Global and Stochastic Analysis
Vol. 7 No. 2 (July-December, 2020)

MUK PUBLICATIONS
Open Access Publisher

THE CANONICAL FORM OF THE MAIN BOUNDARY VALUE PROBLEM OF THE MEMBRANE THEORY OF CONVEX SHELLS

EVGENIY TYURIKOV

Abstract. In this paper we find the geometrical criterion of quasi-correctness for one class of problems in the membrane theory of convex shells.

1. Introduction

In the membrane theory of convex shells, the least studied problems include the problem with *mixed* boundary conditions, the formulation of which originates from I. N. Vekua [1] and A. L. Goldenweiser [2]. The interest in this problem is due to its importance from the point of view of applications in the technical theory of shells. It should also be noted that its geometric counterpart [3] is a generalization of the boundary-value problem of the theory of infinitesimal bendings of regular convex surfaces related to the classical problems of "Geometry in the Large". The first results in this direction were obtained by the author in [4]. Further research on mixed boundary value problems is closely related to the application of Vekua's method to the study of the solvability of the modified Riemann–Hilbert problem, the statement of which is given below.

2. Mathematical statement of the problem

Let S be a simply connected surface with a piecewise-smooth edge $L = \bigcup_{j=1}^{n} L_j$ and corner points of L p_i $(i = 1, \ldots, n)$. We assume that S is the inner part of the surface S_0 of strictly positive Gaussian curvature of regularity class $W^{3,r}$, $r > 2$, and each of the curves L_j belongs to the class $C^{1,\varepsilon}$, $0 < \varepsilon < 1$. We define a piecewise continuous vector field $\boldsymbol{r} = \{\alpha(s), \beta(s)\}$ on S along L, allowing break points of the first kind at p_j, with tangent and normal components $\alpha(s)$, $\beta(s)$ $(\alpha^2 + \beta^2 = 1,\ \beta \geqslant 0)$, where s is a natural parameter, functions $\alpha(s)$, $\beta(s)$ are Hölder ones on every curve L_j. The condition $\beta \geqslant 0$ means that the field $\boldsymbol{r}$ is *admissible* ([1]), i.e. the vector $\boldsymbol{r}$ at each point of the smooth curve L is directed off the surface S.

Date: Date of Submission June 02, 2020; Date of Acceptance June 30, 2020; Communicated by Yuri E. Gliklikh.

2010 *Mathematics Subject Classification.* 35J56; UDC 517.956.223.

Key words and phrases. convex shell, Riman–Gilbert boundary problem, index of the boundary value condition.

We introduce the following notation: Let J be the map of the surface S_0 onto a complex plane $z = x+iy$, determined by the selection of the conformal isometric parameterization (x, y) on S_0; $D = J(S)$ be a simply connected field limited by the complex surface z with the boundary $\Gamma = \bigcup\limits_{j=1}^{n} J(L_j)$ and the corner points $q_i = J(p_i)$.

Let us consider the following problem (problem R): find the complex-valued solution $w(z)$ of the equation (*the function of the bendings* of the surface S) in the field D

$$w_{\bar{z}}(z) - B(z)\bar{w}(z) = F(z), \quad z \in D, \tag{2.1}$$

by the given Riemann–Hilbert boundary condition

$$\mathrm{Re}\big\{\lambda(\zeta)w(\zeta)\big\} = \gamma(\zeta), \tag{2.2}$$

where

$$\lambda(\zeta) = s(\zeta)\big[\beta(\zeta)t(\zeta) - \alpha(\zeta)s(\zeta)\big], \tag{2.3}$$

$s(\zeta) = s_1(\zeta) + is_2(\zeta)$, $t(\zeta) = t_1(\zeta) + it_2(\zeta)$, $i^2 = -1$, s_j $(j = 1, 2)$ are coordinates of the unit vector tangent to Γ at the point ζ, t_j $(j = 1, 2)$ which is J-image of the tangent direction on the surface in the point $J^{-1}(\zeta)$; the values of the functions $\alpha(\zeta)$, $\beta(\zeta)$ coincide with the value of the functions α, β in the corresponding point $c = J^{-1}(\zeta)$; function $\gamma(\zeta)$ is Hölder one on every curve $\Gamma_j = J(L_j)$, $w_{\bar{z}} = \dfrac{1}{2}(w_x + iw_y)$; $B(z)$, $F(z)$ are functions of the class $L_r(D)$, $r > 2$ in the field D. In this case, the $W^{1,r}$-regular solutions $w(z)$ are found that are in the field D and that are continuously extendable to L, with the exception of break points q_j, in the neighborhood of which the following est holds true $\big|w(z)\big| < \mathrm{const}\cdot\big|z-q_j\big|^{-\alpha_j}$, $0 < \alpha_j < 1$.

