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# CHARACTERISTIC SUBGROUPS OF A FINITE ABELIAN P-GROUP $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$

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Abstract. In this paper, we study the following points:- (i) list all characteristic subgroups of a finite abelian 2-group $\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^n}$ when $m, n \in \mathbb{Z}^+$ (ii) list all characteristic subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ when $m, n \in \mathbb{Z}^+$ where p is an odd prime (iii) lattices of characteristic subgroups of a finite abelian 2-group $\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^5}$ and (iv) Lattices of characteristic subgroups of a finite abelian p-group $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^5}$ where p is odd prime.

## 1. Introduction

A subgroup $N$ of a group $G$ is called a characteristic subgroup if $\phi(N) = N$ for all automorphisms $\phi$ of $G$. This term was first used by $Frobenius$ in 1895. In 1939, Baer [2] considered the following question "When do two groups have isomorphic subgroups lattices?" Since this is a very difficult problem.In 2011,Brent L. Kerby and Emma Rode [3] consider the related question "Lattices of characteristic subgroups of $\mathbb{Z}_p \times \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^4}$ isomorphic to lattices of characteristic subgroups of $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^5}$ for any prime p".In 2017, Amit Sehgal and Manjeet Jakhar [7] consider the related question "Lattices of characteristic subgroups of $\mathbb{Z}_n \times \mathbb{Z}_n$ isomorphic to lattices of characteristic subgroups of $\mathbb{Z}_n$".In 2021, Hayder Baqer Shelash and Ali Reza Ashraf [9] consider the related question for group $U_{6n}, V_{8n}, H(n)$. In 2021, Sarita and Manjeet Jakhar [8] consider the related question "Lattices of characteristic subgroups of $\mathbb{Z}_p \times \mathbb{Z}_{p^n}$ ". We will now consider the problem of lattices of characteristic subgroups of a finite abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ when $m, n \in \mathbb{Z}^+$.

In Section 2, we list subgroups of group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$. In Section 3, we list all the automorphism group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ and by using these we list the characteristic subgroups of group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ in section 4 and 5 according as $m = n$ or not. Finally in section 6,lattices of characteristic subgroups of group $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^5}$ is discussed for both the case when p is even or odd prime.

## 2. List of subgroups of group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$

We know that group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} = \{x^i y^j | x^{p^m} = y^{p^n} = e,\ xy = yx, i = 0, 1, \cdots, p^m - 1, j = 0, 1, \cdots, p^{n-1}\}$ is an abelian group of order $p^{m+n}$. So converse of Lagrange's theorem is true for this group. So possible order of subgroup are $1, p, p^2, \cdots, p^{m+n}$. Here, all subgroups are subgroups from abelian group, so they

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must be abelian. Now we have to search for cyclic subgroups whose order are $1, p, p^2, \cdots, p^n$ and abelian subgroups which are not cyclic of order $p^2, p^3, \ldots p^{m+n}$

**2.1. List of cyclic subgroups of order $p^k(1 \leq k \leq m \leq n)$ from group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$.** We count elements of order $p^k$ in $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ as $\sum_{i=0}^{k}$((no. of elements of order $p^i$ from $\mathbb{Z}_{p^m}$) $\times$ (no. of elements order $p^k$ from $\mathbb{Z}_{p^n}$)$+\sum_{i=0}^{k-1}$ (no. of elements order $p^k$ from $\mathbb{Z}_{p^m}$)$\times$ (no. of elements order $p^i$ from $\mathbb{Z}_{p^n}$)) $= p^{2k-2}(p^2-1)$. Hence, the number of cyclic subgroups of order $p^k$ are $\frac{p^{2k-2}(p^2-1)}{\phi(p^k)} = p^{k-1}(p+1)$ . The list of these $p^{k-1}(p+1)$ subgroups is $< x^{p^{m-k}} y^{jp^{n-k}} >$ where $j = 1, 2, \cdots, p^k$ and $< x^{jp^{m-k+1}} y^{p^{n-k}} >$ where $j = 1, 2, \cdots, p^{k-1}$

**2.2. List of cyclic subgroups of order $p^k(1 \leq m < k \leq n)$ from group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$.** We count elements of order $p^k$ in $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ as $\sum_{i=0}^{k}$((no. of elements of order $p^i$ from $\mathbb{Z}_{p^m}$) $\times$ (no. of elements order $p^k$ from $\mathbb{Z}_{p^n}$) $= p^{k+m-1}(p-1)$. Hence, the number of cyclic subgroups of order $p^k$ are $\frac{p^{k+m-1}(p-1)}{\phi(p^k)} = p^m$ . The list of these $p^m$ subgroups are $< x^j y^{p^{n-k}} >$ where $j = 1, 2, \cdots, p^m$ and $k = m+1, m+2, \cdots, n$.

**2.3. List of abelian subgroups which are not cyclic subgroups of order $p^k$ from group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ where $2 \leq k \leq n$.** By fundamental theorem of finite abelian group, we know that number of non-isomorphic abelian groups of order $p^k$ are $p(k)$, out of which only one group is cyclic. So abelian group have $p(k) - 1$ subgroups which are not cyclic.

