(t, \omega))|^2 G_m^2(t, \xi_m^{\kappa_m}, u^\kappa(\xi_m(\omega))\kappa(d\omega) \le$$

$$\le L_\rho^2 e^{2tK_c}\|y - y_1\|^2.$$

To estimate β_2 we apply the Jensen inequality to get

$$\beta_2^2 \le \int_{\mathcal{C}\times\mathcal{C}} |\rho(y_1 - \xi_m(t, \omega))G_m(t, \xi_m^{\kappa_m}, u^\kappa(\xi_m(\omega)))- \tag{2.8}$$

$$-\rho(y_1 - \xi_m(t, \omega_1))G_m(t, \xi_m^{\kappa_m}, u^{\kappa_1}(\xi_m(\omega_1)))|^2\pi(d\omega, d\omega_1)$$

for any $\pi \in \Pi(\kappa, \kappa_1)$, where Π is a set of measures with marginals κ and κ_1. Using Lipschitz continuity of ρ and boundedness of u_m and G_m we may verify that there exists a constant $M > 0$ such that an estimate

$$\beta_2^2 \le M \left[\int_{\mathcal{C}\times\mathcal{C}} [1 + t] \sup_{s\le t} \xi_m(s, (\omega) - \xi_m(s, \omega_1|^2 + \right.$$

$$\left. + \int_0^t |u^\kappa(s, \xi_m(s, \omega)) - u^{\kappa_1}(s, \xi_m(s, \omega_1))|^2 ds\right] \pi(d\omega, d\omega_1).$$

Combining the above estimates we get that

$$\zeta_m(s) = \int_{\mathcal{C}\times\mathcal{C}} ||u^\kappa(s, \xi_m(s, \omega)) - u^{\kappa_1}(s, \xi_m(s, \omega_1))|^2 ds\pi(d\omega, d\omega_1)$$

satisfy the estimate

$$\zeta_m(t) = K(t) \int_0^t \zeta(s)ds + K(t)(t + 2) \int_{\mathcal{C}\times\mathcal{C}} \sup_{s\le t} |\xi_m(s, \omega) - \xi_m(s, \omega_1)|^2 \pi(d\omega, d\omega_1)$$

for $m = 1, 2,\ t \in [0, T]$. Hence by the Gronwall lemma we obtain

$$\zeta_m(t) \leq (t+2)MK_u(t)e^{tK(t)} \int_{\mathcal{C}\times\mathcal{C}} |\sup_{s\leq t} \xi_m(s,\omega) - \xi_m(s,\omega_1)|^2 \pi(d\omega, d\omega_1). \quad (2.9)$$

Summing up the above estimates we deduce (2.7). □

Now we can prove the following statement.

Theorem 2.4. *Assume that* **C 1** *holds. Then there exists a unique solution of the system (2.4) and $u_m(t,y)$ are bounded Lipschitz continuous functions.*

Proof. By definition of the Wasserstein metrics we know that

$$\mathcal{W}_{T_1}(N(\kappa), N(\kappa^1)) \leq E[\sup_{t\leq T} \|\xi^1(t) - \xi(t)\|^2], \quad (2.10)$$

hence

$$\|u^{\kappa}(t,y) - u^{\kappa_1}(t,y_1)\|^2 \leq M_{K_u}(t)[\|y - y_1\|^2 + \mathcal{W}_{T_1}(N(\kappa), N(\kappa^1)). \quad (2.11)$$

Applying lemma 2.3 and we may deduce that

$$E[\sup_{t\leq\tau} |\xi_1(t) - \xi(t)||^2] \leq K_{M,c}\left[\int_0^{\tau} E[\sup_{s\leq t} |\xi_1(s) - \xi(s)|^2\right] dt + \int_0^{\tau} \mathcal{W}_t^2(\kappa, \kappa_1) dt$$

for any $\tau \leq T_1$ and the constant $K_{M,c}$ depending on M and c.