The problem R under the conditions of smoothness of the boundary L and continuity of the vector function $\boldsymbol{r}(M)$ of the point $M \in L$ was posed and investigated by I. N. Vekua [5]. For the arbitrary piecewise continuous vector fields and a spherical surface S_0, the problem R was studied in [6]. The effective formulas for the index of the boundary condition were found in [7] under certain additional conditions on the "geometry" of the boundary and the field of directions in the vicinity of the corner points. The purpose of the work is to obtain meaningful results on the solvability of problem (2.1)–(2.3), abandoning the condition of "local symmetry" (see [5]) at corner points.

3. The problem R for canonical domes

Let p be one of the corner points p_i of the boundary L; $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ are the main directions in this point, k_1, k_2 are the main curvatures corresponding to them $(k_1 > k_2)$, $\delta = \sqrt{\dfrac{k_2}{k_1}}$. Then the point p is s-canonical $(s = 1, 2)$ if the direction of one of the curves converging at this point coincides with the main direction $\boldsymbol{k}_s$ $(s = 1, 2)$. The surface S is the *canonical* dome K if each corner point p is s-canonical with its interior angle less than π. The problem R for the canonical dome K is canonical if the direction of the field $\boldsymbol{r}$ at each point p is the direction of the generalized tangent [7] in this point, that is, $\boldsymbol{r}_1 = \boldsymbol{r}_2$, where $\boldsymbol{r}_i$ $(i = 1, 2)$ are the unilateral limits of the vector function $\boldsymbol{r}$ in the point p.

4. The notation for the singular nodes of problem R

Let us introduce notation to describe the properties of the boundary condition of the problem R: $\sigma^{(k)}$ $(k = 1, 2)$ are unilateral limits in the corner point p of the tangent singular vector $\boldsymbol{\sigma}$ to L; ν is the value of the interior angle at the point p defined by the vectors $(-1)^k\boldsymbol{\sigma}^{(k)}$ $(k = 1, 2)$; $T(\nu)$ is the set (*sector*) of all directions of the generalized tangent at this point, each of which is given by the vector $\boldsymbol{r}$ on S, separating the pair $(-1)^{k+1}\boldsymbol{\sigma}^{(k)}$. Let us also use notation $\boldsymbol{\sigma}_i^{(s)}$ $(s = 1, 2)$ are unilateral limits of the singular vector $\boldsymbol{\sigma}$ in the corner point p_i, ν_i is the value of the interior angle, $T(\nu_i)$ is *the sector* of directions *of the generalized tangent* at this point.

Let us describe the singular nodes q_j $(j = 1,\dots, n)$ of the boundary condition (2.2) (N. I. Muskhelishvili [8]). Let us consider vector function $\boldsymbol{\rho} = \{\rho_1(\zeta), \rho_2(\zeta)\}$, where $\rho_1(\zeta) + i\rho_2(\zeta) = \beta(\zeta)t(\zeta) - \alpha(\zeta)s(\zeta)$, denoting $\boldsymbol{\rho}_j^{(k)}$ $(k = 1, 2)$ its unilateral limits in the point q_j. Let $\boldsymbol{s}_j^{(k)} = J\big(\boldsymbol{\sigma}_j^{(k)}\big)$ be $k = 1, 2$; $j = 1,\dots, n$; φ_j and ψ_j are the values of the angles between the vectors of the pairs $\big(\boldsymbol{s}_j^{(1)}, \boldsymbol{s}_j^{(2)}\big)$ and $\big(\boldsymbol{\rho}_j^{(1)}, \boldsymbol{\rho}_j^{(2)}\big)$ respectively. In such a case $0 < \varphi_j < \pi$, $0 < \psi_j < 2\pi$, and the value ψ_j depends on the direction of the vector $\boldsymbol{r}$ in the point p_j.

Lemma 4.1 ([6])**.** *The index κ of boundary condition* (2.2) *in the class of limited solutions is calculated according to the formula*

$$\kappa = -4 + \sum_{i=1}^{n} \kappa_i, \tag{4.1}$$

where $\kappa_i = \left[\dfrac{1}{\pi}(\varphi_i + \psi_i)\right]$, $[a]$ *is an integer part of* a, $0 < \varphi_i < \pi$, $0 < \psi_i < 2\pi$.

Thus point q_i is *a singular node* (see [8]), if $\varphi_i+\psi_i$ is equal to one of the numbers π, 2π.

In order to simplify the notation and formulations, we consider a canonical dome K with sharp interior angles ν_i at points p_i, respectively $\Big(0 < \nu_i < \dfrac{\pi}{2};\ i = \overline{1, n}\Big)$, denoting the class of such domes by K_0.

