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ON CONDITIONS FOR COMPLETENESS OF FLOWS GENERATED BY STOCHASTIC DIFFERENTIAL-ALGEBRAIC EQUATIONS

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ABSTRACT. This paper deals with stochastic differential-algebraic equations. Among those equations we find a class such that for equations from this class we can obtain a necessary and sufficient condition for completeness of their flows.

Introduction

The work is devoted to investigation of the problem of global in time existence of solutions of stochastic differential-algebraic equations. We deal with the equations of the form

$$\begin{cases} \widetilde{L}D_S\xi(t) = \widetilde{M}\xi(t) + \widetilde{f}(t,\xi(t)) \\ D_2\xi(t) = \bar{\Theta}(\xi) \end{cases} \tag{0.1}$$

where $\widetilde{L}$ is a degenerate matrix, $\widetilde{M}$ is a nondegenerate matrix, D_S is the symmetric mean derivative and D_2 is the quadratic mean derivative. We suppose that the matrix pencil is regular and satisfies the rank-degree condition (all definitions are given below). Among the equations of this type we find a class such that for those equations we can prove the necessary and sufficient conditions for the completeness of their flows.

The research is based on the theory of mean derivatives and conditions for completeness of flows generated by ordinary stochastic equations with mean derivatives, on the theory of matrix pencils and on the theory of ordinary differential-algebraic equations. Thus, the paper includes rather large preliminaries from all those mathematical subjects. It consists of this Introduction and 4 sections. The first one is devoted to brief introduction into the theory of mean derivatives and equations with them. In the second one we give a survey of results of conditions under which the flows generated by equations with symmetric mean derivatives are complete. Then we describe some facts about matrix pencils and, in the last section, we prove the main result of the paper.

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1. Mean derivatives

Consider the L_1 stochastic process given on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with values in $\mathbb{R}^n$. Denote by $\mathcal{N}_t^\xi$ the "present" σ-subalgebra of $\mathcal{F}$ generated by preimages of Borel sets under the mapping $\xi(t) : \Omega \to \mathbb{R}^n$. By E_t^ξ we denote the conditional expectation with respect to $\mathcal{N}_t^\xi$. Followeing Nelson [1, 2, 3] we give the definitions:

Definition 1.1. (i)The forward mean derivative $D\xi(t)$ of $\xi(t)$ at the time instant t is the L_1-random variable of the form

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\Big(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t}\Big)$$

where the limit is supposed to exist in $L_1(\Omega, \mathcal{F}, P)$ and $\Delta t \to +0$ means that $\Delta t \to 0$ $\Delta t > 0$.

(ii) The backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at the time instant t is the L_1-random variable of the form

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\Big(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t}\Big)$$

where (as well as in (i)) the limit is supposed to exist in $L_1(\Omega, \mathcal{F}, P)$ and $\Delta t \to +0$ means that $\Delta t \to 0$ $\Delta t > 0$..

Definition 1.2. The derivative $D_S = \frac{1}{2}(D + D_*)$ is called the symmetric mean derivative. The derivative $D_A = \frac{1}{2}(D - D_*)$ is called the the antisymmetric mean derivative.

Definition 1.3. $v^\xi(t) = v^\xi(t, \xi(t)) = D_S\xi(t)$ is called the current velocity of $\xi(t)$.

Definition 1.4 ([4, 5])**.**

$$D_2\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\Big(\frac{(\xi(t+\Delta t) - \xi(t))(\xi(t+\Delta t) - \xi(t))^*}{\Delta t}\Big) \qquad (1.1)$$

is called the quadratic mean derivative where $(\xi(t+\Delta t) - \xi(t))$ is considered as a column (vector in $\mathbb{R}^n$), $(\xi(t+\Delta t) - \xi(t))^*$ as a row (transposed vector) and the limit is supposed to exist in $L_1(\Omega, \mathcal{F}, P)$.

It is shown that the quadratic mean derivative takes values in the space of symmetric positive semi-definite $(n \times n)$-matrices, that we denote by $\overline{S}_+(n)$. The space of symmetric positive definite $(n \times n)$-rmatrices is denoted by $S_+(n)$ ($\overline{S}_+(n)$ is the closure of $S_+(n)$).