Thus from the Gronwall lemma we obtain

$$E[\sup_{t\leq\tau} |\xi_1(t) - \xi(t)||^2] \leq K_{M,c}e^{K_{M,c}T_1} \int_0^{T_1} \mathcal{W}_s^2(\kappa, \kappa_1) ds.$$

Finally, from (2.10) we deduce the estimate

$$\mathcal{W}_{T_1}(N(\kappa), N(\kappa^1)) \leq K_{M,c}e^{K_{M,c}T_1} \int_0^{T_1} \mathcal{W}_s^2(\kappa, \kappa_1) ds.$$

Iterating this estimate we prove that N is a contraction and by the fixed point theorem we end the proof of existence and uniqueness of a solution to (2.1). Boundedness and Lipschitz continuity of $u_m(t,y)$ were proved above. □

3. Stability and instability of cross-diffusion system solutions

Let us consider some properties of solutions to (1.1) and in particular their stability. Let in the absence of diffusion ($M_m(u) \equiv 0, m = 1, 2,$) the system (1.1) have stable constant solutions. Turing [15] discovered that if $M_1(u) = M_1 \neq M_2(u) = M_2$, it's possible for the system to have a spatially heterogeneous solution. Thus, diffusion terms, which in many cases are the stabilising factors preventing pattern formation, in this case become essential to drive pattern formation.

Assume that the system (1.1) has a stationary solution $u_m^{(s)}$ which is a constant and let $u_m(t) = u_m^{(s)} + v_m(t)$ where $v_m(t)$ is a small deviation from the stationary solution. Then $v_m(t)$ satisfy a linear system

$$\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}[A^2 v] + Bv, \quad u(0,y) = v_0(y), \quad (3.1)$$

in the neighborhood of a stationary point $u_m^{(s)}$ where

$$B = \begin{pmatrix} c_1 - 2c_{11}u_1^{(s)} - c_{12}u_2^{(s)} & c_{12}u_2^{(s)} \\ c_{21}u_1^{(s)} & c_2 - c_{22}u_2^{(s)} - c_{21}u_1^{(s)} \end{pmatrix} = \begin{pmatrix} -b_{11} & -b_{12} \\ -b_{21} & -b_{22} \end{pmatrix} \quad (3.2)$$

and

$$A^2 = \begin{pmatrix} \alpha_1 + 2\alpha_{11}u_1^{(s)} + \alpha_{22}u_2^{(s)} & \alpha_{12}u_2^{(s)} \\ \alpha_{21}u_2^{(s)} & \alpha_2 + 2\alpha_{22}u_2^{(s)} + \alpha_{21}u_1^{(s)} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (3.3)$$

Note that $u_m^{(s)}$ are also equilibrium solutions of the system

$$\frac{df_m}{dt} = c_m(f), \quad f_m(0,y) = u_{m0}(y), \quad m = 1,2. \quad (3.4)$$

Thus we can find them as solutions of the algebraic system

$$c_1(v_1,v_2) = c_1 - c_{11}u_1 - c_{12}u_2 = 0, \quad c_2(v_1,v_2) = c_2 - c_{21}u_1 - c_{22}u_2 = 0. \quad (3.5)$$

The system (3.5) has two solutions, namely $\tilde{u}^{(s)} = (0,0)$ and

$$u^{(s)} = \left(\frac{c_1c_{22} - c_2c_{12}}{det\,C}, \frac{c_2c_{11} - c_1c_{21}}{det\,C} \right) \quad \text{where} \quad C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

Let us examine a linear instability of $v_m = u_m^{(s)}$ satisfying (3.1). To this end we look for a solution to (3.1) of the form

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} exp(ikx + \lambda t), \quad (3.6)$$

where $\lambda \in R$ and $k > 0$.

To derive conditions for Turing instability we assume that (v_1^0, v_2^0) satisfying (3.1) is a linearly stable solution of the form (3.6).

We say condition **C3.1** holds if

$$a_{11} > 0, a_{22} > 0, \quad det\,A^2 = a_{11}a_{12} - a_{12}a_{21} > 0, \quad Tr\,B < 0,\, det\,B > 0.$$

The system (3.1) has a nontrivial solutions of the form (3.6) if determinant of the matrix

$$k^2A + B - \lambda I = \begin{pmatrix} k^2a_{11} - b_{11} - \lambda & k^2a_{12} - b_{12} \\ k^2a_{21} - b_{21} & k^2a_{22} - b_{22} - \lambda \end{pmatrix}$$

is equal to 0, that is

$$det(k^2A+B-\lambda I) = (k^2a_{11}-\lambda-b_{11})(k^2a_{22}-\lambda-b_{22})-(k^2a_{12}-b_{12})(k^2a_{21}-b_{21}) = 0 \quad (3.7)$$