**Theorem 2.1.** [5] *Number of internal direct product of cyclic subgroup of order $p^{k_1}$ with cyclic subgroup of order $p^{k_2}$ with $1 \leq k_1 \leq k_2 \leq m \leq n$ is $p^{k_1+k_2-1}(p+1)$ from group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$.*

**Theorem 2.2.** [5] *Number of internal direct product of cyclic subgroup of order $p^{k_1}$ with cyclic subgroup of order $p^{k_2}$ with $1 \leq k_1 \leq m < k_2 \leq n$ is $p^{m+k_1}$ from group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$.*

**Theorem 2.3.** [5, 8] *Number of internal direct product of cyclic subgroup of order $p^{k_1}$ with subgroup isomorphic to $\mathbb{Z}_{p^{k_2}} \times \mathbb{Z}_{p^{k_3}} \times \cdots \times \mathbb{Z}_{p^{k_s}}$ is 0 from group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ where $s \geq 3$, $k_i \geq 1$ and $\sum_{i=1}^{s} k_i \leq (m+n)$. In other words, group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ does not possess any subgroup of rank more than two.*

*Proof.* If $A$ is any cyclic subgroups of order $p^{k_1}$ from $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ and $B$ is a subgroup isomorphic to $\mathbb{Z}_{p^{k_2}} \times \mathbb{Z}_{p^{k_3}} \times \cdots \times \mathbb{Z}_{p^{k_s}}$ where $s \geq 3$ from $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, then $A \bigcap B \neq \{e\}$ because there are only p+1 cyclic subgroups of order p in $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ and A contains a unique subgroup of order p and B contains at-least $p+1$ cyclic subgroups of order $p$. So cyclic subgroup of order p must contain in B if B is possible. So, internal direct product of cyclic subgroup of order $p^{k_1}$ with subgroup isomorphic to $\mathbb{Z}_{p^{k_2}} \times \mathbb{Z}_{p^{k_3}} \times \cdots \times \mathbb{Z}_{p^{k_s}}$ from $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ is not possible.
Hence, we conclude that the abelian p-group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ does not possess any subgroup of rank three or more. □

**Theorem 2.4.** [5] *Number of subgroups from group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ which are isomorphic to $\mathbb{Z}_{p^k} \times \mathbb{Z}_{p^k}$ where $1 \leq k \leq m \leq n$ is 1.*

Further, only subgroup which is isomorphic to $\mathbb{Z}_{p^k} \times \mathbb{Z}_{p^k}$ where $1 \leq k \leq m \leq n$ is $< x^{p^{m-k}}, y^{p^{n-k}} >$.

**Theorem 2.5.** [5] *Number of subgroups from group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ which are isomorphic to $\mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 < k_2 \leq m \leq n$ is $p^{k_2-k_1-1}(p+1)$.*

Further, subgroups which are isomorphic to $\mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 < k_2 \leq m \leq n$ are $< x^{p^{m-k_1}}, x^{sp^{m-k_2+1}} y^{p^{n-k_2}} >$ where $s = 1, 2, \cdots, p^{k_2-k_1-1}$ and $< x^{p^{m-k_2}} y^{sp^{n-k_2}}, y^{p^{n-k_1}} >$ where $s = 1, 2, \cdots, p^{k_2-k_1}$

**Theorem 2.6.** [5] *Number of subgroups from group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ which are isomorphic to $\mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 \leq m < k_2 \leq n$ is $p^{m-k_1}$.*

Further, subgroups which are isomorphic to $\mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 \leq m < k_2 \leq n$ are $< x^{p^{m-k_1}}, x^s y^{p^{n-k_2}} >$ where $s = 1, 2, \cdots, p^{m-k_1}$.

Finally list of subgroups of group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ given below:-

(i) $< e > \cong \mathbb{Z}_1$

(ii) $< x^{p^{m-k}} y^{jp^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $j = 1, 2, \cdots, p^k$ and $1 \leq k \leq m \leq n$

(iii) $< x^{jp^{m-k+1}} y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $j = 1, 2, \cdots, p^{k-1}$ and $1 \leq k \leq m \leq n$

(iv) $< x^j y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $j = 1, 2, \cdots, p^m$ and $k = m+1, m+2, \cdots, n$.

(v) $< x^{p^{m-k}}, y^{p^{n-k}} > \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^k}$ where $1 \leq k \leq m \leq n$

(vi) $< x^{p^{m-k_1}}, x^{sp^{m-k_2+1}} y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 < k_2 \leq m \leq n$ and $s = 1, 2, \cdots, p^{k_2-k_1-1}$

(vii) $< x^{p^{m-k_2}} y^{sp^{n-k_2}}, y^{p^{n-k_1}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 < k_2 \leq m \leq n$ and $s = 1, 2, \cdots, p^{k_2-k_1}$

(viii) $< x^{p^{m-k_1}}, x^s y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 \leq m < k_2 \leq n$ and $s = 1, 2, \cdots, p^{m-k_1}$.

Hence, we get the list of $\sum_{d=0}^{m} (m-d+1)(n-d+1)\phi(p^d)$ subgroups of group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ which is same as result in [1, 4].