Let $v : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\alpha : \mathbb{R} \times \mathbb{R}^n \to \bar{S}_+(n)$ be Borel measurable mappings. The system of the form

$$\begin{cases} D_S\xi(t) = v(t, \xi(t)) \\ D_2\xi(t) = \alpha(t, \xi(t)) \end{cases} \qquad (1.2)$$

is called the first order equation with current velocities.

Consider a smooth matrix field $\alpha_0 = (\alpha^{ij}(0, x)) \in S_+$. Since it is smooth and not degenerate, there exists the smooth matrix field $(\alpha_{ij}(0, x))$ of inverse matrices. Since, in addition, by construction those matrices are not degenerate, we can consider this field as a new Riemannian metric on $\mathbb{R}^n$. We shall denote this metric by $\alpha_0(\cdot, \cdot)$.

Theorem 1.5 ([6]). *Let $v : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\alpha : [0,T] \times \mathbb{R}^n \to S_+(n)$ be mappings smooth jointly in all variables. Let also the relations*

$$\|v(t,x)\| < K(1 + \|x\|) \tag{1.3}$$

$$tr\ \alpha(t,x) < K(1 + \|x\|^2) \tag{1.4}$$

and

$$\|\bar{\Xi}(x)\| < K(1 + \|x\|) \tag{1.5}$$

for a certain $K > 0$ hold, where $\bar{\Xi}(x)$ is the vector fields with coordinate representation $\frac{\partial \alpha^{ij}}{\partial x^j}\frac{\partial}{\partial x^i}$. Let ξ_0 be a random element with values in $\mathbb{R}^n$, whose probabilistic density ρ_0 with respect to the volume form Λ_{α_0} of metric $\alpha_0(\cdot,\cdot)$ on $\mathbb{R}^n$ is smooth and nowhere equals zero. Then for the initial condition $\xi(0) = \xi_0$ equation (1.2) has a solution well-defined on the entire interval $t \in [0,T]$ and this solution is unique as a diffusion process.

We are interested in the flows generated by equation (1.2). We call such flows as generalized ones since the densities of the initial conditions of their orbits are smooth and nowhere equal zero. We denote such orbit with initial condition ξ_0 as $\xi_{t,\xi_0}(s)$.

To deal with a solution having initial values with densities nowhere equal to zero, we have to modify the notion of local solution. Since the matrices (α^{ij}) are non-degenerate, we can recover the coefficients of equation from the generator. This is the reason why we explain this notion in terms of generators. For simplicity, we consider the case where M is a linear space. The general case of a manifold we leave to the reader as a simple exercise.

Consider in R^n an expanding sequence of compacts V_i with smooth boundaries, such that $V_i \subset V_{i+1}$ $\bigcup\limits_{i=1}^{\infty} V_i = M$. We construct a system of smooth bell-shaped functions φ_i, equal to one in V_i, zero outside V_{i+1}, and having uniformly bounded first partial derivatives in all $V_{i+1} \backslash V_i$. Let α^* be a constant symmetric non-degenerate matrix. Consider the sequence of generators, where the drift has the form $\varphi_i a$ and the matrix of coefficients at the second order derivatives takes the form $\varphi_i(\alpha) + (1 - \varphi_i)(\alpha^*)$. We note that the equations with such generators satisfy the conditions introduced above. Since the coefficients of these equations are smooth and bounded, they possess unique solutions. We call these solutions the local solutions of equations under consideration.

Remark 1.6. Denote by a^i the coordinates of vector $a(t,x)$, by a^i_* the coodinates of vector $a_*(t,x)$ and by α^{ij} the elements of matrix $\alpha(t,x)$. It is easy to see that the generator of generalized flow generated by equation (1.2), takes the form $\mathcal{A} = a^i \frac{\partial}{\partial x^i} + \frac{1}{2}\alpha^{ij}\frac{\partial^2}{\partial x^i \partial x^j}$ while the inverse generator (the generator of inverse generalized flow) takes the form $\mathcal{A}_* = -a^i_* \frac{\partial}{\partial x^i} + \frac{1}{2}\alpha^{ij}\frac{\partial^2}{\partial x^i \partial x^j}$

2. Necessary and sufficient conditions for completeness of stochastic flows

Everywhere below we suppose that all initial values are integrable random variables.

