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DIFFERENTIAL INVARIANTS OF CURVES' FOLIATIONS

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ABSTRACT. In this article, we indicate a method for constructing differential invariants of foliations of curves on a plane in various geometries. It is assumed that the geometric structure of the plane is given by the Lie group of admissible transformations. The foliations of curves on the Euclidean and de Sitter planes are considered separately.

1. Introduction

Let M be an open connected domain of $\mathbb{R}^2$ and let $\gamma : M \longrightarrow \mathbb{R}$ be foliation of curves. Such a foliation can be defined locally using the function $f \in C^\infty(M)$ whose differential does not vanish on M. The level lines of this function coincide with the foliation curves. The function f is defined up to gauge transformations $f \longmapsto F(f)$, $F : \mathbb{R} \longrightarrow \mathbb{R}$, where F is a smooth function.

Consider a three-dimensional manifold $\widetilde{E} = M \times \mathbb{R}$ and one-dimensional locally trivial bundle $\widetilde{\pi} : \widetilde{E} \longrightarrow M$. Let x, y be local coordinates on M and let u be a coordinate on $\mathbb{R}$. Each local section $\widetilde{s_f} : (x, y) \mapsto (x, y, f(x, y))$ of this bundle defines a foliation of the curves $f = c$ under the condition $df \neq 0$.

Let a Lie group G acts on M: $\phi : (x, y) \longmapsto (\phi_1(x, y), \phi_2(x, y))$. Let F be the function defining the gauge transformation. Define a mapping

$$\phi_F : (x, y, u) \longmapsto (\phi_1(x, y), \phi_2(x, y), F(u)).$$

Such transformations considered for different ϕ and F form a Lie pseudogroup $\widetilde{G}$. Differential invariants of this Lie pseudogroup are differential invariants of the foliation of curves. The infinitesimal analogue of the gauge transformation is the vector field $H = h(u)\partial_u$ and the Lie algebra $\widetilde{\mathcal{G}}$ of the Lie pseudogroup $\widetilde{G}$ is generated by vector fields from Lie algebras $\mathcal{G}$ and a vector field H.

In order to get rid of the infinite-dimensional part of the Lie algebra $\widetilde{\mathcal{G}}$, instead of the function f we consider the function f_x/f_y, assuming that the partial derivative f_y does not vanish on the Lie algebra $\mathcal{G}$ and the vector field H. This function is a differential invariant with respect to gauge transformations. This observation makes it possible to pass from an infinite-dimensional Lie algebra to a finite-dimensional one and, therefore, from Lie pseudogroups to Lie groups. Instead of

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the space $\widetilde{E}$, consider the space E with coordinates (x, y, v), and instead of the bundle $\widetilde{\pi}$ we consider the bundle $\pi : (x, y, v) \longmapsto (x, y)$.

2. Jet space $J^k(\pi)$

For a function $f \in C^\infty(M)$ such that $f_y \neq 0$, we define a local section of the bundle π:

$$s_f : (x, y) \mapsto \left(x, y, \frac{f_x(x, y)}{f_y(x, y)}\right). \tag{2.1}$$

Each such section locally defines a foliation of curves. Let $J^k(\pi)$ be the k-jet space [2] of local sections of the bundle π with local coordinates $v_{(n,m)}$ $(0 \le n + m \le k)$, where

$$v_{(n,m)}([s_f]^k_a) = \left.\frac{\partial^{n+m}}{\partial x^n y^m}\right|_a \left(\frac{f_x}{f_y}\right).$$

and $v_{(0,0)} = v$. Here $[s_f]^k_a$ is a k-jet at a point a of the section $s_f : M \to E$. Prolongations of the Lie group G and its Lie algebra $\mathcal{G}$ into the space $J^k(pi)$ we denote by $G^{(k)}$ and $\mathcal{G}^{(k)}$ respectively.