## 3. List of Automorphisms of Group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ where $1 \leq m < n$

We know that $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} = \{x^i y^j | x^{p^m} = y^{p^n} = e, xy = yx, i = 0, 1, \cdots, p^m - 1, j = 0, 1, \cdots, p^{n-1}\}$ is an abelian group of order $p^{m+n}$. This group is generated by two elements $x$ and $y$ where order of $x$ is $p^m$ and $y$ is $p^n$.

We map $y$ into an element of order $p^n$ and elements of order $p^n$ are obtained from product of elements from $\mathbb{Z}_{p^m}$ (say $\alpha$) whose order divides $p^m$ and elements of order $p^n$ from $\mathbb{Z}_{p^n}$ (say $\beta$). Assume $\alpha = x^{i_1}$ with $o(\alpha) | p^m$ and $\beta = y^{j_1}$ with $o(\beta) = p^n$. So, there is no condition on $i_1$ and $(j_1, p^n) = 1$. So $y \mapsto x^{i_1} y^{j_1} \Rightarrow y^{p^m} \mapsto y^{j_1 p^m}$ .So, image of $y$ depends upon values of $i_1$ and $j_1$, here possibilities for $i_1$ and $j_1$ are $p^m$ and $p^{n-1}(p-1)$ respectively. Hence, the possibilities for $y$ are $p^{m+n-1}(p-1)$

So every element of order $p$ of the type $y^{j_1 p^{n-1}}$ from $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ is already mapped. So $x$ maps into an element of order $p^m$ from $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ other than elements of whose $p^{m-1}$ power is $y^{j_1 p^{n-1}}$. Hence, $x$ maps to $x^{i_2} y^{j_2 p^{n-m}}$ where

$(i_2, p^m) = 1$ and there is no condition on $j_2$. So, map of $x$ depends upon values of $i_2$ and $j_2$. Here possibilities for $i_2$ and $j_2$ are $p^{m-1}(p-1)$ and $p^m$ respectively. Hence, the possibilities for $x$ are $p^{m+m-1}(p-1)$.

Finally, we define an automorphism $f : \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} \mapsto \mathbb{Z}_p \times \mathbb{Z}_{p^n}$ as $f(y) = x^{i_1}y^{j_1}$ and $f(x) = x^{i_2}y^{j_2 p^{n-m}}$ where $i_1, i_2, j_2 = 1, 2, \cdots, p^m$ with $(i_2, p^m) = 1$ and $j_1 = 1, 2, \cdots, p^n$ with $(j_1, p^n) = 1$. Hence, the total number of automorphisms are $p^{3m+n+2}(p-1)^{2^2}$ which is same as result in [6].

## 4. List of the Characteristic Subgroups of Group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m}$

From [7], we know that only $m+1$ subgroups of group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^m} = \{x^i y^j | x^{p^m} = y^{p^m} = e,\ xy = yx, i = 0, 1, \cdots, p^m - 1, j = 0, 1, \cdots, p^{m-1}\}$ are characteristic subgroups which are listed below:-

(i) $< e > \cong \mathbb{Z}_1$

(ii) $< x^{p^{m-k}}, y^{p^{m-k}} > \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^k}$ where $1 \leq k \leq m$

## 5. List of the Characteristic Subgroups of Group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ with $m < n$

**Theorem 5.1.** *Let* $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} = \{x^i y^j | x^{p^m} = y^{p^n} = e, xy = yx, i = 0, 1, \cdots, p^m - 1, j = 0, 1, \cdots, p^n - 1\}$*, then list of characteristic subgroups of group* $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ *when* $1 \leq m < n$ *given below:-*

*(i)* $< e > \cong \mathbb{Z}_1$

*(ii)* $< x^{p^{m-k}}, y^{p^{n-k}} > \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^k}$ *where* $1 \leq k \leq m < n$

*(iii)* $< y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ *where* $1 \leq k \leq n - m$

*(iv) (only for case when p is even prime)* $< x^{p^{m-1}} y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ *where* $2 \leq k \leq n - m$

*(v)* $< x^{p^{m-k_1}}, y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ *where* $1 \leq k_1 \leq m < n, k_1 < k_2 \leq n$ *and* $1 \leq k_2 - k_1 \leq n - m$.

*(vi) (only for case when p is even prime)* $< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ *where* $1 \leq k_1 < m < n, k_1 < k_2 \leq n$ *and* $2 \leq k_2 - k_1 \leq n - m$.

*Proof.* Case 1:- Subgroups $< e > \cong \mathbb{Z}_1$ and $< x^{p^{m-k}}, y^{p^{n-k}} > \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^k}$ where $1 \leq k \leq m < n$

Out of $\sum_{d=0}^{m} (m - d + 1)(n - d + 1)\phi(p^d)$ subgroups, $m + 1$ subgroups namely $< e >$ and $< x^{p^{m-k}}, y^{p^{n-k}} >$ where $1 \leq k \leq m \leq n$ have property that they are not isomorphic to any other subgroups of the group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$. Hence, image of these subgroups cannot be changed with any of the group automorphisms of the group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$, so they are characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$.