For specification suppose that p_i is a 2-canonical point. For convenience, we denote the point p_i and its corresponding values φ_i, ψ_j, ν_i, κ_i, $\boldsymbol{\rho}_i^{(k)}$ $(k = 1, 2)$ by φ, ψ, ν, κ, $\boldsymbol{\rho}^{(k)}$, and also introduce the notation θ for the value of the interior angle at the corner point $q = J(p)$, $0 < \theta < \pi$. Note that any vector $\boldsymbol{r} \in T(\nu)$ separates a pair of vectors $(-1)^{k+1}\boldsymbol{\sigma}^{(k)}$ $(k = 1, 2)$, and the direction of the vector $\boldsymbol{\sigma}^{(2)}$ is the main direction $\boldsymbol{k}_2$ on the surface S at the point p.

Lemma 4.2. *If the direction of the vector $\boldsymbol{r}$ at the 2-canonical point p coincides with the direction of the vector $\boldsymbol{\sigma}^{(1)}$, then the point $q = J(p)$ is* a singular node *of the boundary condition* (2.2) *iff*

$$\nu = \arccos\frac{1}{1+\delta}. \tag{4.2}$$

Proof. From lemma, it follows that $\boldsymbol{\rho}^{(1)} = -\boldsymbol{s}^{(1)}$, $\varphi = \nu$, thereby point q as a singular node is determined by equality $\nu + \psi = 2\pi$. Using the well-known properties [5] of the mapping J and obvious geometric considerations, we write this equality as follows: $2\theta + \nu = \pi$ or $\sin 2\theta = \sin\nu$. From here, using the known relation ([5, Ch. 2]) $\sin\theta = \sqrt{\dfrac{K}{k_1 \cdot k_s}} \cdot \sin\nu$, where K is the Gaussian curvature of the surface at the point p, k_s is the normal curvature of the surface in the direction of the vector $\boldsymbol{s}^{(1)}$, as well as the Euler formula for normal curvature, after simple trigonometric transformations, we obtain

$$(1-\delta)\cos^2\nu + 2\sqrt{\delta}\cos\nu - 1 = 0. \tag{4.3}$$

This implies the statement of the lemma.

Remark 4.3. The statement of lemma 4.2 remains valid if, in the formulation, the 2-canonical point is replaced by the 1-canonical one, the direction $\boldsymbol{\sigma}_1$ is replaced by $\boldsymbol{\sigma}_2$, and the value δ by δ^{-1}.

Remark 4.4. The statement of lemma 4.2 remains valid if the corner point with an acute interior angle is replaced by a point with an interior angle ν $\left(\dfrac{\pi}{2} < \nu < \pi\right)$, and the value $\arccos(1+\delta)^{-1}$ by $\pi - \arccos(1+\delta)^{-1}$. It can be verified by passing from the equality $\nu + \psi = \pi$, which defines *a singular node*, to the equality $2\theta + \nu = 2\pi$.

5. Index of the canonical problem R

We introduce the notation: δ_i^2 is the ratio of the corresponding principal curvatures ($0 < \delta_i < 1$) at the corner point ($i = 1,\ldots, n$), $\gamma_i^{(s)} = \arccos\left(1 + \delta_i^{(-1)^s}\right)^{-1}$ ($s = 1, 2$; $i = 1,\ldots, n$). Let m_1, m_2, m_3 $\left(\sum\limits_{k=1}^{3} m_k = n\right)$ be the number of corner points from the set of all corner s-canonical ($s = 1$ or $s = 2$) points p_i ($i = 1,\ldots, n$) which meet the conditions

$$0 < \nu_{i_k} < \gamma_{i_k}^{(s)}, \qquad \gamma_{i_r}^{(s)} < \nu_{i_r} < \pi - \gamma_{i_r}^{(s)}, \qquad \pi - \gamma_{i_m}^{(s)} < \nu_{i_m} < \pi, \tag{5.1}$$
$$(k = 1, \ldots, m_1; \quad r = 1, \ldots, m_2; \quad m = 1, \ldots, m_3; \quad i_k \neq i_r \neq i_m)$$

respectively.

Lemma 5.1. *The index κ of the boundary condition* (2.2)–(2.3) *in the class of solutions, admitting the "integrable infinity" at points p_i, is calculated according to the formula*

$$\kappa = 3m_1 + 2m_2 + m_3 - 4 \tag{5.2}$$

and does not depend on the directions of the generalized tangents from the sector $T(\nu_i)$.