Differential 1-forms

$$\omega_{n,m} = dv_{n,m} - v_{n+1,m}dx - v_{n,m+1}dy, \quad 0 \le n + m \le k \tag{2.2}$$

define the Cartan distribution $C^{(k)}$ on the space $J^k(\pi)$:

$$C^{(k)} : J^k(\pi) \ni \theta \mapsto C^{(k)}(\theta) = \bigcap_{0 \le n+m \le k} \ker \omega_{n,m} \subset T_\theta(J^k(\pi)).$$

The function $h = \omega_{0,0}(X)$ is called a generation function of a vector field X on $J^0(\pi)$.

Let $X^{(k)}$ be a prolongation of the vector field X into the space $J^k(\pi)$:

$$X^{(k)} = -\frac{\partial h}{\partial v_{1,0}} \frac{d}{dx} - \frac{\partial h}{\partial v_{0,1}} \frac{d}{dy} + \sum_{0 \le n+m \le k} \frac{d^{n+m}h}{dx^n dy^m} \frac{\partial}{\partial v_{n,m}}. \tag{2.3}$$

Here $\frac{d}{dx}$ and $\frac{d}{dy}$ are operators of total derivatives.

An evolutionary part [4] of vector field (2.3) is

$$S^{(k)} = \sum_{0 \le n+m \le k} \frac{d^{n+m}h}{dx^n dy^m} \frac{\partial}{\partial v_{n,m}}.$$

Define projections $\pi_{k,r} : J^k(\pi) \to J^r(\pi)$, $(k = 0, 1, 2, \ldots; r \le k)$ where

$$\pi_{k,r} : (x, y, v_{n,m}) \mapsto (x, y, v_{p,q}), 0 \le n + m \le k; 0 \le p + q \le r.$$

3. Differential invariants

A function J on the space $J^k(\pi)$ is called a *differential invariant* of order $\le k$ of the Lie group G_{S}, if $(\varphi^{(k)})^*(J) = J$ for each transformation $\varphi \in G$ (see, for example, [1]).

A differential invariant J of order $\le k$ satisfies to the differential equation

$$X^{(k)}(J) = 0 \tag{3.1}$$

for each vector field $X \in \mathcal{G}$.

The set of all differential invariants forms an algebra with respect to addition and multiplication operations.

Let us describe a method for calculating the dimensions of algebras of differential invariants of order $\leq k$ (see [5]).

Suppose that G acts transitively on M. Let a be some point on the manifold M. Since the Lie group G acts transitively, any point can be taken as a point a. Consider the smooth manifold $N^{(k)}(a) = \pi_{k,0}^{-1}(a)$ with coordinates $v_{n,m}$, $(n+m \leq k)$.

Let $G_a \subset G$ be the isotropy group of the point a. Transformations from the prolonged Lie group $G_a^{(k)}$ preserve the manifold N_a^k.

Let $\mathcal{G}_a \subset \mathcal{G}$ be the corresponding to G_a Lie subalgebra. We call this subalgebra the stabilizer of the point a. Note that the point a is a singular point for each vector field from $\mathcal{G}_a$.

Suppose that vector fields $Y_1^{(k)}, \ldots, Y_r^{(k)}$ form a basis of the stabilizer $\mathcal{G}_a^{(k)}$ and let $S_1^{(k)}, \ldots, S_r^{(k)}$ be their evolutionary parts. Vector fields $S_1^{(k)}, \ldots, S_r^{(k)}$ are tangent to N_a^k and let $\overline{S}_1^{(k)}, \ldots, \overline{S}_r^{(k)}$ be their restrictions on N_a^k.

Note that k-jet of a section s_f crosses $N^{(k)}(a)$ at only one point θ. Therefore the dimension of the algebra of differential invariants of order $\leq k$ at the point θ is equal to the rank of the tangent subspace $\mathrm{Span}(\overline{S}_1^{(k)}, \ldots, \overline{S}_r^{(k))})|_\theta \subset T_\theta N^{(k)}(a)$.