Case 2:- Subgroups $< x^{p^{m-k}} y^{jp^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $j = 1, 2, \cdots, p^k$ and $1 \leq k \leq m < n$

Subgroups $< x^{p^{m-k}} y^{jp^{n-k}} >$with $j = 1, 2, \cdots, p^k$ are not characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ if we choose automorphism $f(x) = xy^{p^{n-m}}$ and $f(y) = y$ then

$f(x^{p^{m-k}} y^{jp^{n-k}}) = x^{p^{m-k}} y^{(j+1)p^{n-k}} \notin < x^{p^{m-k}} y^{jp^{n-k}} >$ because $j \not\equiv j+1 (mod\ p)$

Case 3:- Subgroups $< x^{jp^{m-k+1}} y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $j = 1, 2, \cdots, p^{k-1} - 1$ except $p^{k-2}$ and $2 \leq k \leq m < n$

Subgroups $< x^{jp^{m-k+1}} y^{p^{n-k}} >$ with $j = 1, 2, \cdots, p^{k-1} - 1$ except $p^{k-2}$ are not characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ if we choose automorphism $f(x) = xy^{p^{n-m}}$ and $f(y) = y^2$ then $f(x^{jp^{m-k+1}} y^{p^{n-k}}) = x^{jp^{m-k+1}} y^{(jp+2)p^{n-k}} \notin < x^{jp^{m-k+1}} y^{p^{n-k}} >$ because $1 \not\equiv jp + 2 (mod\ p)$

Case 4:- Subgroups $< x^j y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $j = 1, 2, \cdots, p^m - 1$ except $p^{m-1}$ and $1 \leq m < k \leq n$

Subgroups $< x^j y^{p^{n-k}} >$ with $j = 1, 2, \cdots, p^m - 1$ except $p^{m-1}$ are not characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ if we choose automorphism $f(x) = xy^{p^{n-m}}$ and $f(y) = y^2$ then $f(x^j y^{p^{n-k}}) = x^j y^{(jp^{m-k}+2)p^{n-k}} \notin < x^j y^{p^{n-k}} >$ because $1 \not\equiv jp^{m-k} + 2 (mod\ p)$

Case 5:- Subgroups $< y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $1 \leq k \leq n - m$

By use of concept $1 \leq k \leq n - m$, we have $x^{i_1 p^{n-k}} = e$. Subgroups $< y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $1 \leq k \leq n - m$ are characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ because we can choose any automorphism as $y \mapsto x^{i_1} y^{j_1}$ where $(j_1, p) = 1 \implies y^{p^{n-k}} \mapsto y^{j_1 p^{n-k}}$. So, we get $f(y^{p^{n-k}}) = y^{j_1 p^{n-1}} \in < y^{p^{n-k}} >$ , hence we get $f(< y^{p^{n-k}} >) = < y^{p^{n-k}} >$.

Case 6:- Subgroups $< y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $n - m < k \leq n$

By use of concept $n - m < k \leq n$, we have $x^{i_1 p^{n-k}} \neq e$. Subgroups $< y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $n - m < k \leq n$ are not characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ because we can choose automorphism as $y \mapsto x^{i_1} y^{j_1}$ where $(j_1, p) = 1 \implies y^{p^{n-k}} \mapsto x^{i_1 p^{n-k}} y^{j_1 p^{n-k}}$. So, we get $f(y^{p^{n-k}}) = x^{i_1 p^{n-k}} y^{j_1 p^{n-1}} \notin < y^{p^{n-k}} >$ , hence we get $f(< y^{p^{n-k}} >) \notin < y^{p^{n-k}} >$.

Case 7:- Subgroups $< x^{p^{m-1}} y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $1 \leq k \leq n$ and $p$ is an odd prime

Subgroups $< x^{p^{m-1}} y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $1 \leq k \leq n$ are not characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ because we can choose automorphism as $y \mapsto y^2$ and $x \mapsto xy^{p^{n-m}}$ So, we get $f(x^{p^{m-1}} y^{p^{n-k}}) = x^{p^{m-1}} y^{p^{n-1}} y^{2p^{n-k}} \notin < x^{p^{m-1}} y^{p^{n-k}} >$ , hence we get $f(< x^{p^{m-1}} y^{p^{n-k}} >) \notin < x^{p^{m-1}} y^{p^{n-k}} >$ because $p^{k-1} + 2 \not\equiv 1$ (mod p)

Case 8:- Subgroups $< x^{p^{m-1}} y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $2 \leq k \leq n - m$

By use of concept $2 \leq k \leq n-m$, we have $x^{i_1 p^{n-k}} = e$ and $x^{i_2 p^{m-1}} = x^{p^{m-1}}$ when $(i_2, p) = 1$. Subgroups $< x^{p^{m-1}} y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $2 \leq k \leq n-m$ are characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ because we can choose any automorphism as $y \mapsto x^{i_1} y^{j_1}$ where $(j_1, p) = 1 \implies y^{p^{n-k}} \mapsto y^{j_1 p^{n-k}}$ and $x \mapsto x^{i_2} y^{j_2 p^{n-m}}$ where $(i_2, p) = 1$ . So, we get $f(x^{p^{m-1}} y^{p^{n-k}}) = x^{p^{m-1}} y^{j_2 p^{n-1}} y^{j_1 p^{n-k}} \in < x^{p^{m-1}} y^{p^{n-k}} >$ , hence we get $f(< x^{p^{m-1}} y^{p^{n-k}} >) = < x^{p^{m-1}} y^{p^{n-k}} >$.