To complete the proof, it suffices to note that, by Lemma 4.2 and Remarks 4.3, 4.4, at the corresponding points $\kappa_{i_k} = 2$, $\kappa_{i_r} = 1$, $\kappa_{i_m} = 0$. It is easy to show that these equalities are preserved for any directions of the generalized tangent from the corresponding sectors $T(\nu_i)$. Next we apply formula (4.1), increasing each summand by one (see [7]).

Theorem 5.2. *The canonical problem R for any* dome *K under the condition of stress concentration at the corner points is quasi correct iff* $3m_1+2m_2+m_3 \geqslant 3$.

The proof of the theorem is carried out according to the scheme [1] using the results of the author [4] and formula (5.1).

Corollary 5.3. *If the number n of angular s-canonical points is not more than two, thereby $\pi - \gamma_i^{(s)} < \nu_i < \pi$ ($i = 1, 2$; $s = 1$ or $s = 2$), then the problem R has a unique solution if $3 - n$ conditions of solvability of the integral type are fulfilled* [5].

6. Solvability of the problem R in the class of limited solutions.

Suppose that the angular s-canonical points $p_{i_1},\dots, p_{i_t}$ $(1 \leqslant t \leqslant n;\ s = 1$ or $s = 2)$ hold one of the equalities $\nu_{i_t} = \gamma_{i_t}^{(s)}$, $\nu_{i_t} = \pi - \gamma_{i_t}^{(s)}$, and the remaining $n - t$ points hold the conditions (5.1). According to Lemma 4.2, the point $q_{i_r} = J(p_{i_r})$ $(r = 1,\dots, t)$ is a *singular node* of the problem R if the direction of the vector of the generalized tangent coincides with the direction of the vector defining the interior angle at this point. Since the solution of the problem R is necessarily limited in a neighborhood of the singular node [8], when we pose the problem R in the class of limited solutions, it is possible not to separate the singular and nonsingular nodes. Replacing the first and second strict inequalities in conditions (5.1) with the inequalities $0 < \nu_{i_k} \leqslant \gamma_{i_k}^{(s)}$, $\gamma_{i_r}^{(s)} < \nu_{i_r} \leqslant \pi - \gamma_{i_r}^{(s)}$, we obtain the following formula for the index κ in the class of solutions limited in the domain D:

$$\kappa = 2m_1 + m_2 - 4. \tag{6.1}$$

According to [8], the class of such solutions is denoted by H_0.
From (5.2) we have:

Theorem 6.1. *The canonical problem R for any dome K is quasi correct in the class H_0 iff*

$$2m_1 + m_2 \geqslant 3. \tag{6.2}$$

From (6.2) we have:

Theorem 6.2. *If $\pi - \gamma_i^{(s)} < \nu_i < \pi$ for each s-canonical point p_i ($i = 1,\dots, n$; $s = 1$ or $s = 2$), then the problem R in the class H_0 has a unique solution only if the three solvability conditions of the integral type are satisfied* [5].

Corollary 6.3. *If the values of the interior angles at all the corner points of the canonical dome are sufficiently close to the number π, then the solvability picture of the problem R coincides with the solvability picture of the static boundary-value problem by I. N. Vekua* [5] *for convex shells with a piecewise smooth boundary (see Theorem 6.2).*

References

[1] Vekua I. N.: *Some General Methods for Constructing Versions of Shell Theory.* Fizmatgiz, Moscow, 1982. (in Russian).

[2] Goldenveizer A. L.: On the application of the Riemann-Hilbert problem solutions to the calculation of membrane shells. *Prikl. Mat Mekh.*, **15** (2) (1951), 149–166.

[3] Tyurikov E. V.: Geometric Analogue of the Vekua-Goldenveizer Problem. *Doklady Mathematics*, **79** (1) (2009), 83–86.
[4] Tyurikov E. V.: Bondary problems of the theory of infinitesimal bending of the surfaces of positive curvature with piecewise smooth edge. *Sbornik: Mathematics*, **103** (7) (1977), 445–462.
[5] Vekua I. N.: *Generalized analytic functions.* Fizmatgiz, Moscow, 1959. (in Russian).
[6] Tyurikov E. V. On one boundary problem of the theory of infinitesimal bends of the surfaces. *Vladikavkaz Mathematical Journal*, **9** (1) (2007), 62–68.
[7] Tyurikov E. V.: One case of extended boundary value problem of the membrane theory of convex shells by I. N. Vekua. *Issues of Analysis*, (7) (25), Special Issues, (2018) 153–162.
[8] Muskhelishvili N. I.: *Singular integral equations.* Fizmatgiz, Moscow, 1968. (in Russian).

Evgeniy Tyurikov: Don State Technical University, Rostov-on-Don, 344000, Russia.
E-mail address: etyurikov@hotmail.com