4. Invariant differentiations

An operator ∇ on the space $J^\infty(\pi)$ is called an *invariant differentiation* if it commutes with each prolongation of a vector field from the Lie algebra $\mathcal{G}$.

Invariant differentiations can be used for constructing differential invariants. For example, if J is a differential invariant, then the function $\nabla(J)$ is so too. Indeed, for any vector field $X \in \mathcal{G}^{(\infty)}$ then $X(J) = 0$. On the other hand, since $[\nabla, X] = 0$, then $X(\nabla(J)) = 0$.

Moreover, if ∇_1 and ∇_2 are two invariant differentiations and their commutator

$$[\nabla_1, \nabla_2] = J_1\nabla_1 + J_2\nabla_2, \tag{4.1}$$

then the functions J_1 and J_2 are differential invariants.

5. Euclidean plane

Consider Euclidean plane with the metric $g = dx^2 + dy^2$. The Lie group of motions consists of parallel translations and rotations. Corresponding Lie algebra is generated by the vector fields

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Their prolongations into the 1-jet space are

$$X_1^{(1)} = \frac{\partial}{\partial x}, \qquad X_2^{(1)} = \frac{\partial}{\partial y}, \qquad X_3^{(1)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - f_y\frac{\partial}{\partial f_x} + f_x\frac{\partial}{\partial f_y}.$$

In coordinates $x, y, v = v_{(0,0)}$ on $J^0(\pi)$, these vector fields have the form

$$Y_1 = \frac{\partial}{\partial x}, \qquad Y_2 = \frac{\partial}{\partial y}, \qquad Y_3 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - (v^2+1)\frac{\partial}{\partial v}.$$

Theorem 5.1. *The operators*

$$\nabla_1 = \frac{1}{\sqrt{1+v^2}}\left(-\frac{d}{dx} + v\frac{d}{dy}\right) \quad \text{and} \quad \nabla_2 = \frac{1}{\sqrt{1+v^2}}\left(v\frac{d}{dx} + \frac{d}{dy}\right)$$

are invariant differentiations.

Using formula (4.1) we get two differential invariants of first order

$$J_1 = \frac{(vv_y - v_x)^2}{(v^2+1)^3}, \quad \text{and} \quad J_2 = \frac{(vv_x + v_y)^2}{(v^2+1)^3}.$$

In this case $[\nabla_1, \nabla_2] = -J_1\nabla_1 + J_2\nabla_2$.

We obtain second-order differential invariants by acting on them with invariant differentiations:

$$J_{11} = \nabla_1(J_1), J_{21} = \nabla_2(J_1), J_{12} = \nabla_1(J_2), J_{22} = \nabla_2(J_2). \tag{5.1}$$

We indicate their coordinate representations:

$$\begin{aligned} J_{11} =& \frac{1}{(v^2+1)^3}(v^4 v_{yy} + (-2v_{xy} - 2v_y^2)v^3 + (v_{xx} + 5v_y v_x + v_{yy})v^2 + \\ &+ (-2v_{xy} + v_y^2 - 3v_x^2)v - v_y v_x + v_{xx}), \end{aligned}$$

$$\begin{aligned} J_{21} =& \frac{1}{(v^2+1)^3}(v^4 v_{xy} + (-2v_y v_x - v_{xx} - v_{yy})v^3 + (3v_x^2 + 2v_y^2)v^2 + \\ &+ (4v_y v_x - v_{xx} + v_{yy})v - v_{xy} + v_y^2), \end{aligned}$$

$$\begin{aligned} J_{12} =& \frac{1}{(v^2+1)^3}(v^4 v_{xy} + (-2v_y v_x - v_{xx} + v_{yy})v^3 + (2v_x^2 - 3v_y^2)v^2 + \\ &+ (4v_y v_x - v_{xx} + v_{yy})v - v_x^2 - v_{xy}), \end{aligned}$$

$$\begin{aligned} J_{22} =& \frac{1}{(v^2+1)^3}(v_{xx}v^4 + (2v_{xy} - 2v_x^2)v^3 + (v_{yy} - 5v_y v_x + v_{xx})v^2 + \\ &+ (2v_{xy} + v_x^2 - 3v_y^2)v + v_{yy} + v_y v_x). \end{aligned}$$

Here $v_x = v_{(1,0)}, v_y = v_{(0,1)}, v_{xx} = v_{(2,0)}, \dots$.