Case 9:- Subgroups $< x^{p^{m-1}} y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $n-m < k \leq n$

By use of concept $n-m < k \leq n$, we have $x^{i_1 p^{n-k}} \neq e$ and $x^{i_2 p^{m-1}} = x^{p^{m-1}}$ when $(i_2, p) = 1$. Subgroups $< x^{p^{m-1}} y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $n-m < k \leq n$ are not characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ because we can choose automorphism as $y \mapsto x^{i_1} y^{j_1}$ where $(j_1, p) = 1 \implies y^{p^{n-k}} \mapsto x^{i_1 p^{n-k}} y^{j_1 p^{n-k}}$ and $x \mapsto x^{i_2} y^{j_2 p^{n-m}}$ . So, we get $f(x^{p^{m-1}} y^{p^{n-k}}) = x^{p^{m-1}} y^{j_2 p^{n-1}} x^{i_1 p^{n-k}} y^{j_1 p^{n-k}}$.
If $m-1 > n-k$ and with use of $(j_1, p) = 1$, we get $(j_1 + j_2 p^{k-1})$ is odd. So,$f(x^{p^{m-1}} y^{p^{n-k}}) = x^{p^{m-1}} x^{p^{n-k}} y^{(j_1 + j_2 p^{k-1}) p^{n-k}} \notin < x^{p^{m-1}} y^{p^{n-k}} >$.
If $m-1 = n-k$ and with use of $(j_1, p) = 1$, we get $(j_1 + j_2 p^{k-1})$ is odd. So,$f(x^{p^{m-1}} y^{p^{n-k}}) = y^{(j_1 + j_2 p^{k-1}) p^{n-k}} \notin < x^{p^{m-1}} y^{p^{n-k}} >$.

Case 10:- Subgroups $< x^{p^{m-k_2}} y^{s p^{n-k_2}}, y^{p^{n-k_1}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $s = 1, 2, \cdots, p^{k_2 - k_1}$ and $1 \leq k_1 < k_2 \leq m < n$.

On the basis above 9 cases, we know that group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ has only characteristic subgroup of order $p$ is $y^{p^{n-1}}$ when $1 \leq m < n$, so group $< x^{p^{m-k_2}} y^{s p^{n-k_2}}, y^{p^{n-k_1}} >$ where $s = 1, 2, \cdots, p^{k_2 - k_1}$ and $1 \leq k_1 < k_2 \leq m < n$ has a characteristic subgroup of order $p$ as $< x^{p^{m-1}} y^{s p^{n-1}} >$.

If possible, assume that subgroups $< x^{p^{m-k_2}} y^{s p^{n-k_2}}, y^{p^{n-k_1}} >$ is characteristic subgroup of group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$.By transitivity property of characteristic subgroup, then subgroup $< x^{p^{m-1}} y^{s p^{n-1}} >$ is a characteristic subgroup of order $p$ from group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ which is contradiction. Hence, our supposition is wrong and we say that subgroup $< x^{p^{m-k_2}} y^{s p^{n-k_2}}, y^{p^{n-k_1}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $s = 1, 2, \cdots, p^{k_2 - k_1}$ and $1 \leq k_1 < k_2 \leq m < n$ are not characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$.

Case 11:- Subgroups $< x^{p^{m-k_1}}, x^{s p^{m-k_2+1}} y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $s = 1, 2, \cdots, p^{k_2-k_1-1} - 1$ except $p^{k_2-k_1-2}$ and $1 \leq k_1 < k_2 \leq m < n$

Subgroups $< x^{p^{m-k_1}}, x^{s p^{m-k_2+1}} y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 < k_2 \leq m < n$ and $s = 1, 2, \cdots, p^{k_2-k_1-1} - 1$ except $p^{k_2-k_1-2}$ are not characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ if we choose automorphism $f(x) = x y^{p^{n-m}}$ and $f(y) = y^2$ then $f(x^{s p^{m-k_2+1}} y^{p^{n-k_2}}) = x^{s p^{m-k_2+1}} y^{(sp+2) p^{n-k_2}} \notin < x^{p^{m-k_1}}, x^{s p^{m-k_2+1}} y^{p^{n-k_2}} >$ because $1 \not\equiv sp + 2 (mod\ p)$

Case 12:- Subgroups $< x^{p^{m-k_1}}, x^s y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 \leq m < k_2 \leq n$ and $s = 1, 2, \cdots, p^{m-k_1-1} - 1$ except $p^{m-k_1-2}$.

Subgroups $< x^{p^{m-k_1}}, x^s y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $s = 1, 2, \cdots, p^{m-k_1-1} - 1$ except $p^{m-k_1-2}$ and $1 \leq k_1 \leq m < k_2 \leq n$ are not characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ if we choose automorphism $f(x) = xy^{p^{n-m}}$ and $f(y) = y^2$ then $f(x^s y^{p^{n-k_2}}) = x^s y^{(sp^{k_2-m}+2)p^{n-k_2}} \notin < x^{p^{m-k_1}}, x^s y^{p^{n-k_2}} >$ because $1 \not\equiv (sp^{k_2-m} + 2)(mod\ p)$

Case 13:- Subgroups $< x^{p^{m-k_1}}, y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 \leq m < n$, $k_1 < k_2 \leq n$ and $1 \leq k_2 - k_1 \leq n - m$.