Definition 2.1. The generalized flow is called complete on $[0,T]$, if $\xi_{t,\xi_0}(s)$ a.s. takes values in $\mathbf{R}^n$ for any random initial value ξ_0, initial time instant t (with $0 \le t \le T$) and for all $s \in [t,T]$. The generalized flow $\xi(s)$ is called *complete*, if it is complete on every interval $[0,T] \subset R$.

Note that a particular case of Definition 2.1 is the standard definition of completeness of ordinary flow where only the orbits with (deterministic) initial conditions (points in $\mathbf{R}^n$) are involved.

Definition 2.2. Let X be a topological space. A function $\varphi\colon X \to \mathbf{R}$ is called *proper* (i.e., proper mapping to $\mathbf{R}$), if the preimage of every relatively compact set in $\mathbf{R}$ is relatively compact in X.

Theorem 2.3 ([7]). *Let there exist a smooth positive proper function φ on $\mathbf{R}^n$ such that $\mathcal{L}(t,x)\varphi < C$ for a certain $C > 0$ at all $t \in \mathbf{R}$ and $x \in \mathbf{R}^n$, where $\mathcal{L}$ is the generator of generalized flow $\xi(s)$. Then the generalized flow $\xi(s)$ is complete.*

Note that the complete analogue of Theorem 2.3 is valid also for ordinary stochastic flows.

Definition 2.4 ([7]). The generalized flow $\eta(s)$ is continuous at infinity on the interval $[0,T] \subset \mathbf{R}$, if for all $0 \le t \le T$, for every compact $K \subset \mathbf{R}^n$ and for every orbit $\eta_{t,\eta_i}(s)$ the equality

$$\lim_{\|\mathbf{E}\eta_i\| \to +\infty} \mathbf{P}(\eta_{t,\eta_i}(T) \in K) = 0. \tag{2.1}$$

holds. The generalized flow is continuoua at infinity it this property holds for every $T > 0$.

L. Schwartz has introduced the definition of continuity at infinity for the ordinary flow, i.e., in which only the orbits with deterministic initial values sre involved (see [9, 10]).

Theorem 2.5 ([7]). *Let on $\mathbf{R}^n$ there exist a smooth positive proper function u such than $\widetilde{\mathcal{L}}u < C$ for a certain constant $C > 0$, where $\widetilde{\mathcal{L}}$ is the generator of backward generalized flow $\widetilde{\eta}(t)$. Then the forward generalized flow $\eta(t)$ is continuous at infinity on $[0,T]$.*

Theorem 2.6 ([7]). *The generalized flow $\xi(s)$ on $\mathbf{R}^n$ having smooth and strictly elliptic generator and being continuous at infinity in the sense of Definition 2.4, is complete on $[0,T]$ if and only if there exists a smooth positive proper function $u_+\colon \mathbf{R}^n_+ \to R$ such that $\mathcal{L}_+ u_+ < C$ for a certain constant $C > 0$ at all points $(t,x) \in \mathbf{R}^n_+$.*

Theorem 2.7 ([7]). *The forward generalized flow $\xi(s)$ and the backward generalized flow $\widetilde{\xi}(s)$ generated by equation (1.2), are simultaneously both complete and continuous at infinity if and only if on $\mathbf{R}^n_+$ there exist positive smooth proper functions $u(t,x)$ and $\widetilde{u}(t,x)$ such that the inequalities*

$$\left(\frac{\partial}{\partial t} + \mathcal{A}\right)u < C \quad \text{and} \quad \left(-\frac{\partial}{\partial t} + \widetilde{\mathcal{A}}\right)\widetilde{u} < \widetilde{C}$$

hold for certain positive constants C and $\widetilde{C}$.

3. Matrix pencils

We need also some facts from the theory of matrices. Detailed explanation of this material can be found, e.g., in [8].

Definition 3.1. Let two $n \times n$ constant matrices A and B be given. The expression $\lambda A + B$ where λ is a real or complex valued parameter, is called the matrix pencil. The polynomial $\det(\lambda A + B)$ (with respect to λ) is called the characteristic polynomial of the pencil. If $det(\lambda A + B)$ is not identical zero, the pencil is called regular.