Theorem 5.2. *The dimension of the algebra of differential invariants of order $\leq k$ is $\nu(k) = C^k_{k+2} - 1$, and the number of independent differential invariants whose order is k is $k+1$.*

Proof. For any $k \geq 0$ the dimension of the Lie algebra $\mathcal{G}^{(k)}$, and therefore the dimension of an orbit in general position, is three. The dimension of the space of k-jets is $\dim J^k(\pi) = C^k_{k+2} + 2$. Since the number of independent differential invariants of order $\leq k$ coincides with the codimension of the orbit in general position, it is equal to $\nu(k) = C^k_{k+2} - 1$. Therefore, the number of differential invariants whose order is k is $\mu(k) = \nu(k) - \nu(k-1) = k+1$. $\square$

The constructed second-order differential invariants $J_{11}, J_{21}, J_{12}, J_{22}$ are functionally dependent. Indeed, according to theorem 5.2, there should only be three differential second-order invariants. In our case

$$J_1^2 + J_2^2 - J_{12} + J_{21} = 0. \tag{5.2}$$

Applying invariant differentiations to second-order differential invariants, we obtain third-order invariants: $J_{ijk} = \nabla_i(\nabla_j(J_k)) \quad (i, j, k = 1, 2)$.

Compositions of the operators $\nabla_1 \circ \nabla_2$ and $\nabla_2 \circ \nabla_1$ give the same invariants modulo lower-order invariants. Thus, we get six third-order invariants: J_{111}, J_{112}, J_{211}, J_{212}, J_{221}, J_{222}.

We get 12 invariants whose order is not higher than three. According to the theorem 5.2, independent invariants of order ≤ 3 should be nine. Therefore, there must be three relations between them. The first relation is the relation (5.2). The other two we can obtain by applying the differential operators ∇_1 and ∇_2 to (5.2).

Similarly, we can obtain differential invariants of any order.

Theorem 5.3. *The algebra of differential invariants of the foliation of curves on the Euclidean plane is generated by differential invariants J_1 and J_2 and by two invariant differentiations ∇_1 and ∇_2.*

6. De Sitter plane

Like Minkowski geometry, de Sitter geometry is one of the formalizations of the theory of relativity. Model de Sitter universe was proposed in [6, 7]. Unlike Minkowski geometry, it is better suited to describe a nontrivial gravitational field, in particular, the expansion effect of the universe [8]. The problem of classifying curves on the de Sitter plane is considered in [9].

We consider the upper half-plane

$$M = \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 | y > 0\},$$

with the metric

$$g_{\mathrm{S}} = \frac{dx^2 - dy^2}{y^2}.$$

as a model of this space. this half-plane we call *de Sitter plane.*

The proper motions generate a 3-dimensional Lie group, which we denote by G_{S} and called *de Sitter group.* This group is generated by translations by x, the transformation

$$\Phi_t : x \longmapsto -\frac{2(-ty^2 + tx^2 - 2x)}{t^2x^2 - 4tx - t^2y^2 + 4}, \quad y \longmapsto \frac{4y}{t^2x^2 - 4tx - t^2y^2 + 4}$$

and homotheties.