Subgroups $< x^{p^{m-k_1}}, y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 \leq m < n$,$k_1 < k_2 \leq n$ and $1 \leq k_2 - k_1 \leq n - m$ are characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ because we can choose any automorphism as $y \mapsto x^{i_1} y^{j_1}$ where $(j_1, p) = 1 \implies y^{p^{n-k_2}} \mapsto x^{i_1 p^{n-k_2}} y^{j_1 p^{n-k_2}}$ and $x \mapsto x^{i_2} y^{j_2 p^{n-m}}$ where $(i_2, p) = 1 \implies x^{p^{m-k_1}} \mapsto x^{i_2 p^{m-k_1}} y^{j_2 p^{n-k_1}}$.
So, we get $f(< x^{p^{m-k_1}}, y^{p^{n-k_2}} >) =< x^{i_2 p^{m-k_1}} y^{j_2 p^{n-k_1}}, x^{i_1 p^{n-k_2}} y^{j_1 p^{n-k_2}} >$.
By use of concept $1 \leq k_2 - k_1 \leq n - m$, we have $m - k_2 < m - k_1 \leq n - k_2 < n - k_1$, it is easily to see that $< x^{p^{m-k_1}}, y^{p^{n-k_2}} >=< x^{i_2 p^{m-k_1}} y^{j_2 p^{n-k_1}}, x^{i_1 p^{n-k_2}} y^{j_1 p^{n-k_2}} >$

Case 14:- Subgroups $< x^{p^{m-k_1}}, y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 \leq m < n$, $k_1 < k_2 \leq n$ and $k_2 - k_1 > n - m$.

By use of concept $k_2 - k_1 > n - m$, we have $m - k_1 > n - k_2$ and above case it is easily to see that $< x^{p^{m-k_1}}, y^{p^{n-k_2}} > \neq < x^{i_2 p^{m-k_1}} y^{j_2 p^{n-k_1}}, x^{i_1 p^{n-k_2}} y^{j_1 p^{n-k_2}} >$. Hence, subgroup $< x^{p^{m-k_1}}, y^{p^{n-k_2}} >$ are not characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$.

Case 15:- Subgroups $< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 < m < n$, $k_1 < k_2 \leq n$ and $k_2 - k_1 \geq 2$ for every odd prime.

Subgroups $< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where where $1 \leq k_1 < m < n$, $k_1 < k_2 \leq n$ and $k_2 - k_1 \geq 2$ are not characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ because we can choose automorphism as $y \mapsto y^2$ and $x \mapsto xy^{p^{n-m}}$ So, we get $f(x^{p^{m-k_1-1}} y^{p^{n-k_2}}) = x^{p^{m-k_1-1}} y^{(p^{k_2-k_1-1}+2)p^{n-k_2}} \notin < x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} >$, hence we get $f(< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} >) \notin < x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} >$ because $(p^{k_2-k_1-1} + 2) \not\equiv 1$ (mod p)

Case 16:- Subgroups $< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 < m < n$, $k_1 < k_2 \leq n$ and $2 \leq k_2 - k_1 \leq n - m$ for even prime only.

Subgroups $< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 < m < n$, $k_1 < k_2 \leq n$ and $2 \leq k_2 - k_1 \leq n - m$ are characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ because we can choose any automorphism as $y \mapsto x^{i_1} y^{j_1}$ where $(j_1, p) = 1 \implies$

$y^{p^{n-k_2}} \mapsto x^{i_1 p^{n-k_2}} y^{j_1 p^{n-k_2}}$ and $x \mapsto x^{i_2} y^{j_2 p^{n-m}}$ where $(i_2, p) = 1 \implies x^{p^{m-k_1}} \mapsto x^{i_2 p^{m-k_1}} y^{j_2 p^{n-k_1}}$.

So, we get value of $f(< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} >)$ as
$< x^{i_2 p^{m-k_1}} y^{j_2 p^{n-k_1}}, x^{i_2 + i_1 p^{((n-m)-(k_2-k_1-1))}} y^{(j_1 + j_2 p^{k_2-k_1-1}) p^{n-k_2}} >$.
By use of concept $2 \leq k_2 - k_1 \leq n - m$, we have $m - k_2 < m - k_1 \leq n - k_2 < n - k_1$, it is easily to see that subgroup $< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} >$ is same to subgroup $< x^{i_2 p^{m-k_1}} y^{j_2 p^{n-k_1}}, x^{i_2 + i_1 p^{((n-m)-(k_2-k_1-1))}} y^{(j_1 + j_2 p^{k_2-k_1-1}) p^{n-k_2}} >$
Hence, we get $f(< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} >) = < x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} >$

Case 17:- Subgroups $< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 < m < n$, $k_1 < k_2 \leq n$ and $k_2 - k_1 > n - m$ for even prime only.