Theorem 3.2. *Let the matrix pencil $\lambda A + B$ be regular. Then there exist nondegenerate matrices P and Q such that*

$$P(\lambda A + B)Q = \lambda \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}, \tag{3.1}$$

where I_d and I_{n-d} are unit matrices of the corresponding dimensions, N is an upper triangle matrix consisting of Jordan boxes with zeros on diagonal and J is a certain $d \times d$ block.

Definition 3.3. If the characteristic polynomial satisfies the equality

$$rank(A) = \deg(\det(\lambda A + B)), \tag{3.2}$$

we say that the polynomial satisfies the condition *rank-degree*.

Theorem 3.4. *If the characteristic polynomial satisfies the rank-degree condition, assertion of Theorem 3.2 holds true and formula (3.1) takes the form*

$$P(\lambda A + B)Q = \lambda \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}. \tag{3.3}$$

4. The main result

Consider a regular pencil $\lambda \widetilde{L} + \widetilde{M}$ ($\widetilde{L}$ is degenerate and $\widetilde{M}$ is nondegenerate) and denote by P and Q the nondegenerate matrices from Theorem 3.2 for this pencil.

We suppose that the pencil under consideration satisfies the rank-degree condition. Construct $L = P\widetilde{L}Q$ and $M = P\widetilde{M}Q$ and by Theorem 3.4 obtain $L = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}$ and $M = \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}$, where J is nondegenerate since $\widetilde{M}$ is nondegenerate.

Now consider a certain symmetric positive definite matrix Ξ in $\mathbb{R}^d$. Since it is positive definite, it is non-degenerate. Introduce the matrix $\Theta = \begin{pmatrix} \Xi & 0 \\ 0 & 0 \end{pmatrix}$ and the matrix $\bar{\Theta} = Q\Theta Q^*$ in $\mathbb{R}^n$. The matrix $\bar{\Theta}$ is symmetric and degenerate. Thus, $D_2\xi(t) = \bar{\Theta}(\xi)$ is well-defined.

Consider the equation

$$\begin{cases} \widetilde{L} D_S \xi(t) = \widetilde{M}\xi(t) + \widetilde{f}(t, \xi(t)) \\ D_2\xi(t) = \bar{\Theta} \end{cases}. \tag{4.1}$$

Taking into account the above formulae and the definition of D_2 by formula (1.1), from (4.1) we obtain for $\eta(t) = Q^{-1}\xi(t)$ and $f(t,x) = P\widetilde{f}(t, Q^{-1}x)$ the equation

$$\begin{cases} LD_S\eta(t) = M\eta(t) + f(t) \\ D_2\eta(t) = \Theta \end{cases} \tag{4.2}$$

in $\mathbb{R}^n = \mathbb{R}^d \oplus \mathbb{R}^{n-d}$ and equation (4.2) decomposes into two independent equations in $\mathbb{R}^d$ and $\mathbb{R}^{n-d}$, respectively:

$$\begin{cases} D_S\eta^{(1)}(t) = J\eta^{(1)}(t) + f^{(1)}(t, \eta(t)) \\ D_2\eta^{(1)}(t) = \Xi \end{cases} \tag{4.3}$$

in $\mathbb{R}^d$ and

$$\begin{cases} \eta^{(2)}(t) + f^{(2)}(t, \eta(t)) = 0 \\ D_2\eta^{(2)}(t) = 0 \end{cases} \tag{4.4}$$

in $\mathbb{R}^{n-d}$.

Condition 1. *We suppose that for every point $(t, y^{(1)}) \in \mathbb{R} \times \mathbb{R}^d$ there exists a unique point $y^{(2)} = \Phi(t, y^{(1)}) \in \mathbb{R}^{n-d}$ that is continuos jointly in $(t, y^{(1)})$, coercive (i.e. $\Phi(t, y^{(1)}) \to \infty$ as $(t, y^{(1)}) \to \infty$) and such that $-f^{(2)}(t, y^{(1)} + \Phi(t, y^{(1)})) = \Phi(t, y^{(1)}) = y^{(2)}$.*