From the last equality deduce a quadratic equation

$$\lambda^2 + p\lambda + q = 0$$

with respect to λ, where

$$p = -k^2Tr\,A + TrB, \quad q = k^4detA + k^2m(A,B) + detB = detL_k$$

and

$$m(A,B) = -a_{11}b_{22} + a_{21}b_{12} + a_{12}b_{21} - b_{11}a_{22}.$$

If (3.1) is a linearly unstable system, its solution $v(t,y)$ has to go to infinity as $t \to \infty$. This means that one of roots of (3.7) must have a positive real part or that one of the eigenvalues of the matrix $L_k = -k^2 A - B$ has a positive real part. Since $\lambda_{1,2} = \frac{-p \pm \sqrt{p^2-4q}}{2}$, one of $\lambda_{1,2}$ should have a positive real part provided $q < 0$.

By assumption $TrB < 0$, then $TrL_k = TrB - k^2 Tr\, A < 0$ is always true since by assumption $a_{11} > 0$ and $a_{22} > 0$. Hence if L_k has an eigenvalue with a positive real part then the other eigenvalue has to be a negative real one.

For instability we need an estimate $q = detL_k < 0$ for some $k > 0$. But the function $det\, L_k$ has a minimum value

$$\min_k detL_k = -\frac{m^2(A,B)}{4detA} + detB \quad \text{at critical point } k_*^2 = -\frac{m(A,B)}{2detA}. \tag{3.8}$$

Hence if $m(A,B) < 0$ and $\min_k det\, L_k < 0$, then $(u_1^{(s)}, u_2^{(s)})$ is an unstable equilibrium of (3.1). Now we can state the following result.

Theorem 3.1. *Assume that* $(u_1^{(s)}, u_2^{(s)})$ *are constant equilibrium solutions of (1.1), matrices* B *and* A *are defined by (3.2), (3.3) and* $det\, A > 0, \quad TrB < 0,\ a_{12} = a_{21} = 0$ *and* $det\, B > 0$. *Then there exists an unbounded region* $G = \{a_{11} > 0, \quad a_{22} > 0, a_{22} = \kappa a_{11}\}$, *for some* $\kappa > 0$ *such that for any* $(a_{11}, a_{22}) \in D$ *the point* $(v_1^{(s)}, v^{(s)})$ *is an unstable equilibrium solution of (3.1). If*

$$\min_{k \in R} det\, L_k = -\frac{m^2(A,B)}{4detA} + detB < 0 \tag{3.9}$$

and

$$k_*^2 = -\frac{m(A,B)}{2det\, A} > 0, \tag{3.10}$$

then $(u_1^{(s)}, u_2^{(s)})$ *is unstable equilibrium solution of (3.1) but a stable solution of the ODE (3.4).*

Theorem 3.1 gives a general conditions that imply that adding self- or cross diffusion to a dynamical system (3.4) yields an instability of the system. It allows to analyse separately the effect of adding self and cross-diffusion.

We say condition **C 3.2** holds if

$$detA > 0, a_{11} > 0, a_{22} > 0, \quad b_{11} > 0, b_{22} < 0, Tr\, B < 0, det\, B > 0.$$

In the absence of cross- diffusion we obtain conditions of classical Turing instability.

Theorem 3.2. *Assume that* $(u_1^{(s)}, u_2^{(s)})$ *is a constant stable equilibrium solution of (1.1),* **C 3.2** *holds and consider a system*

$$\frac{\partial u_m}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial y^2}[\alpha_m + \alpha_{m1} u_1] u_1 + c_m(u_1,u_2) u_m, \quad u_m(0,y) = u_{m0}(y),\ m = 1,2. \tag{3.11}$$

Then there exists an unbounded region $G_1 = \{(a_{11}, a_{22}) : a_{11} > 0, a_{22} > \theta\}$ *for some* $\theta > 0$, *such that* $(u_1^{(s)}, u_2^{(s)})$ *is an unstable equilibrium solution of (3.11).*