By use of concept $n - m < k_2 - k_1$, we have $m - k_1 > n - k_2$. Subgroups $< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 < m < n$, $k_1 < k_2 \leq n$ and $k_2 - k_1 > n - m$ are not characteristic subgroups of $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ because we can choose automorphism as $y \mapsto x^{i_1} y^{j_1}$ where $(j_1, p) = 1$ and $x \mapsto x$ . So, we get $f(< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} >) = < x^{p^{m-k_1}}, x^{p^{m-k_1-1}} x^{p^{n-k_2}} y^{p^{n-k_2}} >$.
If $m - k_1 - 1 = n - k_2$, we get
$< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} x^{p^{n-k_2}} y^{p^{n-k_2}} > \neq < x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} >$.
If $m - k_1 - 1 > n - k_2$, we get $(p^{(k_2-k_1)-(n-m)-1} + 1)$ is odd.
So,$< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} x^{p^{n-k_2}} y^{p^{n-k_2}} > \neq < x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} >$.

□

Finally, we combine the results of section 4 and 5 in next theorem

**Theorem 5.2.** *Let $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n} = \{x^i y^j | x^{p^m} = y^{p^n} = e, xy = yx, i = 0, 1, \cdots, p^m - 1, j = 0, 1, \cdots, p^n - 1\}$, then list of characteristic subgroups of group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ when $1 \leq m \leq n$ given below:-*
*(i) $< e > \cong \mathbb{Z}_1$*
*(ii) $< x^{p^{m-k}}, y^{p^{n-k}} > \cong \mathbb{Z}_{p^k} \times \mathbb{Z}_{p^k}$ where $1 \leq k \leq m \leq n$*
*(iii) $< y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $1 \leq k \leq n - m$*
*(iv) (only for case when p is even prime) $< x^{p^{m-1}} y^{p^{n-k}} > \cong \mathbb{Z}_{p^k}$ where $2 \leq k \leq n - m$*
*(v) $< x^{p^{m-k_1}}, y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 \leq m < n$,$k_1 < k_2 \leq n$ and $1 \leq k_2 - k_1 \leq n - m$.*
*(vi) (only for case when p is even prime) $< x^{p^{m-k_1}}, x^{p^{m-k_1-1}} y^{p^{n-k_2}} > \cong \mathbb{Z}_{p^{k_1}} \times \mathbb{Z}_{p^{k_2}}$ where $1 \leq k_1 < m < n$,$k_1 < k_2 \leq n$ and $2 \leq k_2 - k_1 \leq n - m$.*

## 6. Lattice of Characteristic Subgroups of group $\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^5}$ where p is prime

Now we write one already known results which are very useful to form characteristic subgroup lattice for group $\mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ where p may be even or odd prime.

**Theorem 6.1.** [10] *Characteristic property is transitive. That is, if N is characteristic subgroup of K and K is characteristic subgroup of G, then N is characteristic subgroup of G.*