Remark 4.1. Under Condition 1 one can easily see that $y^{(2)} = \Phi(t, y^{(1)}) \in \mathbb{R}^{n-d}$ is a unique fixed point of the operator $-f^{(2)}(t, y^{(1)} + (\cdot)) : \mathbb{R}^{n-d} \to \mathbb{R}^{n-d}$. This fixed point exists, e.g., if $f^{(2)}(t, y^{(1)} + (\cdot))$ is Lipschitz continuous, i.e., if there exists $k \in (0, 1)$ such that for every pair of points $z_1, z_2 \in \mathbb{R}^{n-d}$ and every $(t, y^{(1)}) \in \mathbb{R} \times \mathbb{R}^d$ the inequality $\|f^{(2)}(t, y^{(1)} + z_1) - f^{(2)}(t, y^{(1)} + z_2)\| < k\|z_1 - z_2\|$ holds. Note that k may be a continuous function depending on $(t, y^{(1)})$. A condition of this sort in the language of functions $y^{(1)}(t)$ and $y^{(2)}(t)$ is used in [11, 12].

Under Condition 1 $\eta^{(2)}(t) = \Phi(t, \eta^{(1)}(t))$, $f^{(1)}(t, \eta(t))$ is represented in the form

$$f^{(1)}(t, \eta(t)) = f^{(1)}(t, \eta^{(1)}(t) + \eta^{(2)}(t)) = f^{(1)}(t, \eta^{(1)}(t) + \Phi(t, \eta^{(1)}(t))). \tag{4.5}$$

Taking into account formula (4.5) and Remark 1.6, we see that for the flow generated by equation (4.2) on $\mathbb{R}^d$, the generator $\mathcal{A}$ and the backward generator $\widetilde{\mathcal{A}}$ have the forms

$$\mathcal{A} = Jx^{(1)} + f^{(1)}(t, x^{(1)} + \Phi(t, x^{(1)})) + \frac{1}{2}\Xi^{ij}\frac{\partial^2}{\partial q^i \partial q^j} \tag{4.6}$$

and

$$\widetilde{\mathcal{A}} = -Jx^{(1)} - f^{(1)}(t, x^{(1)} + \Phi(t, x^{(1)})) + \frac{1}{2}\Xi^{ij}\frac{\partial^2}{\partial q^i \partial q^j}, \tag{4.7}$$

respectively.

Note that in (4.6) and (4.7) we consider $Jx^{(1)}$ and $f^{(1)}(t, x^{(1)} + \Phi(t, x^{(1)}))$ as vector fields, i.e., the fields of first order differential operators on $\mathbb{R}^d$.

Theorem 4.2. *Let the matrix pencil in equation (4.1) be regular and satisfy the rank-degree condition and Condition 1. The flow $\xi(t)$ and the backward flow $\widetilde{\xi}(t)$,* *generated on $\mathbb{R}^n$ by equation (4.1), are simultaneously both complete and continuous at infinity if and only if on $\mathbf{R}^d_+$ there exist positive smooth proper functions $u(t,x)$ and $\widetilde{u}(t,x)$ such that the inequalities*

$$(\frac{\partial}{\partial t} + Jx^{(1)} + f^{(1)}(t, x^{(1)} + \Phi(t, x^{(1)})) + \frac{1}{2}\Xi^{ij}\frac{\partial^2}{\partial q^i \partial q^j})u < C$$

and

$$(-\frac{\partial}{\partial t} - Jx^{(1)} - f^{(1)}(t, x^{(1)} + \Phi(t, x^{(1)})) + \frac{1}{2}\Xi^{ij}\frac{\partial^2}{\partial q^i \partial q^j})\widetilde{u} < \widetilde{C}$$

hold for certain positive constants C $\widetilde{C}$.

Proof. Indeed, by Theorem 2.7 the assertion of the theorem holds for the flow $\eta^{(1)}(t)$ and the backward flow $\widetilde{\eta}^{(1)}(t)$ on $\mathbb{R}^d$ generated by equation (4.3). Thus the assertion of the theorem for the flows $\xi(t)$ and $\widetilde{\xi}(t)$ on $\mathbb{R}^n$ follows from the formula $\eta^{(2)}(t) = \Phi(t, \eta^{(1)}(t))$ and from the fact that $\Phi(t, y^{(1)})$ is coercive. □

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