Proof. We consider the linearised system associated with (3.11) and deduce from (3.9) that

$$\min_{k\in R_+} detL_k = -\frac{(a_{22}b_{11}+a_{11}b_{22})^2}{2a_{11}a_{22}} \quad \text{at}\, k_*^2 = \frac{a_{22}b_{11}+a_{11}b_{22}}{2a_{11}a_{22}}.$$

Set

$$\kappa(a_{11},a_{22}) = -(a_{22}b_{11}+a_{11}b_{22})^2 + 4a_{22}a_{11}(b_{11}b_{22}-b_{12}b_{21}) =$$
$$= 2[det\, B - b_{12}b_{21}]a_{11}a_{22} - b_{22}^2a_{11}^2 - a_{22}^2b_{11}^2$$

and

$$\beta(a_{11},a_{22}) = b_{11}a_{22} + a_{11}b_{22}.$$

Since **C 3.2** holds we deduce that $\kappa(a_{11},a_{22})<0$ and $\beta(a_{11},a_{22})>0$. Set $\theta=\frac{a_{22}}{a_{11}}$, then

$$\kappa(a_{11},a_{22})=0 \quad \text{iff} \quad b_{22}^2 - 2\theta[detB - b_{12}b_{21}] + \theta^2b_{11}^2 = 0 \tag{3.12}$$

and

$$\beta(a_{11},a_{22})=0, \quad \text{iff} \quad \theta = -\frac{b_{22}}{b_{11}} = \hat\theta. \tag{3.13}$$

Keeping in mind **C 3.2** we deduce that roots $\theta_\pm$ of (3.12) have the form

$$\theta_\pm = \frac{det\, B - b_{12}b_{21} \pm \sqrt{(detB - b_{11}b_{22})^2 - b_{11}^2b_{22}^2}}{b_{11}^2}. \tag{3.14}$$

Since due to **C 3.2** we have

$$(det\, B - b_{11}b_{22})^2 - b_{11}^2b_{22}^2 = det\, B[det\, B - 2b_{11}b_{22}] > 0,$$

we deduce that $\theta_+ > \hat\theta > \theta_- > 0$. Finally, we have $\kappa(a_{11},a_{22})>0$ between the line $a_{22}=\theta_+a_{11}$ and the line $a_{22}=\theta_+a_{22}$ and $\beta(a_{11},a_{22})>0$ between the line $a_{22}=\theta a_{11}$ and the b_{22}-axis. Hence the region G_1 between the line $a_{22}=\theta_+a_{11}$ and a_{22}-axis is an unstable region that is for any $(a_{11},a_{22})\in G_1$ the solution $(u_1^{(s)},u_2^{(s)})$ is an unstable equilibrium of (3.11). □

Let us show that Turing instability can arise in a system with cross-diffusion in the case when in the corresponding system without cross diffusion there was no such instability.

Theorem 3.3. *Let $(u_1^{(s)},u_2^{(s)})$ be a constant equilibrium of (1.1) and* **C 3.2** *holds. In addition we assume that $(u_1^{(s)},u_2^{(s)})$ is a stable constant equilibrium of (3.11). Given G_1 defined in theorem 3.2 for fixed a_{11},a_{22} which are not in $\bar G_1$ there exists an unbounded region $G_2=\{a_{21},a_{12}\}$ defined by*

$$G_2 = \{(a_{21},a_{12})\in R^2 : (a_{21}b_{11}+a_{12}b_{22}) < a_{22}b_{11}+a_{11}b_{22} - 2\sqrt{det\, A}\sqrt{det\, B}\} \tag{3.15}$$

such that for any point $(a_{21},a_{12})\in G_2$ the solution $(u_1^{(s)},u_2^{(s)})$ is an unstable equilibrium solution of (1.1).