The corresponding Lie algebra $\mathcal{G}_{\mathrm{S}}$ is generated by the vector fields

$$X = \frac{\partial}{\partial x}, \qquad Y = \left(\frac{x^2}{2} + \frac{y^2}{2}\right)\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \qquad H = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Their prolongations into the 1-jet space are

$$\begin{aligned}
X^{(1)} =&\frac{\partial}{\partial x},\\
Y^{(1)} =&\left(\frac{x^2}{2}+\frac{y^2}{2}\right)\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}+(-xu_x-yu_y)\frac{\partial}{\partial u_x}+(-yu_x-xu_y)\frac{\partial}{\partial y},\\
H^{(1)} =&x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-u_x\frac{\partial}{\partial u_x}-u_y\frac{\partial}{\partial y}.
\end{aligned}$$

In coordinates $x, y, v = v_{(0,0)}$ on $J^0(\pi)$, these vector fields have the form

$$\begin{aligned}
Y_1 =&\frac{\partial}{\partial x},\\
Y_2 =&\left(\frac{x^2}{2}+\frac{y^2}{2}\right)\frac{\partial}{\partial x}+xy\frac{\partial}{\partial y}+y(-1+v^2)\frac{\partial}{\partial v},\\
Y_3 =&x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}.
\end{aligned}$$

Theorem 6.1. *The operators*

$$\nabla_1=\frac{y}{\sqrt{1-v^2}}\left(v\frac{d}{dx}-\frac{d}{dy}\right)\quad \textit{and}\quad \nabla_2=\frac{y}{\sqrt{1-v^2}}\left(-\frac{d}{dx}+v\frac{d}{dy}\right)$$

are invariant differentiations.

Using formula (4.1), we get first order differential invariants

$$J_1=\frac{v-v^3+yvv_x-yv_y}{(-1+v^2)^{3/2}},\qquad J_2=\frac{yvv_y+v^2-1-yv_x}{(-1+v^2)^{3/2}}.$$

As above, we obtain second-order differential invariants by acting on them operators ∇_1 and ∇_2:

$$\begin{aligned}
J_{11} =&\frac{1}{\sqrt{1-v^2}(-1+v^2)^{5/2}}(v_{xx}v^4+(-2v_xy-2v_x^2)v^3+(5v_xv_y-v_{xx}+v_{yy})v^2+\\
&(-v_x^2-3v_y^2+2v_{xy})v-v_{yy}+v_xv_y)y^2,\\
J_{21} =&-\frac{1}{\sqrt{1-v^2}(-1+v^2)^{5/2}}(((yv_{xy}+v_x)v^4-(2(v_xv_y+\frac{1}{2}v_{yy}+\frac{1}{2}v_{xx}))yv^3+\\
&((2v_x^2+3v_y^2)y-2v_x)v^2-(4(v_xv_y-\frac{1}{4}v_{xx}-\frac{1}{4}v_{yy}))yv+(v_x^2-v_{xy})y+v_x)y),\\
J_{12} =&\frac{1}{\sqrt{1-v^2}(-1+v^2)^{5/2}}(((yv_{xy}-v_x)v^4-(2(v_xv_y+\frac{1}{2}v_{yy}+\frac{1}{2}v_{xx}))yv^3+\\
&((2v_y^2+3v_x^2)y+2v_x)v^2-(4(v_xv_y-\frac{1}{4}v_{xx}-\frac{1}{4}v_{yy}))yv+(-v_{xy}+v_y^2)y-v_x)y),\\
J_{22} =&-\frac{1}{\sqrt{1-v^2}(-1+v^2)^{5/2}}y^2(v_{yy}v^4+(-2v_{xy}-2v_y^2)v^3+(5v_xv_y+v_{xx}-v_{yy})v^2+\\
&(-3v_x^2+2v_{xy}-v_y^2)v+v_xv_y-v_{xx}).
\end{aligned}$$

Theorem 5.2 is true for de Sitter plane too.

Theorem 6.2. *The algebra of differential invariants of the foliation of curves on the de Sitter plane is generated by differential invariants J_1 and J_2 and by two invariant differentiations ∇_1 and ∇_2.*

Remark 6.3. The described method for finding differential invariants can be applied to problems of classifying curves foliations in other geometries. In addition, it is applicable to the classification problems of certain classes of differential equations (see, for example, [3]).

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