**6.1. Lattice of Characteristic Subgroups of group $\mathbb{Z}_9 \times \mathbb{Z}_{243}$.**

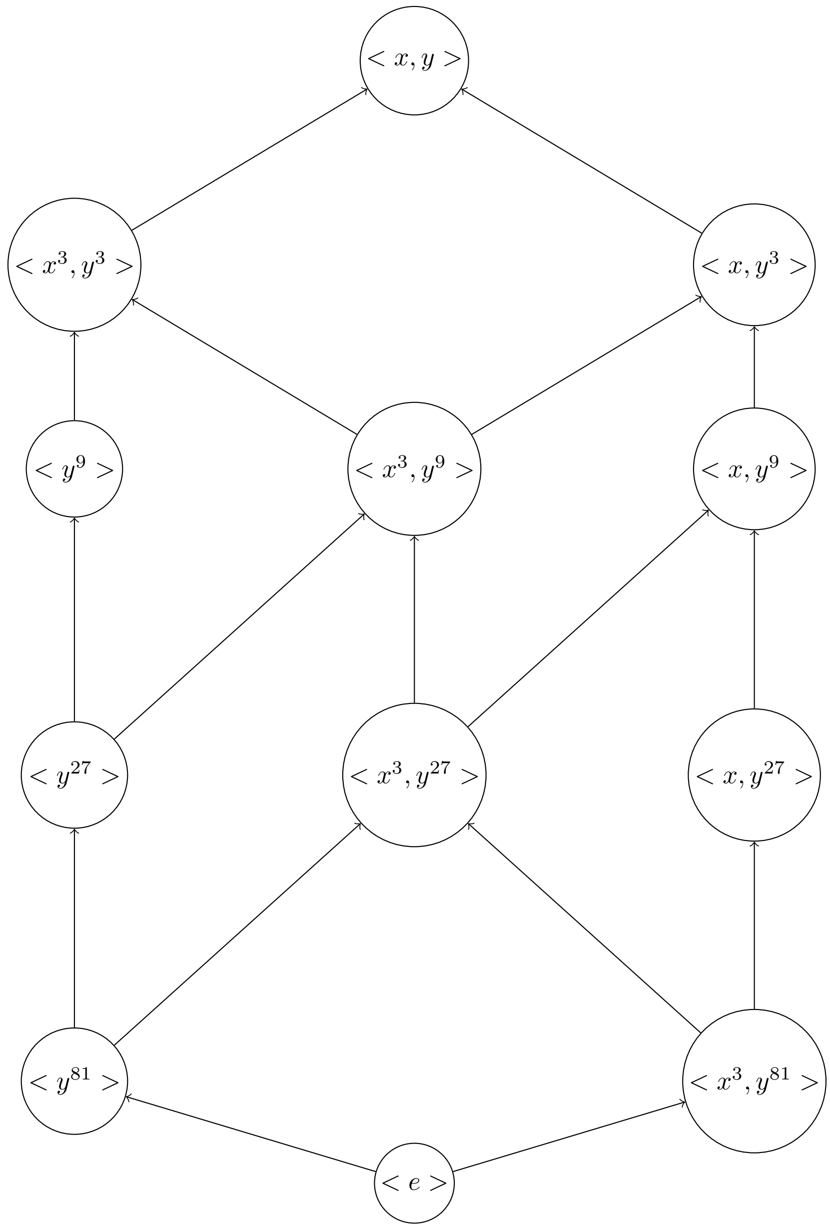


Fig-1 Lattices of characteristic subgroups $\mathbb{Z}_9 \times \mathbb{Z}_{243}$

**6.2. Lattice of Characteristic Subgroups of group $\mathbb{Z}_4 \times \mathbb{Z}_{32}$.**

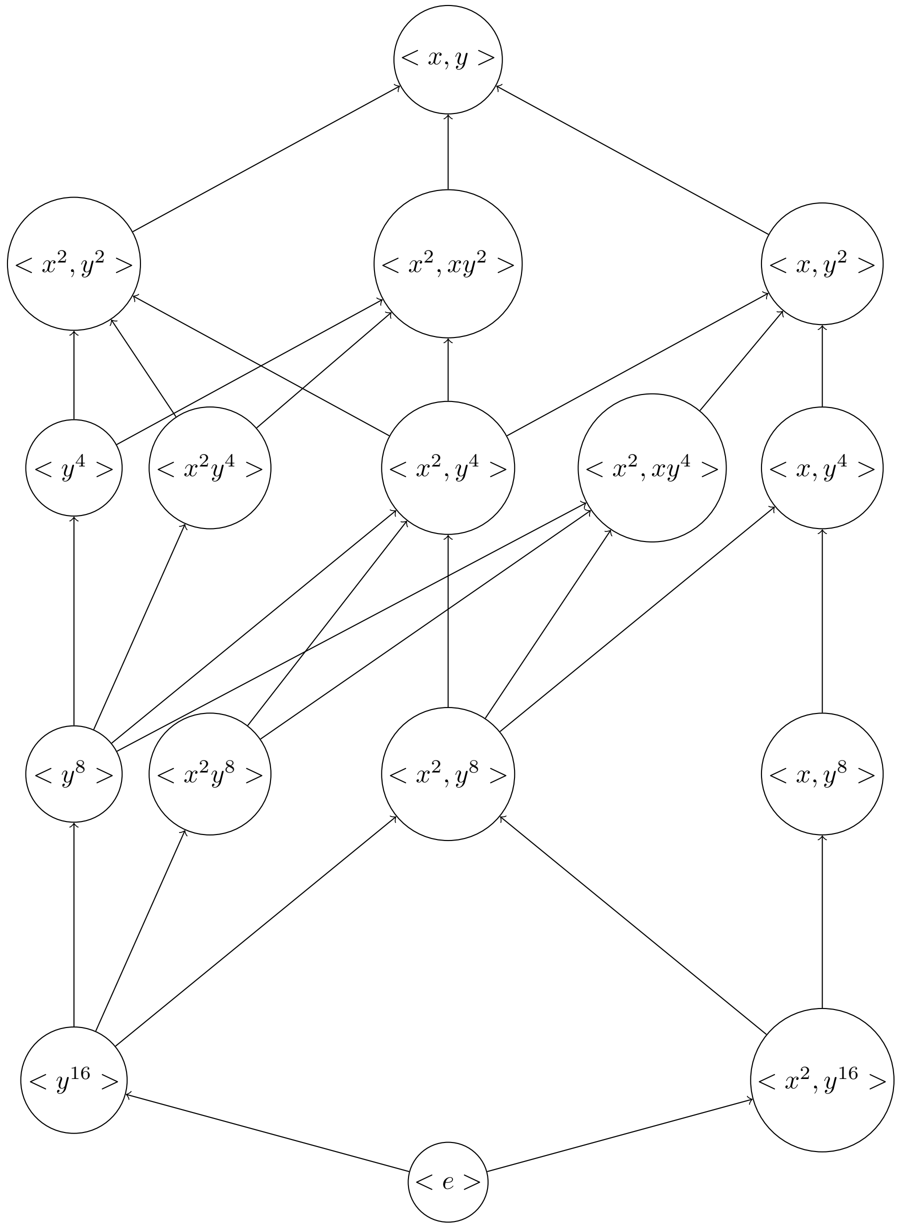


Fig-2 Lattices of characteristic subgroups $\mathbb{Z}_4 \times \mathbb{Z}_{32}$

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