Proof. By assumption $detA > 0$ that yields $a_{11}a_{22} > a_{12}a_{21}$. Consider (a_{11}, a_{22}) inside a region between two components of the hyperbola $a_{11}a_{22} = a_{11}a_{22}$ and recall that from theorem 3.1 we know the explicit expression for $\mu_k det\, L_k$ which is attained at the point k_* (see (3.9) and (3.10)). Then keeping in mind **C 3.2** we can verify that (3.9) and (3.10) are equivalent to inequalities

$$\kappa_1(a_{21}, a_{12}) = -(a_{21}b_{11} + a_{12}b_{22} - l)^2 + 4det\, Adet\, B < 0, \tag{3.16}$$

$$\beta_1 = \gamma - a_{21}b_{12} - a_{12}b_{21} > 0, \quad \gamma = a_{22}b_{11} + a_{11}b_{22}. \tag{3.17}$$

Let us show that the equations $\kappa_1(a_{21}, a_{12}) = 0$ and $a_{12}a_{21} = a_{11}a_{22}$ define an ellipse and a hyperbola in the plane (a_{12}, a_{21}) with intersection points a, b which belong to the line defined by the equation $\beta_1(a_{21}, a_{12}) = 0$. Set

$$\Theta = a_{21}b_{12} + a_{12}b_{21}, \quad R = a_{21}b_{12} - a_{12}b_{21}, \quad a_{21} = \frac{\Theta - R}{2b_{21}}, \quad a_{12} = \frac{\Theta + R}{2b_{12}}.$$

Substituting the derived expressions for a_{12}, a_{21} into equation $\kappa_1(a_{21}, a_{12}) = 0$ we derive an equation

$$\left(1 + \frac{det\, B}{b_{12}b_{21}}\right) - 2\gamma\Theta - \frac{det\, B}{b_{12}b_{21}}R^2 - 4a_{11}a_{22}det\, B + \gamma^2 = 0. \tag{3.18}$$

Since $det\, B > 0, b_{11} > 0, b_{22} < 0$ we get $b_{12}b_{21} < 0$. Thus,

$$\frac{det\, B}{b_{12}b_{21}} < 0 \quad \text{and} \quad -\left(1 + \frac{det\, B}{b_{12}b_{21}}\right) = -\frac{b_{11}b_{22}}{b_{21}b_{12}} < 0$$

and we may rewrite (3.18) in the form

$$\frac{b_{11}b_{22}}{b_{21}b_{12}}\left(\Theta - \frac{b_{12}b_{21}\gamma}{b_{11}b_{22}}\right)^2 - \frac{det\, B}{b_{12}b_{21}}R^2 = -\gamma^2\frac{det\, B}{b_{11}b_{22}} + 4a_{11}a_{22}det\, B. \tag{3.19}$$

Since the right hand side of (3.18) is positive, the equation $\kappa_1(a_{21}, a_{12}) = 0$ defines an ellipse in (b_{12}, b_{21})-plane.

Let $a_{21}a_{12}$ be a point of the hyperbola $a_{21}a_{12} = a_{11}a_{22}$, then $\kappa_1(a_{21}, a_{12}) = 0$ if and only if $\gamma - a_{12}b_{21} - a_{21}b_{12} = 0$. Thus, the hyperbola $a_{21}a_{12} = a_{11}a_{22}$, the ellipse $\kappa_1(a_{21}, a_{12}) = 0$ and the line $\gamma - a_{12}b_{21} - a_{21}b_{12} = 0$ intersect at two points. One can easily verify that if a point a_{21}, a_{12} is outside the ellipse, then $\kappa_1(a_{21}, a_{12}) < 0$. □

Thus, we have shown that there are cases when Turing instability is the result of the addition of cross-diffusion. On the other hand if the equilibrium is lost due to self-diffusion it can be restored by adding the suitable cross-diffusion.

Theorem 3.4. *Let a couple $(u_1^{(s)}, u_2^{(s)})$ be a stable solution of (1.1), but is an unstable solution of (3.11) that is Turing unstable. Then for fixed $(a_{11}, a_{22}) \in G_1$ defined in theorem 3.2 there exists an unbounded region Q in the plane (d_{12}, d_{21}) defined by*

$$Q = \{(a_{21}, a_{12}) \in R^2 : a_{12}a_{21} < a_{11}a_{22}, \quad \kappa_1(a_{21}, a_{12}) > 0, \quad \beta_1(a_{21}, a_{12}) < 0\},$$

where κ_1 and β_1 are given by (3.16), (3.17) such that for any point $(a_{21}, a_{12}) \in Q$ the point $(u_1^{(s)}, u_2^0(s))$ is a stable equilibrium solution of (1.1).

The proof of this theorem is similar to the proof of the previous theorem. Note that the region Q is bounded by the short arc of the ellipse and the adjacent hyperbola branches and its points are on the upper -left side of the boundary.

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